Exceptional Tensor Categories in Subfactor Theory

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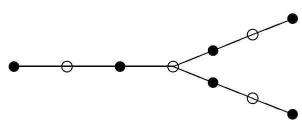
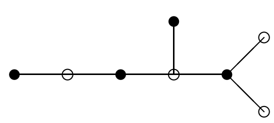
In subfactor theory, some exceptional tensor categories have been constructed by Asaeda, Haagerup and Izumi in [1] and Section 7 of [3]. These examples are not known in other approaches to CFT.

Tensor categories arise from subfactors in the following way: Given a finite-index inclusion of II_1 subfactors $N \subset M$, one of its most important invariants is the *principal graph*. The principal graph is a bipartite graph; it has even vertices $\mathcal{E} = \{$ (isomorphism classes of) irreducible $N-N$ bimodules ${}_N X_N$ which appear in the decomposition of $({}_N L^2(M)_N)^{\otimes n}$ for some $n \in \{0, 1, 2, \ldots\}\}$, and odd vertices $\mathcal{O} = \{$ (isomorphism classes of) irreducible $N-M$ bimodules ${}_N Y_M$ which appear in the decomposition of $({}_N L^2(M)_N)^{\otimes n} \otimes {}_N L^2(M)_M$ for some $n \in \{0, 1, 2, \ldots\}\}$, with all above tensors being taken over N. The even vertex ${}_N X_N$ connects to the odd vertex ${}_N Y_M$ with k edges if $k \cdot {}_N Y_M \subset {}_N X_N \otimes {}_N L^2(M)_M$. The bimodules of $\mathcal{E}$ are a tensor category, and the bimodules of $\mathcal{O}$ are a module category over $\mathcal{E}$. Similarly, one can define the *dual principal graph* of a subfactor as the inclusion/reduction graph of irreducible $M-M$ and $M-N$ bimodules contained in tensor powers (over M) of $L^2(M)$; this gives another pair of tensor and module categories. (See [4] for more details.)

If $N \subset M$ is a II_1 subfactor and has finite principal graph Λ, then one can show that the index $[M : N] = \|\Lambda\|^2$ (the norm of a bipartite graph is the operator norm of its adjacency matrix). It is a theorem of Jones ([5]) that if $N \subset M$ is a II_1 subfactor, then $[M : N] \in \{4\cos^2(\pi/n) | n \geq 3\} \cup [4, \infty]$. Haagerup proved in [2] that if additionally, $N \subset M$ is irreducible (i.e., the bimodule ${}_N L^2(M)_M$ is irreducible), and the principal graph of $N \subset M$ is finite, then

$$[M : N] \notin (4, \frac{5+\sqrt{13}}{2} \approx 4.303\ldots).$$

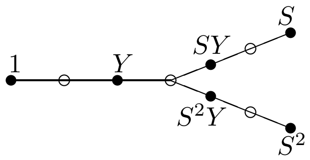
His proof relies on information about the Perron-Frobenius eigenvector of a principal graph, the fact that there are not so many bipartite graphs with index in this interval, and the relation between the dual and principal dual graph. Later, Asaeda and Haagerup proved in [1] that there are exactly two non-isomorphic subfactors of the hyperfinite II_1 factor having index $\frac{5+\sqrt{13}}{2}$; they each have their principal graph as one of

 , 

and their dual principal graph the other one. These subfactors are especially interesting because the tensor category they give is not known to come from anywhere else. Unlike most or all previously constructed subfactors, their construction does not start with a group or a quantum group or another known tensor category.

Asaeda and Haagerup's proof has three steps: First, they guess the fusion rules of $N-N$ subfactors by using the symmetries of the first graph. Then they construct a bimodule ${}_N X_M$ satisfying these rules (this is the difficult step); finally, they get a subfactor from ${}_N X_M$ by considering $L_X(N) \subset R_X(M)'$.

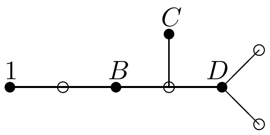
Here are the tensor categories that these graphs are describing. If we label the even vertices of the first graph like so



then the tensor category of even vertices has fusion rules

	1	S	S^2	Y	SY	S^2Y
1	1	S	S^2	Y	SY	S^2Y
S	S	S^2	1	SY	S^2Y	Y
S^2	S^2	1	S	S^2Y	Y	SY
Y	Y	S^2Y	SY	$Y+SY$ $+S^2Y+1$	$Y+SY$ $+S^2Y+S^2$	$Y+SY$ $+S^2Y+S$
SY	SY	Y	S^2Y	$Y+SY$ $+S^2Y+S$	$Y+SY$ $+S^2Y+1$	$Y+SY$ $+S^2Y+S^2$
S^2Y	S^2Y	SY	Y	$Y+SY$ $+S^2Y+S^2$	$Y+SY$ $+S^2Y+S$	$Y+SY$ $+S^2Y+1$

and if we label the even vertices of the second graph like so



then the tensor category of even vertices has fusion rules

	1	B	C	D
1	1	B	C	D
B	B	$1+B+C+D$	$B+D$	$B+C+2D$
C	C	$B+D$	$1+D$	$B+C+D$
D	D	$B+C+2D$	$B+C+D$	$1+2B+C+2D$

References

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[5] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25