1. Introduction

The absolute Galois group $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ has played a fundamental and oftentimes surprising part in the advancement of number theory and arithmetic: from its first injection into the study of roots of quintic polynomials, to its prominent role in the proof of Fermat’s Last Theorem. Accordingly Galois representations have become an important object of study, both as they afford us a glimpse into the nature of this mysterious group and as they arise frequently and naturally as useful tools for arithmetic problems.

Elliptic curves provide an excellent first example. The action of Galois on the ℓ-Tate modules of an elliptic curve gives rise to a family of 2-dimensional ℓ-adic representations. On the one hand, as the image of these representations is often large and nonabelian, the representations furnish a wealth of explicit nonabelian quotients of $G_{\mathbb{Q}}$ (or equivalently, explicit field extensions of $\mathbb{Q}$ with nonabelian Galois groups), thus shedding some light on the structure of $G_{\mathbb{Q}}$. On the other hand, the representations also encode much arithmetic information about the elliptic curve itself; they tell us about the rational torsion subgroup, for example, and detect the curve’s reduction type and isogeny class. Lastly, our understanding of these Galois representations can in turn be applied to Diophantine equations, usually to exclude the existence of nontrivial solutions. The trick here typically is to associate to a hypothetical integer solution of our given equation a tailor-made elliptic curve, and to then show that the Galois representations associated to this curve satisfy some impossible property. This was used most famously in the solution to Fermat’s Last Theorem, as well as in some close variants of the Fermat equation, including Dénès’ equation (cf. [DM97]).

From this first example we can spring in many different directions. We can play the same game, for example, with Tate modules of higher dimensional abelian varieties. And in fact, we can view all of these representations as members of a much larger class of representations “coming from geometry”: representations arising from the Galois action on the ℓ-adic étale cohomology of a $\mathbb{Q}$-variety. On the other hand, our discussion of the solution to the conjectures of Fermat and Dénès leads us in quite another direction. A major factor in the trick employed there actually coming off is the fact that the representations involved are modular: that is, isomorphic to ℓ-adic representations arising from modular forms. We know now more generally that all such representations associated to elliptic curves defined over $\mathbb{Q}$ fit into this larger family of modular representations. Amazingly these jumps into seemingly wildly different regions of Galois representations are intimately connected, as articulated by various conjectures by Serre, Fontaine-Mazur and Langlands. This is a very rich and exciting area of research, as witnessed by the recent explosion of activity which, among other advances, has resulted in a full proof of Serre’s conjecture.

2. Serre’s open image theorem

My research stems from Serre’s seminal work on representations arising from the action of Galois on the torsion points of abelian varieties. Given a principally polarized abelian variety $A/K$, with $K$ a number field, the action of the Galois group $G_K = \text{Gal}(\overline{K}/K)$ on the torsion points of $A/K$ gives
rise to continuous representations

\[ \rho_{A,\ell} : G_K \to \GSp_{2d}(\mathbb{Z}_\ell) \]
\[ \rho_A : G_K \to \GSp_{2d}(\hat{\mathbb{Z}}). \]

We refer to \( \rho_{A,\ell} \) and \( \rho_A \) respectively as the \( \ell \)-adic and adelic representations associated to \( A/K \). The images of these linear representations land inside \( \GSp \subseteq \GL \) thanks to the Galois-equivariance of the Weil pairing. As \( \rho_A \) is continuous and \( G_K \) is compact, the image \( \rho_A(G_K) \) is always a closed subgroup of \( \GSp_{2d}(\hat{\mathbb{Z}}) \). Serre proves in [Ser00] (cf. Théorème 3 in Lettre à Marie-France Vignéras) that if \( d \in \{2,6\} \cup \{1+2\mathbb{Z}\} \) and if \( \text{End}_{K}(A) = \mathbb{Z} \), then \( \rho_A(G_K) \) is in fact open in \( \GSp_{2d}(\hat{\mathbb{Z}}) \). We will refer to this result as Serre’s open image theorem (SOIT). There are two equivalent formulations of SOIT which highlight respectively its adelic and \( \ell \)-adic flavors. The first (adelic) version states that if \( A/K \) satisfies the conditions above, then \( \rho_A(G_K) \) is of finite index in \( \GSp_{2d}(\hat{\mathbb{Z}}) \). The second (\( \ell \)-adic) version states that for such \( A/K \), the \( \ell \)-adic image \( \rho_{A,\ell}(G_K) \) is open in \( \GSp_{2d}(\mathbb{Z}_\ell) \) for all \( \ell \) and equal to \( \GSp_{2d}(\mathbb{Z}_\ell) \) for almost all \( \ell \).

The adelic and \( \ell \)-adic formulations of SOIT in turn give rise to two basic categories of questions. Since an open subgroup of a profinite group is of finite index, the adelic version of SOIT naturally leads us to ask what this index is. For example, are there abelian varieties for which the index is 1? Similarly, from the \( \ell \)-adic version it follows that for \( A/K \) satisfying the conditions of SOIT there are only finitely many \( \ell \) for which \( \rho_\ell \) is not surjective. Let us restrict for the moment to the \( d=1 \) case and consider an elliptic curve \( E/K \) without complex multiplication (\( \text{End}_{\mathbb{R}}(E) = \mathbb{Z} \)). For each such \( E/K \) there is a smallest positive integer \( c(E/K) \) such that \( \rho_\ell \) is surjective for all \( \ell > c(E/K) \). The question then arises: how does \( c(E/K) \) vary with \( E \) and \( K \)? More to the point, we can ask whether \( c(E/K) \) is bounded for a fixed number field \( K \). In other words, is there a constant \( S(K) \) (the Serre constant) such that for all non-CM elliptic curves \( E/K \) and all \( \ell > S(K) \) the \( \ell \)-adic representation \( \rho_\ell \) is surjective?

This last question, the so-called Serre uniformity problem, has generated a lot of research in the \( K = \mathbb{Q} \) case suggesting that in fact such a constant does exist. Serre himself has proposed 37 as a candidate for \( S(\mathbb{Q}) \), and his guess is borne out by all elliptic curves \( E/\mathbb{Q} \) with conductor less than 40,000. The general strategy here, initiated by B. Mazur, is to translate everything into a problem about points on modular curves. If \( \ell \geq 17 \) is an exceptional prime of an elliptic curve \( E/\mathbb{Q} \), then the mod \( \ell \) image \( \overline{\rho}_\ell(G_{\mathbb{Q}}) \subset \GL_2(F_\ell) \) is contained in a Borel subgroup or in the normalizer of a split or nonsplit Cartan subgroup. For each of these three subgroup types \( T_i \) with \( i \in \{1,2,3\} \), and for each prime \( \ell \), there is a modular curve \( X_{T_i}(\ell) \) parametrizing the elliptic curves whose mod \( \ell \) image is contained in the corresponding subgroup. We can solve the uniformity problem by showing that for \( \ell \gg 0 \) the only rational points on the curves \( X_{T_i}(\ell) \) are CM or cuspidal. Using this strategy, Mazur showed in [Maz78] that if \( \rho_\ell(G_{\mathbb{Q}}) \) is contained in a Borel subgroup, then \( \ell \leq 37 \). In an exciting recent development ([BPR11]), Bilu, Parent and Rebolledo neatly do away with the split Cartan subcase of the uniformity problem for \( K = \mathbb{Q} \). Namely, they prove that for all non-CM elliptic curves \( E/\mathbb{Q} \) and all primes \( \ell \geq 11, \ell \neq 13 \), the image of the \( \ell \)-adic representation \( \rho_\ell \) is not contained in the normalizer of a split Cartan subgroup. There are as yet no analogous results for the nonsplit situation, where the geometry of the relevant modular curve creates particular difficulties. Other approaches to the uniformity problem exploit the modularity of elliptic curves over \( \mathbb{Q} \). For example, using modular forms, A. C. Cojocaru ([Coj05]) was able to produce an explicit formula for a bound \( S(N,\mathbb{Q}) \) such that for any elliptic curve \( E/\mathbb{Q} \) with conductor \( N \), if \( \ell \) is exceptional for \( E \), then \( \ell \leq S(N,\mathbb{Q}) \).

3. Adelic images

3.1. Elliptic curves. When is the adelic representation surjective? We will continue to consider the \( d = 1 \) case for the moment. This means that our abelian variety is just an elliptic curve, and...
that $\text{GSp}_{2d} = \text{GSp}_2 = \text{GL}_2$. Motivated by the $\ell$-adic formulation of SOIT, we will call a prime $\ell$ an exceptiona

Given an elliptic curve $E/K$, an obvious necessary condition for $\rho$ to be surjective is that $\rho_{E,\ell}(G_K) = \text{GL}_2(\mathbb{Z}_\ell)$ for all $\ell$; i.e., the elliptic curve must have no exceptional primes. This condition is in no way sufficient. Indeed, William Duke has shown that “most” elliptic curves $E/\mathbb{Q}$ have no exceptional primes, and yet as Serre showed, for any elliptic curve $E/\mathbb{Q}$ the corresponding adelic image $\rho_E(G_\mathbb{Q})$ is contained in a subgroup $S_E \subseteq \text{GL}_2(\hat{\mathbb{Z}})$ of index 2, the so-called Serre subgroup of $E$. In other words, over $\mathbb{Q}$ there are no elliptic curves admitting a surjective adelic Galois representation.

In the paper [Gre10], I investigate the possibility of $\rho$ being surjective for elliptic curves defined over number fields other than $\mathbb{Q}$. I first consider the related (profinite) group theoretic question of determining when a closed subgroup $H \subseteq \text{GL}_2(\hat{\mathbb{Z}})$ is in fact all of $\text{GL}_2(\hat{\mathbb{Z}})$. As above, there is the obvious necessary condition that all the projections $\pi_\ell : H \rightarrow \text{GL}_2(\mathbb{Z}_\ell)$ must be surjective. It turns out that this condition is not so far from being sufficient; we need only add the condition that $H$ surjects onto the abelianization of $\text{GL}_2(\hat{\mathbb{Z}})$. By translating this group theory into arithmo-geometric terms, we derive simple necessary and sufficient conditions for an elliptic curve $E/K$ to have a surjective adelic representation: namely,

**Theorem 3.1.** Let $E/K$ be an elliptic curve defined over a number field $K$. Let $\Delta_E$ be the discriminant of $E/K$ with respect to some Weierstrass model. Then $\rho_E(G_K) = \text{GL}_2(\hat{\mathbb{Z}})$ if and only if

(i) $\rho_{E,\ell}(G_K) = \text{GL}_2(\mathbb{Z}_\ell)$ for all $\ell$,

(ii) $K \cap \mathbb{Q}^{\text{cyc}} = \mathbb{Q}$ and

(iii) $\sqrt{\Delta} \notin K^{\text{cyc}}$.

With this theorem in hand, it is not difficult to find examples of number fields $K$ and elliptic curves $E/K$ which admit surjective adelic representations. The most challenging task involved is developing techniques for computing the set of exceptional primes of elliptic curves defined over number fields other than $\mathbb{Q}$. By considering the restriction of Galois representations to inertia subgroups, I was able to extend some techniques in [Ser72] for semistable curves over $\mathbb{Q}$ to certain cubic extensions $K$, allowing one to easily come up with examples of elliptic curves $E/K$ with surjective adelic representations. In [Gre10] one finds a fully computed example of a curve over $K = \mathbb{Q}(\alpha)$, where $\alpha$ is the real root of $x^3 + x + 1$, whose adelic representation is surjective. One application of this result, which may be of interest in the context of the inverse Galois problem, is that elliptic curves with surjective adelic representations give rise to explicit Galois extensions of $\mathbb{Q}(\alpha)$ with Galois group isomorphic to $\text{GL}_2(\hat{\mathbb{Z}})$.

While investigating this subject in Ph.D. thesis ([Gre07]) I also asked whether for appropriate number fields $K$ it was the case that in fact “most” (in the spirit of W. Duke) elliptic curves $E/K$ admitted a surjective adelic Galois representation. David Zywina ([Zyw08]) has since answered this in the affirmative using sieve techniques.

### 3.2. Abelian varieties

Much of the profinite group theory concerning $\text{GL}_2(\hat{\mathbb{Z}})$ that went into my work with elliptic curves generalizes nicely to $\text{GSp}_{2d}$ for larger $d$. In fact in some ways the group theory becomes easier as $d$ grows. For $d > 2$, for example, I can show that for a closed subgroup $H \subset \text{GSp}_{2d}(\hat{\mathbb{Z}})$, we have $H = \text{GSp}_{2d}(\hat{\mathbb{Z}})$ if and only if $H$ surjects onto $\text{GSp}_{2d}(\mathbb{Z}_\ell)$ for all $\ell$ and the multiplier map $m : \text{GSp}_{2d}(\hat{\mathbb{Z}}) \rightarrow \hat{\mathbb{Z}}^*$ is surjective when restricted to $H$. This is translated into arithmetic as follows.

**Proposition 3.2.** Let $K$ be a number field and let $A/K$ be a principally polarized abelian variety of dimension $d > 2$. Then $\rho_A$ is surjective if and only if

(i) $\rho_{A,\ell}(G_K) = \text{GSp}_{2d}(\mathbb{Z}_\ell)$ for all $\ell$ and
The $d = 2$ case bears closer resemblance to the $d = 1$ case in that the surjectivity of the multiplier map $m$ is no longer sufficient for $H$ to be equal to $\text{GSp}_{2d}(\hat{\mathbb{Z}})$; this is a result of the fact that the commutator subgroup of $\text{GSp}_{2d}(\hat{\mathbb{Z}})$ is an index-2 subgroup of $\text{Sp}_{2d}(\mathbb{Z})$ for $d \leq 2$. There is a corresponding similarity between the arithmetic of abelian surfaces and elliptic curves. A principally polarized abelian surface $A/K$ is isomorphic over $\overline{K}$ either to the product of two elliptic curves or to the Jacobian of a smooth projective curve of genus 2. Since we demand $\text{End}_{\overline{K}}(A) = \mathbb{Z}$, we are interested in the latter surfaces, and one can show that in this case $A$ is in fact isomorphic over $K$ to a Jacobian. We therefore consider abelian varieties $A = J(C)$. Without loss of generality, we may take $C/K$ to be hyperelliptic with affine model $y^2 = f(x)$, where $f \in K[x]$ is separable and $\deg f = 6$. Then, exactly as with elliptic curves, our group theory results yield the following proposition.

**Proposition 3.3.** Let $K$ be a number field and let $A/K$ be the Jacobian of a genus 2 hyperelliptic curve with affine model $y^2 = f(x)$, where $f \in K[x]$ is separable and $\deg f = 6$. Let $\Delta = \text{disc}(f)$. Then $\rho_A$ is surjective if and only if

1. $\rho_{A,\ell}(\hat{G}_K) = \text{GSp}_{2d}(\mathbb{Z}_\ell)$ for all $\ell$,
2. $K \cap \mathbb{Q}^{\text{cyc}} = \mathbb{Q}$ and
3. $\sqrt{\Delta} \notin K^{\text{cyc}}$.

**Corollary 3.4.** The adelic representation of a principally polarized abelian surface $A/\mathbb{Q}$ is never surjective.

From the proposition we see that the main work involved in showing that a given abelian surface $A/K$ has surjective adelic representation $\rho_A$ consists in showing that all the $\ell$-adic representations are surjective. Thanks to the corollary, we have the added difficulty that $A$ cannot be defined over $\mathbb{Q}$. There is currently an algorithm due to Dieulefait ([Die02]) which, given a principally polarized abelian surface $A/\mathbb{Q}$ with $\text{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$, produces a finite set of primes $S$ containing all the primes $\ell$ for which $\rho_{A,\ell}$ is not surjective. Using this algorithm and some additional surjectivity tests for the finite set of primes in the output set $S$, one can easily provide examples of abelian surfaces $A/\mathbb{Q}$, all of whose representations $\rho_\ell$ are surjective (cf. [JR09, Ex. 6.4]). The only detail of Dieulefait’s algorithm preventing its use over other number fields is its reliance on Serre’s conjecture (now a theorem) to exclude certain types of nonsurjective representations for all but finitely many $\rho_{A,\ell}$. Currently I am attempting to work around this by excluding these same representations through reduction-type conditions imposed on $A/K$. Once I have managed to extend Dieulefait’s algorithm in this manner, I am confident that it will not be difficult to give explicit examples of abelian surfaces with surjective adelic representations.

Analogues of Serre’s open image theorem have also been proved for abelian varieties with larger endomorphism rings $\mathcal{R} := \text{End}_{\overline{K}}(A)$: e.g., by Ribet for abelian varieties with real multiplication ([Rib76]), and more recently by G. Banaszak, W. Gajda and P. Krasoń ([BGK2, BGK3]), who prove open image theorems for a large class of simple abelian varieties whose $\mathbb{Q}$-endomorphism algebras $D := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q}$ fall into type I, II or III in the Albert list of division algebras with involution. In all of the covered cases the image $\rho_\ell(G_K)$ is shown to be open inside the subgroup $C_{\mathcal{R}}(\text{GSp}_{2d}(\mathbb{Z}_\ell))$ of elements of $\text{GSp}_{2d}$ commuting with the action of $\mathcal{R}$. These results open up an extended setting in which to examine adelic questions. Namely we can ask how big $\rho_A(G_K) = (\prod \rho_{A,\ell}(G_K))$ can be inside $C_{\mathcal{R}}(\text{GSp}_{2d}(\hat{\mathbb{Z}}))$ and endeavor to come up with fully computed examples. As a first interesting case we can consider a simple abelian surface $A/K$ for which $D$ is either a real quadratic field or an indefinite quaternion algebra.

Lastly, it is also worth mentioning a more recent open image result due to Chris Hall in [Hal11], where he considers principally polarized abelian varieties $A/K$ of dimension $d$, whose endomorphism
There is a recipe due to Deligne which, for each prime \( \ell \) \( \epsilon \) Dirichlet character \( \ell \) Furthermore for each \( p \) Modular forms.

3.3. Mumford-Tate and étale cohomology. We can give a general description of the foregoing open image results as follows. For each prime \( \ell \) let \( G_\ell /\Q_\ell \) be the algebraic subgroup of \( \GL_{2d}/\Q_\ell \) associated to the Zariski closure of \( \rho_{A,\ell}(G_K) \) in \( \GL_{2d}(\Q_\ell) \). Then there is an algebraic group \( MT(A) \subseteq \GL_{2g} \) defined over \( \Q \) such that

\[
\begin{align*}
(\text{i}) \quad & G_\ell = MT(A)_{\Q_\ell} \text{ for all } \ell \text{ (after a choice of basis),} \\
(\text{ii}) \quad & \rho_{A,\ell}(G_K) \text{ is open inside } G_\ell(\Z_\ell) = MT(A)(\Z_\ell) \text{ for all } \ell, \text{ and} \\
(\text{iii}) \quad & \rho_{A,\ell}(G_K) = G_\ell(\Z_\ell) = MT(A)(\Z_\ell) \text{ for almost all } \ell.
\end{align*}
\]

For the abelian varieties covered by SOIT we have \( MT(A) = \GSp_{2d} \); for those treated in [BGK2,BGK3] we have \( MT(A) = CR(\GSp_{2d}) \). As suggested by the notation, the group \( MT(A) \) is in fact the Mumford-Tate group of \( A/K \). The cases described above not only illustrate how the Mumford-Tate group naturally bounds the size of the \( \ell \)-adic image of Galois, they are also examples where the Mumford-Tate conjecture is known to be true; i.e., we have \( G^\circ_\ell \simeq MT(A)_{\Q_\ell} \), where \( G^\circ_\ell \) denotes the connected component containing the identity.

Now recall that Tate module representations are just examples of representations arising from \( \ell \)-adic étale cohomology groups. Given a smooth projective variety \( X/K \) with \( K \) a number field, the \( \ell \)-adic cohomology groups \( H^k(X_\overline{K}, \Z_\ell) \) admit a Galois action, producing a family \( \rho = (\rho_\ell) \) of \( \ell \)-adic representations \( \rho_\ell : G_K \to \GL_{n(k)}(\Z_\ell) \), where the dimension \( n(k) \) is a constant independent of \( \ell \). In the special case where \( X/K = A/K \) is an abelian variety, we have \( H^1(A_\overline{K}, \Z_\ell) \) isomorphic to the dual of \( T_1(A) \) as \( G_K \)-modules.

How do we formulate an open image conjecture in this more general setting? Define \( G_\ell \) again to be the Zariski closure of \( \rho_\ell(G_K) \) in \( V_\ell := H^k(X_\overline{K}, \Z_\ell) \otimes_{\Z_\ell} \Q_\ell \). Then exactly as above it is conjectured that \( G^\circ_\ell \simeq MT(\rho)_{\Q_\ell} \), where \( MT(\rho) \) is the Mumford-Tate group associated to \( H^m(X(\C), \Q) \). Concerning the size of the \( \ell \)-adic images, one has to be slightly more careful; it is possible for example, that \( \rho_\ell(G_K) \) sits inside the \( \Q_\ell \)-points of an algebraic group related to \( G_\ell \) by a nontrivial isogeny. However, Serre conjectures at least ([Ser77,Ser94]) that if we assume additionally that \( G_\ell \) is connected for all \( \ell \), then \( \rho_\ell(G_K) \) is open in \( MT(\rho)_{\Q_\ell}(\Z_\ell) \) for all \( \ell \), and of bounded index.

Michael Larsen and Richard Pink have made a surprising amount of progress on this problem simply by axiomatizing the situation: extracting from our family of cohomological representations the two salient properties that it forms a (strictly) compatible system in the sense of Serre and that the representations are pure of weight \( k \). For example, assuming all the \( G_\ell \) are reductive, they can show that the quotients \( G_\ell/G^\circ_\ell \) are all isomorphic, and that after a finite extension of \( K \) the groups \( G_\ell \) are all connected ([LP92]); and in the case of abelian varieties they can prove ([LP95]) that if there exists a single prime \( \ell_0 \) for which \( G^\circ_{\ell_0} \simeq MT(A)_{\Q_{\ell_0}} \), then \( G^\circ_\ell \simeq MT(A)_{\Q_\ell} \) for all primes \( \ell \).

3.4. Modular forms. Let \( f = \sum_{n=1}^{\infty} a_n(f) q^n \) be a newform in \( S_k(\Gamma_1(N)) \) of weight \( k > 1 \) and Dirichlet character \( \epsilon \). Let \( K_f = \Q(a_n(f) : n \geq 1) \) be its coefficient field, with ring of integers \( \O_f \).

There is a recipe due to Deligne which, for each prime \( \ell \), attaches to \( f \) a continuous \( \ell \)-adic Galois representation

\[
\rho_{f,\ell} : G_\Q \to \GL_2(\O_f \otimes \Z_\ell).
\]

Furthermore for each \( p \nmid lN \) and each Frobenius element \( \Frob_p \), we have \( \text{tr}(\rho_\ell(\Frob_p)) = a_p \) and \( \text{det}(\rho_\ell(\Frob_p)) = \epsilon(\Frob_p) p^{k-1} \), where we think of \( \epsilon \) as a character \( \epsilon : G_\Q \to K^*_f \). The latter property,
Recall that a simple abelian variety \( A / \mathbb{Q} \) is a field of dimension of abelian varieties over \( \rho : \text{End} A \rightarrow \mathbb{Q} \) the image \( \rho \). Rational points on \( \text{BPR11} \) Yu. Bilu, P. Parent, and M. Rebolledo, Rational points on \( X_0^+(p') \) (2011). arXiv:1104.4641v1 [math.NT].

Consider first the \( N = 1 \) case. Here the Dirichlet character \( \epsilon \) is trivial, and for all \( \ell \) we have the image \( \rho_f,\ell(G\mathbb{Q}) \) contained in the subgroup \( A_\ell = \{ g \in \text{GL}_2(\mathcal{O}_f \otimes \mathbb{Z}_\ell) : \det g \in (\mathbb{Z}_\ell^\times)^{k-1} \} \). Ribet was able to prove in this case that in fact

1. \( \rho_f,\ell(G\mathbb{Q}) \) is open in \( A_\ell \) for all \( \ell \), and
2. \( \rho_f,\ell(G\mathbb{Q}) = A_\ell \) for \( \ell \gg 0 \).

We can package these \( \ell \)-adic representations into one adelic representation \( \rho_f := \prod \rho_f,\ell. \) Since each \( \rho_f,\ell \) is unramified outside \( \ell \), it follows that \( \rho_f(G\mathbb{Q}) = \prod \rho_f,\ell(G\mathbb{Q}) \). From results (i) and (ii) above it now follows that \( \rho_f(G\mathbb{Q}) \) is an open subgroup of the adelic group \( \prod_{\ell \text{ prime}} A_\ell. \)

This result has been generalized to all levels \( N \geq 1 \), thanks to additional work by Ribet, F. Momose and E. Papier (cf. \[\text{Rib85}\]). In higher levels the situation is complicated by the potential existence of complex multiplication and inner twists. The natural analogue of the \( N = 1 \) theorem actually restricts the representations to a certain open subgroup \( H \subset G\mathbb{Q} \) and considers the \( \ell \)-adic images inside \( (D \otimes \mathbb{Q}_\ell)^* \), where \( D \) is a central simple algebra over a certain subfield \( F_f \subset K_f \).

It would be interesting to formulate and prove open image theorems about the corresponding adelic representation in this more general setting. For example, if the form \( f \) happens to have no CM and no inner twists (in which case the Dirichlet character \( \epsilon \) is trivial), then \( H = G\mathbb{Q}, F_f = K_f \) and \( D = M_2(K_f) \), and after a choice of basis, properties (i) and (ii) are true of the \( \rho_f,\ell \) exactly as in the \( N = 1 \) case. In the yet more specialized case of a non-CM newform \( f \) without inner twists and of weight \( k = 2 \) there is an algorithm due to Dieulefait and Vila ([DV00]) that computes a finite set \( S \) containing all primes \( \ell \) for which \( \rho_f,\ell(G\mathbb{Q}) \neq A_\ell \). However when \( N > 1 \) the question remains as to what the adelic image \( \rho_f(G\mathbb{Q}) \subset \prod \rho_f,\ell(G\mathbb{Q}) \subset \prod A_\ell = A \) is. Indeed we can ask if the group theory of the \( A_\ell \) and properties (i) and (ii) are enough to ensure \( \rho_f(G\mathbb{Q}) \) is open in \( A \).

In another direction, if we allow inner twists but fix \( k = 2 \), an open image theorem regarding \( \rho_f \) for a non-CM newform \( f \in S_2(\Gamma_1(N)) \) would translate directly to a result about a certain class of abelian varieties over \( \mathbb{Q} \). Associated to such an \( f \) is a simple abelian variety \( A_f / \mathbb{Q} \) of \( \text{GL}_2 \)-type. Recall that a simple abelian variety \( A / \mathbb{Q} \) is of \( \text{GL}_2 \)-type if the \( \mathbb{Q} \)-endomorphism algebra \( \text{End}_\mathbb{Q}(A) \otimes \mathbb{Q} \) is a field of dimension \( d = \dim A \) over \( \mathbb{Q} \). In the present situation we have \( \dim A_f = \dim \mathbb{Q} K_f \) and \( \text{End}_\mathbb{Q}(A_f) \otimes \mathbb{Q} \approx K_f \). Furthermore since the action of \( G\mathbb{Q} \) on the the \( \ell \)-Tate module \( T_\ell(A_f) \) is \( \mathcal{O}_f \otimes \mathbb{Z}_\ell \)-invariant, the \( \ell \)-adic representation \( \rho_{A_f,\ell} \) maps to \( \text{GL}_2(\mathcal{O}_f \otimes \mathbb{Z}_\ell) \) and is in fact isomorphic to the modular representation \( \rho_{f,\ell} \). Conversely any simple \( A / \mathbb{Q} \) of \( \text{GL}_2 \)-type is isogenous to \( A_f \) for some \( N \geq 1 \) and some \( f \in S_2(\Gamma_1(N)) \), as Ribet proves in \[\text{Rib04}\] assuming Serre’s conjecture. Under this correspondence between newforms and abelian varieties, the simple abelian varieties \( A / \mathbb{Q} \) of \( \text{GL}_2 \)-type arising from a CM newform are those which contain a subvariety of CM-type, or equivalently, those for which \( A_{\mathbb{Q}} \) is isogenous to a power of a CM elliptic curve. Thus an open image theorem about the adelic representations associated to non-CM newforms \( f \in S_2(\Gamma_1(N)) \) would yield an open image theorem about the adelic representations associated to abelian varieties \( A / \mathbb{Q} \) of \( \text{GL}_2 \)-type which are not isogenous over \( \mathbb{Q} \) to a power of a CM elliptic curve.

References


RESEARCH STATEMENT


