Instructions. There are no special instructions for these practice problems. The abbreviation ‘CF’ means context-free.

1. Be sure to know how to define (or state in the case of theorems) each of the following. (You must give definitions and statements like the ones presented in class. Check your notes!)
   - left- and right-linear grammars
   - regular grammar
   - CF grammar
   - CF language
   - yield of a derivation tree
   - leftmost and rightmost derivations of a string
   - contrapositive of the Pumping Lemma for CF languages
   - nondeterministic pushdown automaton $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$
   - instantaneous description $(q, w, u)$ of an npda
   - $L(M)$ for an npda $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$
   - $N(M)$ for an npda $M = (Q, \Sigma, \Gamma, \delta, q_0, z, \varnothing)$
   - deterministic npda (see problem 7)
   - instantaneous description $uqv$ of a Turing machine
   - language $L(M)$ of a Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, F)$

2. Let $\Sigma$ be an input alphabet. Write down a CF grammar for the set of all regular expressions over $\Sigma$. This proves that the set of all regular expressions over an input alphabet $\Sigma$ is a CF language.

3. Consider the grammar $G = (\{S, A, B\}, \{0, 1\}, P)$ where $P = \{S \rightarrow A1B, A \rightarrow 0A | \lambda, B \rightarrow 0B | 1B | \lambda\}$.
   (a) Determine $L(G)$.
   (b) Construct a left-linear grammar that is equivalent to $G$.

4. Prove that the language $L = \{a^ib^jc^k : i < j < k\}$ is not CF by using the Pumping Lemma for CF languages.

5. Consider the language $L = \{0^n1^n : n \geq 0\}$. Show that when we try to apply the Pumping Lemma for CF languages to $L$, the adversary always wins regardless of what actions we take. Explain?
6. Prove that the language \( L = \{a^3b, a^9b^3, a^{15}b^5, a^{21}b^7, \ldots \} \) is CF by completing the following three steps.

(a) Prove that the language \( L_1 = \{w \in \{a,b\}^* : w \) contains an odd number of \( a \} \) is regular.

(b) Prove that the language \( L_2 = \{a^{3i}b^i : i \geq 0 \} \) is CF.

(c) Using steps (a) and (b) above, prove that \( L \) is CF.

7. For the npda \( M = (\{q_0, q_1\}, \{0, 1\}, \{A, B, z\}, \delta, q_0, z, \emptyset) \) accepting the language \( L = \{ww^R : w \in \{0, 1\}^* \} \) discussed in class, do a complete trace (like the diagram distributed in class) of the input string 011110.

8. Construct an npda \( M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \) to recognize the language \( L = \{w \in \{a, b\}^* : n_a(w) = n_b(w) + 1 \} \) by final state. That is, \( L(M) = L \).

9. An npda \( M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \) is called deterministic if it satisfies the following two conditions:

- \( \delta(q, a, b) \) has at most one member (remember that \( \delta(q, a, b) \) is, in general, a set) for any \( q \in Q, a \in \Sigma \cup \{\lambda\} \) and \( b \in \Gamma \), and
- if \( \delta(q, a, b) \) is nonempty for some \( a \in \Sigma \), then \( \delta(q, \lambda, b) \) is empty.

Deterministic npdas are called often called dpda. Explain what these conditions mean in English.

10. Prove that if \( L \) is a regular language, then \( L = L(M) \) for some dpda \( M \).

11. Let \( L = \{wcw^R \in \{0, 1, c\}^* : w \in \{0, 1\}^* \} \). Prove that \( L = L(M) \) for some dpda \( M \) (see problem 9).

12. For the Turing machine constructed in class that recognizes the language \( L = \{a^n b^n : n \geq 0 \} \), show the complete list of ID, for the input string 00111.

13. Construct a Turing machine \( M = (Q, \{a, b\}, \{a, b, \square\}, \delta, q_0, \square, F) \), called a shift left machine, that simply shifts the input string one space to the left on the tape. An example input is shown below.

\[
\text{before:} \\
\begin{array}{cccccccc}
\square & \square & a & b & b & a & b & \square & \square & \square
\end{array}
\]

\[
\text{after:} \\
\begin{array}{cccccccc}
\square & a & b & b & a & b & \square & \square & \square & \square
\end{array}
\]

Complete the description of \( M \). In particular, specify \( Q, F \) and \( \delta \).

14. Design a Turing machine \( M = (Q, \Sigma, \Gamma, \delta, q_0, \square, F) \) that recognizes the language \( L = \{w \in \{a, b\}^* : n_a(w) = n_b(w) \} \). To answer the question properly, you must identify each component of the 7-tuple!
15. Turing machines not only recognize languages but also can do arithmetic and other
types of computations. Design a Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$
that takes an input number $N$, represented in binary, and adds +1 to it. Specifically, the tape
initially contains the symbol $\$ followed by the number $N$ in binary. The tape head is
initially scanning the $\$ in state $q_0$. Your Turing machine should halt with $N + 1$ in
binary, on its tape, scanning the leftmost symbol of $N + 1$, in a final state $q_f$. You may
destroy the symbol $\$ in creating $N + 1$, if necessary. For example, $q_0\$10011$ $\&$ $q_f10100$
and $q_0\$11111$ $\&$ $q_f100000$. (Essentially, the symbol $\$ is needed as a placeholder for a
possible carry out of 1.)

16. Consider the Turing machine $M = (\{q_0, q_1, q_2, q_f\}, \{0, 1\}, \{0, 1, \Box\}, \delta, q_0, \Box, \{q_f\})$ where
$\delta(q_0, 0) = (q_1, 1, R), \delta(q_1, 1) = (q_0, 0, R)$ and $\delta(q_1, \Box) = (q_f, \Box, R)$. Describe $L(M)$. 