Instructions. Problems 2 through 5 are Comp 211 material.

1. Be sure to know how to define (or state in the case of theorems) each of the following. (You must give definitions and statements like the ones presented in class.)
   - Hilbert’s Tenth Problem
   - alphabet
   - length of a string
   - formal language
   - complement of a language
   - *-closure of a language
   - concatenation of two languages
   - difference of two languages
   - reverse of a language
   - a finite automaton (i.e., the 5-tuple definition including the meaning of each member of the tuple)
   - the function $\delta^*$
   - language of a finite automaton $M$
   - regular language
   - Pumping Lemma for regular languages
   - contrapositive of the Pumping Lemma for regular languages

2. Let $A$, $B$ and $C$ be sets. If $C \subseteq A$, prove that $(A \cap B) \cup C = A \cap (B \cup C)$ by completing the following steps. ($\textbf{Hint:}$ For (a) and (b), take an $x$ in the left-hand side and show it is also in the right-hand side.)
   (a) Prove $(A \cap B) \cup C \subseteq A \cap (B \cup C)$
   (b) Prove $A \cap (B \cup C) \subseteq (A \cap B) \cup C$
   (c) Is the result still true if $C \not\subseteq A$? If not, give a counterexample.

3. Prove that the sets $X = \{n^3 + 3n^2 + 3n : n \geq 0\}$ and $Y = \{n^3 - 1 : n > 0\}$ are equal.

4. Prove the following formula for all integers $n$, $n \geq 1$, by induction by completing the steps below.
   \[
   \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}
   \]
   (a) State and prove the \textit{BASE CASE}.
(b) State the **INDUCTION HYPOTHESIS**.
(c) Make the **INDUCTION STEP**.

5. A binary tree is called *strict* if every node is either a leaf node (no children) or has exactly two children. Prove by induction on the number of leaf nodes than a strict binary tree with \( n \) leaf nodes has \( 2n - 1 \) nodes in total.

6. If \( L = \{a, b, c\} \), how many strings are there in \( L^3 \)? In \( L^9 \)? In \( L^n \)? In \( L^* \)?

7. If \( L = \{aa, bb, ab, ba\} \), describe in words the language \( L^* \).

8. Construct a finite automaton \( M \) such that \( L(M) = L \) where \( L \) is the language below.

\[ L = \{w \in \{a,b\}^* : w \text{ contains the substring } bb \text{ or does not contain the substring } aa\} \]

9. Construct a finite automaton \( M \) such that \( L(M) = L \) where \( L \) is the language below.

\[ L = \{w \in \{a,b\}^* : w \text{ contains the substring } ab \text{ and the substring } ba\} \]

Be careful about overlapping substrings. For example, the automaton should accept the string \( w = aba \).

10. Construct finite automata that accept each of the following languages:
(a) \( L_1 = \{w \in \{a,b\}^* : \text{each } a \text{ is preceded by at least one } b\} \)
(b) \( L_2 = \{w \in \{a,b\}^* : w \text{ has } abab \text{ as a substring}\} \)
(c) \( L_3 = \{w \in \{a,b\}^* : w \text{ has neither } aa \text{ nor } bb \text{ as a substring}\} \)

11. Prove that the language of ‘balanced parentheses’ is not regular using the Pumping Lemma. More precisely, if \( \Sigma = \{(,\}) \), the language of balanced parentheses is \( L = \{w \in \Sigma^* : w \text{ is balanced}\} \). For example, the string \((())(()) \in L \) but \(((((() \notin L.\)

12. Use the Pumping Lemma to prove that the language \( L = \{0^{2^n} : n \geq 0\} \) is not regular. **(Hint:** Given a positive integer \( m \), let \( w = 0^{2^m} \). Now show for any decomposition \( w = xyz \) such that \(|y| \geq 1 \) and \(|xy| \leq m \), the string \( y \) cannot be ‘pumped up’. You might find the relation \( 2^m > m \) useful in your proof.)**