Math 315 Practice Problems Since Test #2
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Dr. John G. Del Greco

Instructions. There are no special instructions for these practice problems.

1. Be sure you know about each of the following.
   - dot product in \( \mathbb{R}^n \) and \( \mathbb{C}^n \)
   - inner product
   - positivity, definiteness, additivity, homogeneity, conjugate symmetry, conjugate homogeneity
   - norm of a vector
   - orthogonal vectors
   - Pythagorean Theorem
   - Cauchy-Schartz Inequality
   - Parallelogram Law
   - Triangle Inequality
   - orthonormal set of vectors
   - Gram-Schmidt Orthonormalization Process
   - orthonormal basis
   - linear functional
   - orthogonal complement of a subspace
   - orthogonal projection \( p_U \) of \( V \) onto \( U \)
   - adjoint
   - conjugate transpose
   - Hermitian operator
   - normal operator
   - Complex Spectral Theorem
   - Real Spectral Theorem

2. True or false?
   (a) If \((x, y) = 0\) for all \( x \) in an inner-product space, then \( y = 0 \).
   (b) Every orthogonal set of vectors is linearly independent.
   (c) The adjoint of a linear operator is unique.

3. Let \( V \) be an inner-product space. If \( T \in \mathcal{L}(V) \) and \( \|T(v)\| = \|v\| \) for each \( v \in V \), prove that \( T \) is injective.
4. Why isn’t \((a, b), (c, d) = ac - bd\) an inner product on \(\mathbb{R}^2\)?

5. Use The Gram-Schmidt procedure to find an orthonormal set \(S\) of vectors in \(\mathbb{R}^4\) such that \(\text{Span}(S) = \text{Span}\{(1, 0, 1, 0), (1, 1, 1, 1), (0, 1, 2, 1)\}\).

6. Let \(B\) be a basis for a subspace \(W\) in a finite-dimensional inner-product space \(V\). Prove that \(z \in W^\perp\) if and only if \((z, v) = 0\) for every \(v \in B\).

7. Let \(V = \varphi_2(\mathbb{R})\) where \((f, g) = \int_{-1}^{1} f(t)g(t)dt\). It is easy to show that

\[
B = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} t, \sqrt{\frac{5}{8}} (3t^2 - 1) \right\}
\]

is an orthonormal basis for \(V\).

(a) Write \(h(t) = 1 + 2t + 3t^2\) as a linear combination of the vectors in \(B\).

(b) Let \(W = \varphi(\mathbb{R})\). Find the orthogonal projection of the vector \(v(x) = t^3\) onto the subspace \(V\).

8. Let \(V = C[0, 1]\), the set of continuous functions on the interval \([0, 1]\). Define \((f, g) = \int_{0}^{1} f(t)g(t)dt\) for \(f, g \in V\). Let \(W = \text{Span}\{t, \sqrt{t}\}\).

(a) Find an orthonormal basis for \(W\).

(b) Find \(h(t) \in W\) such that \(\|t^2 - h(t)\|\) is a minimum.

9. Let \(V = \varphi_2(\mathbb{R})\) where \((f, g) = \int_{0}^{1} f(t)g(t)dt\). Define a linear functional \(\varphi : V \to \mathbb{R}\) by \(\varphi(f(t)) = f(0) + f'(1)\). Find \(h \in \varphi_2(\mathbb{R})\) such that \(\varphi(f) = (f, h)\).

10. If \(T\) is any operator, show that \(T + T^*\) and \(TT^*\) are both Hermitian.

11. Show that if \(TT^* = 0\), then \(T = 0\).

12. Let \(V\) be an inner-product space, and suppose \(T : V \to V\) defined by \(T(v) = (v, w)z\) where \(w\) and \(z\) are fixed elements of \(V\).

(a) Show \(T\) is linear.

(b) Find an explicit expression for \(T^*\).

13. Suppose that \(T\) and \(U\) are Hermitian. Show that \(TU\) is Hermitian if and only if \(TU = UT\).

14. Suppose that \(V = \mathbb{R}^2\), and define \(T : \mathbb{R}^2 \to \mathbb{R}^2\) by \(T(a, b) = (2a - 2b, -2a + 5b)\). Is \(T\) Hermitian, normal, neither?