Vector Spaces Every Student Should Know

The following vector spaces are important examples every student should know after having taken a first course in linear algebra. In particular, you should be able to verify that all eight properties of a vector space are satisfied! In each case, we use $\oplus$ and $\odot$ to denote addition and scalar multiplication merely to emphasize that these operations do not necessarily correspond to the usual vector addition and scalar multiplication in $\mathbb{R}^n$!

• **Positive Real Numbers**

Let $\mathbb{R}^+$ denote the positive real numbers. For $x, y \in \mathbb{R}^+$, define $x \oplus y = xy$ and for $c \in \mathbb{R}$, define $c \odot x = x^c$. That is, to add two vectors (positive real numbers in this case), we multiply them as ordinary real numbers, and to perform scalar multiplication, we use exponentiation.

1. Are all the vector space properties satisfied?
2. What is the $\mathbf{0}$ of this vector space?
3. For a positive real number $x$, what is $-x$?
4. What is a finite basis for this vector space?
5. What are some interesting subspaces?

• **$\mathbb{R}^n$**

For $x, y \in \mathbb{R}^n$, let $x \oplus y = z$ where $z_i = x_i + y_i$ (componentwise addition), and for $c \in \mathbb{R}$, $c \odot x = z$ where $z_i = cx_i$. This vector space is the prototype vector space from which much of the theory has been developed.

1. Are all the vector space properties satisfied?
2. What is the $\mathbf{0}$ of this vector space?
3. For a vector $x$, what is $-x$?
4. What is a finite basis for this vector space? (the unit vectors \(\{e_1, e_2, \ldots, e_n\}\), for example, is one)

5. What are some interesting subspaces?
   - \(\mathbb{C}^n\)
     
     For \(x, y \in \mathbb{C}^n\), let \(x \oplus y = z\) where \(z_i = x_i + y_i\) (componentwise addition), and for \(c \in \mathbb{C}\), \(c \odot x = z\) where \(z_i = cx_i\). (Recall how addition and multiplication of complex number are defined: if \(a + bi, c + di \in \mathbb{C}\), then \((a + bi) + (c + di) = (a + c) + (b + d)i\) and \((a + bi)(c + di) = (ac - bd) + (ad + bc)i\).)

1. Are all the vector space properties satisfied?

2. What is the 0 of this vector space?

3. For a vector \(x\), what is \(-x\)?

4. What is a finite basis for this vector space?

5. What are some interesting subspaces?
   - \(m \times n\) Matrices with Entries in \(\mathbb{R}\)
     
     Let \(\mathcal{M}_{m,n}(\mathbb{R})\) denote the set of all \(m \times n\) matrices with entries in \(\mathbb{R}\). For \(A, B \in \mathcal{M}_{m,n}(\mathbb{R})\), define \(A \oplus B = C\) where \(c_{ij} = a_{ij} + b_{ij}\) (componentwise addition) and for \(c \in \mathbb{R}\), \(c \odot A = D\) where \(d_{ij} = ca_{ij}\).

1. Are all the vector space properties satisfied?

2. What is the 0 of this vector space?

3. For a matrix \(A\), what is \(-A\)?

4. What is a finite basis for this vector space?

5. What are some interesting subspaces?
• **Polynomials**

Let \( \mathcal{P}(\mathbb{R}) \) denote the set of all polynomials in the variable \( t \) (or \( x \), etc). For \( p(t), q(t) \in \mathcal{P}(\mathbb{R}) \), define \((p \oplus q)(t) = p(t) + q(t)\), and for \( c \in \mathbb{R} \), define \((c \odot p)(x) = cp(x)\). That is, to add two vectors (polynomials in this case), we add them as ordinary functions. To perform scalar multiplication, just multiply the polynomial through by the scalar. For example, \((6t^2 - t + 1) \oplus (t^3 - t^2 + 2t - 1) = t^3 + 5t^2 + t\) and \(4 \odot (t^2 - 7t + 3) = 4t^2 - 28t + 12\).

1. Are all the vector space properties satisfied?
2. What is the \( \mathbf{0} \) of this vector space?
3. For a polynomial \( p(t) \), what is \((-p)(t)\)?
4. What is a finite basis for this vector space? Does it have one?
5. What are some interesting subspaces?

• **Polynomials with Degree \( \leq n \)**

Let \( \mathcal{P}_n(\mathbb{R}) \) denote the set of all polynomials in the variable \( t \) having degree at most \( n \). Define \( \oplus \) and \( \odot \) as for the vector space \( \mathcal{P}(\mathbb{R}) \).

1. Are all the vector space properties satisfied?
2. What is the \( \mathbf{0} \) of this vector space?
3. For a polynomial \( p(t) \), what is \((-p)(t)\)?
4. What is a finite basis for this vector space?
5. What are some interesting subspaces?

• **Continuous Functions on an Interval \([a, b]\) or \((-\infty, +\infty)\)**

Let \( \mathcal{C}[a, b] \) and \( \mathcal{C}(\infty, +\infty) \) denote the set of all continuous functions defined on a fixed interval \([a, b]\) and \(\infty, +\infty\) respectively. Define \( \oplus \) and \( \odot \) as for the vector space \( \mathcal{P}(\mathbb{R}) \).
1. Are all the vector space properties satisfied? In particular, are the closure properties satisfied?

2. What is the 0 of these vector spaces?

3. For a continuous function $f(x)$, what is $(-f)(x)$?

4. What is a finite basis for this vector space? Does it have one?

5. What are some interesting subspaces? (Note that $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}_n(\mathbb{R})$ are subspaces of $\mathcal{C}(-\infty, +\infty)$.)

- **Functions with Continuous First Derivatives on $(-\infty, +\infty)$**

  Let $\mathcal{C}^1(-\infty, +\infty)$ denote the set of all functions with continuous first derivatives on $(-\infty, +\infty)$. Define $\oplus$ and $\odot$ as for the vector space $\mathcal{C}(\mathbb{R})$.

  1. Are all the vector space properties satisfied? In particular, are the closure properties satisfied?

  2. What is the 0 of this vector space?

  3. For a function $f(x) \in \mathcal{C}^1(-\infty, +\infty)$, what is $(-f)(x)$?

  4. What is a finite basis for this vector space? Does it have one?

  5. What are some interesting subspaces? (Note that $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}_n(\mathbb{R})$ and are subspaces of $\mathcal{C}^1(-\infty, +\infty)$.)

- **Smooth Functions on $(-\infty, +\infty)$**

  Let $\mathcal{C}^\infty(-\infty, +\infty)$ denote the set of all functions with continuous derivatives of all orders on $(-\infty, +\infty)$. This class of functions are often referred to as smooth functions. Define $\oplus$ and $\odot$ as for the vector space $\mathcal{C}(\mathbb{R})$.

  1. Are all the vector space properties satisfied? In particular, are the closure properties satisfied?

  2. What is the 0 of this vector space?

  3. For a function $f(x) \in \mathcal{C}^\infty(-\infty, +\infty)$, what is $(-f)(x)$?
4. What is a finite basis for this vector space? Does it have one?

5. What are some interesting subspaces? (Note that $\wp(\mathbb{R})$ and $\wp_0(\mathbb{R})$ are subspaces of $C^\infty(-\infty, +\infty)$.)

• **Linear Transformations From $\mathbb{R}^n$ to $\mathbb{R}^m$**

Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. (This vector space is important in advanced algebra.) For $T_1, T_2 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, define $(T_1 \oplus T_2)(x) = T_1(x) + T_2(x)$, and for $c \in \mathbb{R}$, define $(c \odot T_1)(x) = cT_1(x)$ (note that $x$ is a vector in $\mathbb{R}^n$ and $T_1(x)$ and $T_2(x)$ are vectors in $\mathbb{R}^m$).

1. Are all the vector space properties satisfied? In particular, are the closure properties satisfied?

2. What is the $\mathbf{0}$ of this vector space?

3. For a linear transformation $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, what is $(-T)(x)$?

4. What is a finite basis for this vector space? Does it have one?

5. What are some interesting subspaces?

• **Linear Transformations From $(V, +, \cdot)$ to $(W, \boxplus, \boxdot)$**

(This vector space is nothing more than a direct generalization of the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$). Let $(V, +, \cdot)$ and $(W, \boxplus, \boxdot)$ be any two arbitrary vector spaces. Let $\mathcal{L}(V, W)$ denote the set of all linear transformations from $V$ to $W$. (As before, this vector space is important in advanced algebra.) For $T_1, T_2 \in \mathcal{L}(V, W)$, define $(T_1 \oplus T_2)(v) = T_1(v) \boxplus T_2(v)$, and for $c \in \mathbb{R}$, define $(c \odot T_1)(v) = c \boxdot T_1(v)$.

1. Are all the vector space properties satisfied? In particular, are the closure properties satisfied?

2. What is the $\mathbf{0}$ of this vector space?

3. For a linear transformation $T \in \mathcal{L}(V, W)$, what is $(-T)(x)$?

4. What is a finite basis for this vector space? Does it have one?

5. What are some interesting subspaces?