Hello students:

I would like to review the last example I gave in class. I am sorry I rushed through it, but I was nervous about the time. In my haste, I think I just managed to confuse everyone.

Example: Let $z = a$ be a complex number. Find all integrals of the form $\int_C \frac{dz}{z-a}$ where $C$ is a contour (oriented positively) not containing the point $z = a$.

Case I: $z = a$ is not in the interior of the contour.

In this case, the function $\frac{1}{z-a}$ is analytic on and throughout the interior of $C$. Therefore, by Cauchy’s Theorem $\int_C \frac{dz}{z-a} = 0$.

Case II: $z = a$ is in the interior of $C$.

First, let me remark that the antiderivative theorem cannot be used. The integrand $f(z) = \frac{1}{z-a}$ is certainly continuous on the domain $D = \mathbb{C} - \{a\}$. (\(\mathbb{C}\) is the complex numbers.) However, we need $f(z)$ to have an antiderivative on that domain. The natural choice is $\text{Log}(z-a)$. The problem is that $\text{Log}(z-a)$ has $\{x + iy : x \leq a_1 \text{ and } y = a_2\}$ as a branch cut where $a = a_1 + ia_2$. The branch cut (and in fact the branch cut of every branch of $\text{log}(z-a)$) intersects the domain $D$.

So, we have to use Cauchy’s Theorem to solve this integral. To do this, position a circle, call it $D$, centered about the point $z = a$ so that $D$ is completely interior to $C$. Consider the situation in Figure 1 (attached). Let

- $C_1$ = section of $C$ from $z_1$ to $z_2$ following the direction of $C$
- $C_2$ = section of $C$ from $z_2$ to $z_1$ following the direction of $C$
- $L_1$ = linear segment from $z_3$ to $z_1$
- $L_2$ = linear segment from $z_2$ to $z_4$
- $L_3$ = linear segment from $z_4$ to $z_2$
- $L_4$ = linear segment from $z_1$ to $z_3$
- $D_1$ = section of $D$ from $z_4$ to $z_3$ positively oriented
- $D_2$ = section of $D$ from $z_3$ to $z_4$ positively oriented

Consider the contour $F = (C_1, -L_3, -D_2, -L_4)$. The function $f(z) = \frac{1}{z-a}$ is analytic on and throughout the interior of this contour. Thus, by Cauchy’s Theorem,

$$\int_F \frac{dz}{z-a} = 0$$
However, observe that
\[
0 = \int_{C_1} \frac{dz}{z-a} + \int_{-L_3} \frac{dz}{z-a} + \int_{-D_2} \frac{dz}{z-a} + \int_{-L_4} \frac{dz}{z-a}
\]
\[
\Rightarrow 0 = \int_{C_1} \frac{dz}{z-a} - \int_{L_3} \frac{dz}{z-a} - \int_{D_2} \frac{dz}{z-a} - \int_{L_4} \frac{dz}{z-a}
\]
\[
\Rightarrow \int_{C_1} \frac{dz}{z-a} = \int_{L_3} \frac{dz}{z-a} + \int_{D_2} \frac{dz}{z-a} + \int_{L_4} \frac{dz}{z-a}
\]
Therefore, there is independence of path! Similarly, we have that
\[
\int_{C_2} \frac{dz}{z-a} = \int_{L_2} \frac{dz}{z-a} + \int_{D_1} \frac{dz}{z-a} + \int_{L_1} \frac{dz}{z-a}
\]
Therefore,
\[
\int_{C} \frac{dz}{z-a} = \int_{C_1} \frac{dz}{z-a} + \int_{C_2} \frac{dz}{z-a}
\]
\[
= \left[ \int_{L_3} \frac{dz}{z-a} + \int_{D_2} \frac{dz}{z-a} + \int_{L_4} \frac{dz}{z-a} \right] + \left[ \int_{L_2} \frac{dz}{z-a} + \int_{D_1} \frac{dz}{z-a} + \int_{L_1} \frac{dz}{z-a} \right]
\]
Now, here’s where the payoff is. Note that
\[
\int_{L_4} \frac{dz}{z-a} = -\int_{L_1} \frac{dz}{z-a}
\]
\[
\int_{L_3} \frac{dz}{z-a} = -\int_{L_2} \frac{dz}{z-a}
\]
Therefore, the above sum reduces to
\[
\int_{C} \frac{dz}{z-a} = \int_{D_2} \frac{dz}{z-a} + \int_{D_1} \frac{dz}{z-a}
\]
\[
= \int_{D} \frac{dz}{z-a}
\]
So we have reduced the integral over $C$ to an integral over a circle of fixed radius!

This was the content of problem 13(a) on p. 103. Just in case you did not get this problem, let us evaluate the integral $\int_{D} \frac{dz}{z-a}$. The circle has some fixed radius, say $R$. An admissible parametrization of $D$ is therefore $z(t) = a + R e^{it}$, $0 \leq t \leq 2\pi$. Therefore,
\[
\int_{D} \frac{dz}{z-a} = \int_{0}^{2\pi} \frac{dz}{a + R e^{it} - a} (R i e^{it})dt = \int_{0}^{2\pi} \frac{dz}{R e^{it}} (R i e^{it})dt = \int_{0}^{2\pi} i dt = 2\pi i
\]
Therefore, the final answer is
\[
\int_{C} \frac{dz}{z-a} = 2\pi i
\]
Let us review. We have found that
\[
\int_C \frac{dz}{z - a} = \begin{cases} 
0, & \text{if } z = a \text{ lies in the exterior of } C \\
2\pi i, & \text{if } z = a \text{ lies in the interior of } C
\end{cases}
\]

**Example:** Let’s do a very similar one. Evaluate
\[
\int_C \frac{3z - 2}{z^2 - z} \, dz
\]
where \( C \) is any positively-oriented contour containing the points \( z = 0, 1 \) in its interior.

Notice that \( f(z) = \frac{3z - 2}{z^2 - z} \) is analytic everywhere except at \( z = 0, 1 \). So, we center two circles, called \( D \) and \( F \), at \( z = 0, 1 \) respectively contained entirely in the interior of \( C \). Please see Figure 2 for all the labeling. \( C_1 \) is that part of \( C \) directed from \( z_1 \) to \( z_6 \), and \( C_2 \) is that part of \( C \) directed from \( z_6 \) to \( z_1 \). Note also that \( D \) is composed of two paths, \( D_1 \) and \( D_2 \), and \( F \) is composed of two paths, \( F_1 \) and \( F_2 \). So, reasoning as in the last example,
\[
\int_C \frac{3z - 2}{z^2 - z} \, dz = \int_{C_1} \frac{3z - 2}{z^2 - z} \, dz + \int_{C_2} \frac{3z - 2}{z^2 - z} \, dz
\]
\[
= \int_D \frac{3z - 2}{z^2 - z} \, dz + \int_F \frac{3z - 2}{z^2 - z} \, dz \quad \text{(after all the cancellations along the linear segments)}
\]

Using the technique of partial fractions (I bet you thought you were done with that stuff!), we see that
\[
\frac{3z - 2}{z^2 - z} = \frac{2}{z} + \frac{1}{z - 1}
\]
So we have that
\[
\int_C \frac{3z - 2}{z^2 - z} \, dz = \int_D \frac{2}{z} \, dz + \int_F \frac{1}{z - 1} \, dz
\]
\[
= \int_D \left( \frac{2}{z} + \frac{1}{z - 1} \right) \, dz + \int_F \left( \frac{2}{z} + \frac{1}{z - 1} \right) \, dz
\]
\[
= \int_D \frac{2}{z} \, dz + \int_D \frac{dz}{z - 1} + \int_F \frac{2}{z} \, dz + \int_F \frac{dz}{z - 1}
\]
\[
= 2(2\pi i) + 0 + 2\pi i
\]
\[
= 6\pi i
\]

Notice that the second and third integrals are zero since the points \( z = 1, 0 \) lie outside of \( D \) and \( F \) respectively. So our answer is
\[
\int_C \frac{3z - 2}{z^2 - z} \, dz = 6\pi i
\]
Example: One last example... Find
\[ \int_C \frac{dz}{z^2 - 1} \]
where \( C \) is any contour containing \( z = -1 \) but not \( z = 1 \) oriented positively.

Center a circle \( D \) at \( z = -1 \) totally contained in the interior of \( C \). See Figure 3 for all the labeling. \( C_1 \) is that part of \( C \) from \( z_1 \) to \( z_4 \), and \( C_2 \) is that part of \( C \) from \( z_4 \) to \( z_1 \). So,
\[
\int_C \frac{dz}{z^2 - 1} = \int_{C_1} \frac{dz}{z^2 - 1} + \int_{C_2} \frac{dz}{z^2 - 1} \\
= \int_{D_1} \frac{dz}{z^2 - 1} + \int_{D_2} \frac{dz}{z^2 - 1} \quad \text{(after linear segment cancellations)} \\
= \int_{D} \frac{dz}{z^2 - 1} \\
= \int_{D} \frac{dz}{2(z - 1)} - \int_{D} \frac{dz}{2(z + 1)} \quad \text{(using partial fractions)} \\
= 0 - \frac{1}{2}(2\pi i) \\
= -\pi i
\]
The first integral is 0 since \( D \) does not contain \( z = 1 \).