Instructions. There are no special instructions for these practice problems.

1. Be sure to know how to define (or state in the case of theorems) each of the following. (You must give definitions and statements like the ones presented in class.)

- Cauchy-Riemann equations
- Cauchy-Riemann equations in polar form
- $f(z)$ analytic at a point $z = z_0$
- entire function
- $e^z$
- $\sin(z)$, $\cos(z)$, etc.
- $\sinh(z)$, $\cosh(z)$, etc.
- $\log(z)$
- $\text{Log}(z)$
- branch of a multivalued function $f(z)$ in a domain $D$
- complex exponentiation
- smooth simple arc
- smooth closed arc
- admissible parametrization $z(t)$ for a smooth arc $C$
- contour $C = (C_1, C_2, \ldots, C_n)$
- admissible parametrization for a contour $C = (C_1, C_2, \ldots, C_n)$
- initial and terminal points of a contour $C = (C_1, C_2, \ldots, C_n)$
- closed contour
- simple closed contour
- $-C$ where $C$ is a contour
- $\int_C f(z)dz$ where $C$ is a directed smooth arc with admissible parametrization $z(t)$ on an interval $[a,b]$
- $\int_C f(z)dz$ where $C = (C_1, C_2, \ldots, C_n)$ is a contour
- ‘ML’ Theorem
- Antiderivative Theorem
- Green’s Theorem
- Cauchy’s Theorem
- Cauchy-Goursat Theorem
2. Use the Cauchy-Riemann equations to show that the function \( f(z) = 2y - ix \) is nowhere differentiable where \( z = x + iy \).

3. Discuss the differentiability and analyticity of the function \( f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y) \) where \( z = x + iy \).

4. Prove that \( \cos(z_2) - \cos(z_1) = 2\sin\frac{z_1 + z_2}{2} \sin\frac{z_1 - z_2}{2} \), and then use this result to show that \( \cos(z_2) = \cos(z_1) \) if and only if \( z_2 = \pm z_1 + 2\pi k, k \in \mathbb{Z} \). \( \text{Hint: Start with the right-hand side and derive the left-hand side.} \)

5. Solve the equation \( e^{2z} + e^z + 1 = 0 \). \( \text{Hint: The equation is quadratic.} \)

6. Let \( \mathcal{L}_a(z) = \ln(|z|) + i\theta, \alpha < \theta < \alpha + 2\pi \). So \( \mathcal{L}_a(z) \) is the branch of \( \log(z) \) that is analytic except at \( z = 0 \) and on the ray \( \theta = \alpha \). \( \text{Note that in this notation, } \log(z) = \mathcal{L}_{-\pi}(z). \)
   (a) Show that the function \( \mathcal{L}_\pi(z^2 + 1) \) is analytic at \( z = 0 \) and has value \( 2\pi i \) there.
   (b) Show that the function \( \mathcal{L}_{-\pi}(z^2 + 2z + 3) \) is analytic at \( z = -1 \). Find its derivative at \( z = -1 \).
   (c) Discuss the analyticity of \( \mathcal{L}_{\pi/2}(2z - 1) \).

7. Solve the equation \( \sin(z) = \cos(z) \). \( \text{Hint: } \tan(z) = 1. \)

8. Find \( f'(i) \) where \( f(z) \) is the principal branch of \( z^{1+i} \).

9. Evaluate \( \int_C (x - 2xyi) \, dz \) where \( C \) is the smooth directed arc with parametrization \( z(t) = t + it^2, 0 \leq t \leq 1 \), where \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \).

10. Prove that \( \left| \int_C \frac{e^{3z}}{1+e^z} \right| \leq \frac{2\pi e^{3R}}{e^R - 1} \) where \( C \) is the vertical line segment from \( z = R > 0 \) to \( z = R + 2\pi i \).

11. Suppose \( f(z) \) is an analytic function with a continuous derivative satisfying \( |f'(z)| \leq M \) for all \( z \) in the open disk \( D = \{ z : |z| < 1 \} \). Prove that \( |f(z_1) - f(z_2)| \leq M |z_1 - z_2| \) for each \( z_1, z_2 \in D \). \( \text{Hint: Observe that } f(z_2) - f(z_1) = \int_C f'(z) \, dz \) where \( C \) is the line segment from \( z_1 \) to \( z_2 \). Why?

12. For each of the following functions, determine the domain of analyticity and explain why \( \int_C f(z) \, dz = 0 \) where \( C \) is the positively-oriented circle \( |z| = 2 \).
   (a) \( f(z) = \frac{z}{z^2 + 25} \)
   (b) \( f(z) = e^{-z}(2z + 1) \)
   (c) \( f(z) = \frac{\cos(z)}{z^2 - 6z + 10} \)

13. Explain why the function \( f(z) = e^{z^2} \) has an antiderivative in the whole complex plane. \( \text{Use the Antiderivative Theorem and Cauchy's Theorem in your answer.} \)

14. Evaluate \( \int_C \frac{z}{(z + 2)(z - 1)} \, dz \) where \( C \) is the circle \( |z| = 4 \) traversed twice in the positive direction.