Some (Discrete) Joint Distributions
Stat 304  Fall 2005

Example 1  Suppose three balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls.
Let \( R \) and \( W \) denote the number of red and white balls chosen respectively. The joint probability density of \( R \) and \( W \) is given by \( f_{R,W}(i,j) = P(R = i, W = j) \) where \( i = 0, 1, 2, 3 \) and \( j = 0, 1, 2, 3 \). Computing, we obtain the following values for \( f_{R,W}(i,j) \).

\[
\begin{align*}
&f_{R,W}(0,0) = \binom{3}{3} \binom{7}{0} = \frac{10}{220} \\
f_{R,W}(1,0) = \binom{3}{2} \binom{7}{1} = \frac{40}{220} \\
f_{R,W}(0,1) = \binom{3}{3} \binom{7}{2} = \frac{30}{220} \\
f_{R,W}(2,0) = \binom{3}{2} \binom{7}{3} = \frac{10}{220} \\
f_{R,W}(0,2) = \binom{3}{3} \binom{7}{4} = \frac{15}{220} \\
f_{R,W}(1,1) = \binom{3}{2} \binom{7}{5} = \frac{60}{220} \\
f_{R,W}(2,1) = \binom{3}{2} \binom{7}{6} = \frac{12}{220} \\
f_{R,W}(3,0) = \binom{3}{3} \binom{7}{7} = \frac{1}{220} \\
\end{align*}
\]

The above values are summarized in the following table.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( j )</th>
<th>( f_R(i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>\frac{10}{220}</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>\frac{40}{220}</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>\frac{30}{220}</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>\frac{10}{220}</td>
</tr>
<tr>
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<td>1</td>
<td>\frac{15}{220}</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>\frac{60}{220}</td>
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<tr>
<td>3</td>
<td>1</td>
<td>\frac{12}{220}</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>\frac{1}{220}</td>
</tr>
</tbody>
</table>

Note that the row and column sums are given in the margins of the table. They are often referred to as the 'marginal pdf's' of \( R \) and \( W \) respectively.

Example 2  A balanced die is thrown. Then, a fair coin is tossed the number of times that the die showed. Let \( X \) be the outcome of the die toss, and \( Y \) be the number of tails that appeared when the coin was tossed. The joint probability density of \( X \) and \( Y \) is given by \( f_{X,Y}(i,j) = P(X = i, Y = j) \) where \( i = 1, 2, 3, 4, 5, 6 \) and \( j = 1, 2, 3, 4, 5, 6 \).

\[
\begin{align*}
&f_{X,Y}(1,0) = P(Y = 0 | X = 1)P(X = 1) = (1/6)(1/2) = 1/12 \\
f_{X,Y}(1,1) = P(Y = 1 | X = 1)P(X = 1) = (1/6)(1/2) = 1/12 \\
f_{X,Y}(2,0) = P(Y = 0 | X = 2)P(X = 2) = (1/4)(1/6) = 1/24 \\
f_{X,Y}(2,1) = P(Y = 1 | X = 2)P(X = 2) = (1/2)(1/6) = 1/12 \\
f_{X,Y}(2,2) = P(Y = 2 | X = 2)P(X = 2) = (1/4)(1/6) = 1/12 \\
&f_{X,Y}(3,0) = P(Y = 0 | X = 3)P(X = 3) = \frac{3}{7}(1/2)^0(1/2)(1/6) = 1/8 \\
f_{X,Y}(3,1) = P(Y = 1 | X = 3)P(X = 3) = \frac{3}{7}(1/2)(1/2)(1/6) = 3/48 \\
f_{X,Y}(3,2) = P(Y = 2 | X = 3)P(X = 3) = \frac{3}{7}(1/2)^2(1/2)(1/6) = 3/48 \\
f_{X,Y}(3,3) = P(Y = 3 | X = 3)P(X = 3) = \frac{3}{7}(1/2)^3(1/2)(1/6) = 1/48 \\
\end{align*}
\]

The computation of the remaining probabilities are similar. The results are summarized in the table below.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( j )</th>
<th>( f_X(i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>\frac{7}{120}</td>
</tr>
<tr>
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<td>1</td>
<td>\frac{1}{120}</td>
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<td>\frac{1}{120}</td>
</tr>
<tr>
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<td>\frac{1}{120}</td>
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<tr>
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<td>\frac{1}{120}</td>
</tr>
<tr>
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<td>5</td>
<td>\frac{1}{120}</td>
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<tr>
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<td>6</td>
<td>\frac{1}{120}</td>
</tr>
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<td>0</td>
<td>\frac{5}{84}</td>
</tr>
<tr>
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<td>1</td>
<td>\frac{1}{84}</td>
</tr>
<tr>
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<td>2</td>
<td>\frac{1}{84}</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>\frac{1}{84}</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>\frac{1}{84}</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>\frac{1}{84}</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>\frac{1}{84}</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
f_Y(j) &= \frac{1}{84}, \frac{1}{84}, \frac{1}{84}, \frac{1}{84}, \frac{1}{84}, \frac{1}{84} \\
\end{align*}
\]
Example 3 Suppose that 15% of the families in a certain community have no children, 20% have one, 35% have two, and 30% have three. Suppose further that in each family, each child is equally likely to be a boy or a girl. If a family is chosen at random from this community, then $B$, the number of boys, and $G$, the number of girls, in this family will have a joint probability distribution $f_{B,G}(i,j) = P(B = i, G = j)$.

\begin{align*}
    f_{B,G}(0,0) &= P(\text{no children}) = .15 \\
    f_{B,G}(0,1) &= P(\text{one girl and a total of one child}) \\
    &= P(\text{one girl | one child})P(\text{one child}) = (.50)(.20) = .10 \\
    f_{B,G}(0,2) &= P(\text{two girls and a total of two children}) \\
    &= P(\text{two girls | two children})P(\text{two children}) = (.50)^2(.35) = .0875 \\
    f_{B,G}(0,3) &= P(\text{three girls and a total of three children}) \\
    &= P(\text{three girls | three children})P(\text{three children}) = (.50)^3(.30) = .0375 \\
    f_{B,G}(1,0) &= P(\text{one boy and a total of one child}) \\
    &= P(\text{one boy | one child})P(\text{one child}) = (.50)(.20) = .10 \\
    f_{B,G}(1,1) &= P(\text{one boy, one girl, and a total of two children}) \\
    &= P(\text{one boy and one girl | two children})P(\text{two children}) = (.50)(.35) = .175 \\
    f_{B,G}(1,2) &= P(\text{one boy, two girls, and a total of three children}) = (3)(.50)^2(.30) = .1125
\end{align*}

The remainder of the probabilities are just as easily computed. They are summarized in the following table.

\[
\begin{array}{c|cccc|c}
  i & 0 & 1 & 2 & 3 & f_B(i) \\
  \hline
  0 & .15 & .10 & .0875 & .0375 & .3750 \\
  1 & .10 & .175 & .1125 & 0 & .3875 \\
  2 & .0875 & .1125 & 0 & 0 & .2000 \\
  3 & .0375 & 0 & 0 & 0 & .0375 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
  j & f_G(j) & .3750 & .3875 & .2000 & .0375 \\
\end{array}
\]