Finding an Optimal Solution
Stat 305 Spring 2006

The purpose of this handout is to verify the optimality of the MLEs for \( \mu \) and \( \sigma^2 \). Recall that, given the data \( x_1, x_2, \ldots, x_n \),
\[
\begin{align*}
\mu_0 &= \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \\
\sigma_0^2 &= \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2
\end{align*}
\]
was the candidate optimal solution derived in class.

The Conditions for Optimality

To verify that the point \((\mu_0, \sigma_0^2)\) is a maximum, the following conditions must be checked.

1. \[
\frac{\partial}{\partial \mu} \ln f_x(x_1, x_2, \ldots, x_n | \mu, \sigma^2) \bigg|_{(\mu_0, \sigma_0^2)} = \frac{\partial}{\partial \mu} L(\mu, \sigma^2) \bigg|_{(\mu_0, \sigma_0^2)} = 0
\]
\[
\frac{\partial}{\partial \sigma^2} \ln f_x(x_1, x_2, \ldots, x_n | \mu, \sigma^2) \bigg|_{(\mu_0, \sigma_0^2)} = \frac{\partial}{\partial \sigma^2} L(\mu, \sigma^2) \bigg|_{(\mu_0, \sigma_0^2)} = 0
\]

2. At least one of the following must hold.
\[
\frac{\partial^2}{\partial \mu^2} L(\mu, \sigma^2) \bigg|_{(\mu_0, \sigma_0^2)} < 0
\]
\[
\frac{\partial^2}{\partial (\sigma^2)^2} L(\mu, \sigma^2) \bigg|_{(\mu_0, \sigma_0^2)} < 0
\]

3. \[
\det \left( \begin{array}{cc}
\frac{\partial^2}{\partial \mu L(\mu, \sigma^2)} & \frac{\partial}{\partial \mu \sigma} L(\mu, \sigma^2) \\
\frac{\partial}{\partial \mu \sigma} L(\mu, \sigma^2) & \frac{\partial^2}{\partial (\sigma^2)^2} L(\mu, \sigma^2)
\end{array} \right)_{(\mu_0, \sigma_0^2)} > 0
\]

Verifying the Conditions

We first take the required partial derivatives.
\[
\begin{align*}
\frac{\partial^2}{\partial \mu^2} L(\mu, \sigma^2) &= -\frac{n}{\sigma^2} \\
\frac{\partial^2}{\partial (\sigma^2)^2} L(\mu, \sigma^2) &= \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^{n} (x_i - \mu)^2 \\
\frac{\partial^2}{\partial \mu \partial \sigma^2} L(\mu, \sigma^2) &= -\frac{1}{\sigma^4} \sum_{i=1}^{n} (x_i - \mu)
\end{align*}
\]

Observe that conditions (1.) and (2.) are satisfied since the point \((\mu_0, \sigma_0^2)\) was obtained by forcing condition (1.) and
\[
\frac{\partial^2}{\partial \mu^2} L(\mu, \sigma^2) = -\frac{n}{\sigma^2} < 0.
\]
Now we must evaluate the determinant in condition (3.).

$$\det \left( -\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) \quad -\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right)_{(\mu_0, \sigma_0^2)}$$

$$= \frac{1}{\sigma_0^6} \left[ -\frac{n^2}{2} + \frac{n}{\sigma_0^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 - \frac{1}{\sigma_0^2} \left( \sum_{i=1}^{n} (x_i - \mu_0)^2 \right)^2 \right]_{(\mu_0, \sigma_0^2)}$$

$$= \frac{1}{\sigma_0^6} \left[ -\frac{n^2}{2} + \frac{n}{\sigma_0^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 - \frac{1}{\sigma_0^2} \left( \sum_{i=1}^{n} (x_i - \mu_0)^2 \right)^2 \right]$$

$$= \frac{1}{\sigma_0^6} \left[ -\frac{n^2}{2} + \frac{n}{\sigma_0^2} (n\sigma_0^2) - \frac{1}{\sigma_0^2} (0)^2 \right]$$

$$= \frac{1}{\sigma_0^6} \left[ \frac{n^2}{2} \right]$$

$$= \frac{1}{2} \frac{n^2}{\sigma_0^6} > 0$$

Therefore, $(\mu_0, \sigma_0^2)$ is a maximum, and the MLEs for $\mu$ and $\sigma^2$ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and}$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$