An UMP Test for the Exponential Distribution

Stat 305 Spring Semester 2006

A random sample of size $n$ is taken from an exponential random variable $X$ with pdf $f_X(x) = \lambda e^{-\lambda x}$ where $x, \lambda > 0$. Formulate $H_0$ and $H_1$ as follows.

\[ H_0: \lambda = \lambda_0 \]
\[ H_1: \lambda = \lambda_1 \quad (\text{where } \lambda_1 > \lambda_0) \]

We will reject $H_0$ if

\[
\frac{f_X(x_1, \ldots, x_n | \lambda_0)}{f_X(x_1, \ldots, x_n | \lambda_1)} = \frac{\lambda_0^n \exp \left( -\lambda_0 \sum_{i=1}^{n} x_i \right)}{\lambda_1^n \exp \left( -\lambda_1 \sum_{i=1}^{n} x_i \right)} = \left( \frac{\lambda_0}{\lambda_1} \right)^n \exp \left( (\lambda_1 - \lambda_0) \sum_{i=1}^{n} x_i \right) < \frac{1}{k}.
\]

Therefore, we reject $H_0$ if

\[
\exp \left( (\lambda_1 - \lambda_0) \sum_{i=1}^{n} x_i \right) \left( \frac{\lambda_1}{k^{1/n} \lambda_0} \right)^n < 1.
\]

Taking natural logarithms of both sides,

\[
(\lambda_1 - \lambda_0) \sum_{i=1}^{n} x_i < \ln \left( \frac{\lambda_1}{k^{1/n} \lambda_0} \right)^n.
\]

or

\[
\sum_{i=1}^{n} x_i < \ln \left( \frac{\lambda_1}{k^{1/n} \lambda_0} \right)^{n - \frac{n}{\lambda_1 - \lambda_0}} = k'.
\]

Thus, the critical region has the form $\sum_{i=1}^{n} x_i < k'$. If we choose that $\alpha(\delta^*) = \alpha_0$,

\[
\alpha_0 = \pi(\delta^*, \lambda_0) = P(\sum_{i=1}^{n} X_i < k' | \lambda = \lambda_0).
\]

Notice that $\sum_{i=1}^{n} X_i$ has a gamma distribution with first parameter $n$ and second parameter $\lambda_0$ since it is the sum of $n$ independent exponential random variables each with parameter $\lambda_0$. We could now formulate the decision rule in terms of the gamma distribution, but with a little more work, we can state the decision rule so that it is more useful. To this end, note that

\[
\sum_{i=1}^{n} X_i = n\bar{X}_n.
\]

By properties of moment-generating functions, the moment-generating function, $\psi_{n\bar{X}_n}(t)$, of $n\bar{X}_n$ is

\[
\psi_{n\bar{X}_n}(t) = \left( \frac{\lambda_0}{\lambda_0 - t} \right)^n.
\]
since $n \overline{X}_n$ is the sum of $n$ independent exponential random variables each with parameter $\lambda_0$. Therefore,

\[
\psi_{2\lambda_0(n\overline{X}_n)}(t) = \left( \frac{\lambda_0}{\lambda_0 - 2\lambda_0 t} \right)^n
\]

\[
= \left( \frac{1}{1 - 2t} \right)^n
\]

\[
= \left( \frac{\frac{1}{2}}{\frac{1}{2} - t} \right)^n.
\]

Now recall that a random variable is called $\chi^2_n$ (chi-squared with $n$ degrees of freedom) if it is a gamma random variable with first parameter $n/2$ and second parameter $1/2$. Therefore, a $\chi^2_{2n}$ random variable is gamma with first parameter $n$ and second parameter $1/2$. Consequently,

\[
2\lambda_0 n \overline{X}_n \sim \chi^2_{2n}.
\]

We obtained that

\[
\sum_{i=1}^{n} x_i < \ln \left( \frac{\lambda_1}{k^{1/n} \lambda_0} \right)^{\frac{n}{\lambda_1 - \lambda_0}}.
\]

Therefore,

\[
2\lambda_0 n \overline{X}_n < \ln \left( \frac{\lambda_1}{k^{1/n} \lambda_0} \right)^{\frac{2n\lambda_0}{\lambda_1 - \lambda_0}} = k^*.
\]

The critical region has the form $2\lambda_0 n \overline{X}_n < k^*$. If we choose that $\alpha(\delta^*) = \alpha_0$,

\[
\alpha_0 = \pi(\delta^*, \lambda_0)
\]

\[
= P(2\lambda_0 n \overline{X}_n < k^* \mid \lambda = \lambda_0).
\]

Therefore, $k^* = \chi^2_{1-\alpha_0, 2n}$. We have derived the standard form of the decision rule for this test.

**Decision Rule**

\[
\delta^*:
\]

Reject $H_0$ if $2\lambda_0 n \overline{Y} < \chi^2_{1-\alpha_0, 2n}$.

Note that the decision rule does not depend on $\lambda_1$. Therefore, any $\lambda_1 > \lambda_0$ would result in the exactly same critical region. Thus, above decision rule constitutes a *uniformly most powerful* (UMP) test for the parameter $\lambda$.

The power of the test can be easily calculated. At $\lambda = \lambda_1 > \lambda_0$, the power is given by

\[
\pi(\delta^*, \lambda_1) = P \left( 2n \overline{X}_n < \frac{\chi^2_{1-\alpha_0, 2n}}{\lambda_0} \mid \lambda = \lambda_1 \right)
\]

\[
= P \left( 2n \lambda_1 \overline{X}_n < \left( \frac{\lambda_1}{\lambda_0} \right) \chi^2_{1-\alpha_0, 2n} \right)
\]

\[
= P \left( \chi^2_{2n} < \left( \frac{\lambda_1}{\lambda_0} \right) \chi^2_{1-\alpha_0, 2n} \right).
\]
Therefore, $\beta(\delta^*) = 1 - P\left(\chi^2_{2n} < \left(\frac{\lambda_1}{\lambda_0}\right)\chi^2_{1-\alpha_0,2n}\right)$. Notice that when $\lambda_1 = \lambda_0$, the power is

$$P\left(\chi^2_{2n} < \left(\frac{\lambda_1}{\lambda_0}\right)\chi^2_{1-\alpha_0,2n}\right) = P\left(\chi^2_{2n} < \chi^2_{1-\alpha_0,2n}\right) = \alpha_0 = \alpha(\delta^*)$$

as expected. Also, as $\lambda_1 \to \infty$, the power of the test approaches 1.