Research Statement

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1. General Overview

My research is in the area of algebraic combinatorics. I find the most captivating aspect of combinatorics is the approachability and elegance of its questions. In combinatorics one finds both open problems which are accessible to undergraduate students, and simply stated problems which prove to have deep and sophisticated solutions. In addition, combinatorial problems are integral to many other disciplines; for example my research deals with enumerative questions related to geometry and algebra.

Suppose there are n points in the plane. Three natural questions one might ask are:

* How many lines do the points determine?

* How many triangular shaped regions are formed by the determined lines?

* What is the maximum number of lines through a point?

All of the information about how an arrangement of points and lines breaks up space is encoded by the flag f- and flag h-vectors, which are combinatorial invariants of a partially ordered set associated to the arrangement. While these vectors are easily defined, they have eluded complete characterization. For example it in not known how to calculate the number of lines which can be determined by n points. My work establishes inequalities which the flag f- and h- vectors satisfy.

A descent of a permutation σ occurs at position i if $\sigma(i) > \sigma(i+1)$. Solomon [11] showed that grouping together permutations with descents at common locations resulted in an algebra. Peaks of a permutation are related to descents in that a peak occurs when there is an ascent followed by a descent. There has been increased interest in peaks recently as connections of peaks in permutations and quasisymmetric functions have been found to such diverse topics as Hopf algebras, random walks and the combinatorics of polytopes. I showed the existence of an algebra whose elements are sums of permutations with a common peak set, and I am currently exploring properties of this algebra.

2. geometric lattice invariants

Consider a finite set of points in Euclidean space and all of the affine subspaces that these points span. If one partially orders these subspaces by inclusion, for example a line is greater than a point if the point is contained in the line, the resulting partially ordered set is a geometric lattice. More generally, any partially ordered set which satisfies properties inherent to these geometric arrangements (for example a pair of points will lie on exactly one line) is a geometric lattice. One particularly important geometric lattice arises from the *near pencil* arrangement on n points in

$\mathbb{R}^{r-1}$ which consists of $(n-r+2)$ points on a line with the remaining $(r-2)$ points in general position (points are in *general position* if every set of $k+1$ points spans a k-dimensional subspace). This lattice is illustrated in Figure 1 with $n=4$ and $r=3$, and is called the rank r near pencil on n atoms. Another notable lattice is the *Boolean algebra*, denoted $\mathcal{B}_r$, which corresponds to r points in general position in $\mathbb{R}^{r-1}$. Similarly, the truncated Boolean algebra $\mathcal{B}_{r,n}$ for $n>r$ corresponds to n points in general position in $\mathbb{R}^{r-1}$.

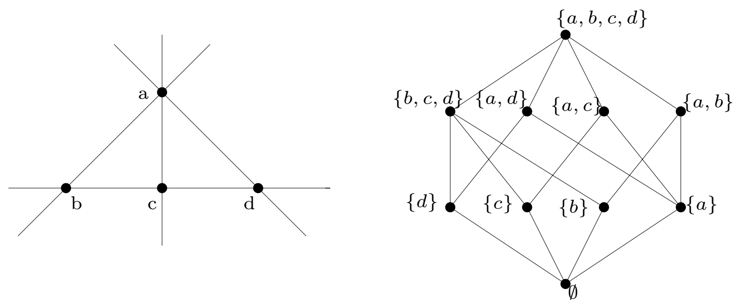

FIGURE 1. The near pencil arrangement on 4 points in the plane with its corresponding geometric lattice.

A *chain*, $x_0 < x_1 < \cdots < x_r$, in a poset P is a totally ordered subset of elements of P. A poset is *graded* if all maximal chains have the same length and this length is the *rank* of P. A graded poset has an associated rank function ρ which assigns to each element y of P a positive integer such that $\rho(y)=k$, where k is the length of the longest chain of the form $y_0 < y_1 < \cdots < y_k = y$. An element with rank 1 is called an *atom*. For example, in the lattice of Figure 1, $\emptyset < \{c\} < \{a,c\} < \{a,b,c,d\}$ is a chain, $\rho(\{c\})=1$, and $\rho(\{a,c\})=2$.

Given a rank r poset P, and $S \subseteq [r-1]$, where $[r-1]=\{1,2,\ldots,r-1\}$, $f_S(P)$ counts the number of chains in P in which the set of the ranks of the elements in the chain is equal to the set S. The collection $\{f_S\}$, for $S \subseteq [r-1]$, is known as the *flag f-vector* of the poset. It is sometimes more natural to work with the *flag h-vector* of a poset, defined by $h_S(P)=\sum_{T\subseteq S}(-1)^{|S|-|T|}f_T(P)$ for $S\subseteq[r-1]$, as it has several nice combinatorial properties.

It is currently not known how to determine which sequences of numbers arise as the flag f-vector of a poset. One step toward obtaining a characterization of these vectors is to determine the linear inequalities that they satisfy. This has been done for the flag f-vector of several classes of posets including all graded posets [4], Cohen-Macaulay posets [12], and face lattices of simplicial polytopes [13].

In my thesis [10] I gave the minimal list of linear inequalities satisfied by the flag f-vector of geometric lattices of rank 3. The result proved to be surprising since in all previously known cases a finite number of inequalities are sufficient to imply all possible linear inequalities satisfied by the flag f-vectors, but in the rank 3 geometric lattice case an infinite family of linear inequalities are necessary. I also extended a result of Dowling and Wilson [7] by giving an explicit formula for f_S of the near pencil lattice, for all $S \subseteq [r-1]$. Furthermore I proved that the near pencil lattice minimizes f_S for all geometric lattices of rank r with n atoms [10].

Recent work with E. Swartz [8] culminated in a stronger result through inequalities on the flag h-vector of geometric lattices. We are currently looking for inequalities on the flag h-vector (using a topological decomposition of lattices) in an effort to explain newly discovered inequalities satisfied by another lattice invariant (the h-vector).

Theorem 2.1 (N-, Swartz). *Let $\mathcal{B}_{r,n}$ and $\mathcal{P}_{r,n}$ be the truncated Boolean algebra and the near pencil lattice, both of rank r on n atoms. Then for L a rank r geometric lattice on n atoms and $S \subseteq [r-1]$*

$$h_S(\mathcal{P}_{r,n}) \leq h_S(L) \leq h_S(\mathcal{B}_{r,n}).$$

3. THE PEAK ALGEBRA

In 1976 Solomon [11] introduced a collection of algebras associated to finite Coxeter groups. In the case of the symmetric group the elements of the associated algebra are sums (in the group algebra of S_n over a field k) of permutations with a common descent set. My work [9] shows that taking sums of permutations with a common peak set results in a subalgebra of Solomon's descent algebra.

Denote a permutation $\gamma \in S_n$ by $(\gamma(1)\gamma(2)\cdots\gamma(n))$. Define the *descent set* of γ, $D(\gamma) := \{i : \gamma(i) > \gamma(i+1)\}$, and the *peak set* of γ, $\Lambda(\gamma) := \{i : \gamma(i-1) < \gamma(i) > \gamma(i+1)\}$. For example when $\gamma = (325461) \in S_6$, $D(\gamma) = \{1,3,5\}$ and $\Lambda(\gamma) = \{3,5\}$ since $\gamma(2) < \gamma(3) > \gamma(4)$ and $\gamma(4) < \gamma(5) > \gamma(6)$. The subsets of the form $\Lambda(\gamma)$ are called peak sets.

Theorem 3.1. *In the group algebra $\mathbb{Q}S_n$ of S_n, define for each peak set $\Gamma \subset [n-1]$*

$$P_\Gamma = \sum_{w:\Lambda(w)=\Gamma} w$$

where $\Lambda(w)$ is the peak set of $w \in S_n$. Then the subspace $\mathcal{P}_n$ spanned by the P_Γ's is a subalgebra of Solomon's descent algebra of S_n.

Example 3.2. *Consider the algebra $\mathcal{P}_3$. There are two basis elements: $P_\emptyset = (123)+(312)+(213)+(321)$ and $P_{\{2\}} = (132)+(231)$. The product $P_\emptyset \times P_{\{2\}} = P_\emptyset + 2P_{\{2\}}$. Notice $\mathcal{P}_3$ is a subalgebra of the descent algebra since each element is a sum of descent classes; for example, $P_\emptyset$ is the sum of permutations with descent sets $\emptyset$, $\{1\}$, and $\{1,2\}$.*

The algebra of Theorem 3.1 does not have an identity. In more recent work, with M. Aguiar and N. Bergeron [1] we find that this algebra is actually a two sided ideal in a larger, unital algebra called the *peak algebra*. The main difference between the two algebras is that a descent at $i = 1$ of a permutation is considered a peak at position 1 in the peak algebra while it is not considered a peak in the smaller algebra.

This new work was inspired by the observation that peaks of ordinary permutations are closely related to descents of signed permutations. B_n is the group of *signed permutations* on $[n]$. Write a permutation $w \in B_n$ as $w = (w_1 \cdots w_n)$ and take $w_0 = 0$. For example $w = (\bar{4}2\bar{3}\bar{1}) \in B_4$ where the bars indicate an element is negative. We use the standard ordering, $\cdots < -2 < -1 < 0 < 1 < 2 < \cdots$, so in the above example the descent set of w is $\{0, 2\}$ since $0 > -4$ and $2 > -3$. Notice if we "forget" the signs, the underlying permutation $|w| = (4231)$ has peak set $\{1, 3\}$ (in the extended sense of [1]). The shifting of the descent set of w by one to obtain the peak set of $|w|$ gives a hint as to the connection between descent sets of signed permutations and peak sets of unsigned permutations.

Our approach to a variety of questions is to obtain properties that hold at the level of signed permutations and then map them down, using the map that forgets signs, to ordinary permutations. Indeed the image of the descent algebra defined for signed permutations under this forgetful map is the peak algebra. In our work we show that the algebra of Theorem 3.1 is the image of a canonical ideal of the descent algebra of signed permutations. We also obtain a commutative semisimple subalgebra of the descent algebra by grouping permutations according to their number of peaks. We are currently exploring further properties of the peak algebra including idempotents, eigenvalues of a descents-to-peaks transform and a multiplication rule for basis elements.

References

[1] M. Aguiar, N. Bergeron and K. Nyman, The peak algebra and the descent algebras of types B and D, *Trans. Amer. Math. Soc.*, to appear.

[2] M. Bayer, A. Klapper, A new index for polytopes, *Discrete and Comput. Geom.*, 6: 33–47, 1991.

[3] L.J Billera, R. Ehrenborg, and M. Readdy, The **c**-2**d**-index of oriented matroids, *J. Combin. Theory Ser. A*, 80: 79–105, 1997.

[4] L.J. Billera and G. Hetyei, Linear inequalities for flags in graded partially ordered sets, *J. Comb. Theory Ser. A*, 89: 77–104, 2000.

[5] L.J. Billera, S.K. Hsiao and S. van Willigenburg, Peak quasisymmetric functions and Eulerian enumeration, *Adv. Math*, to appear.

[6] S. Billey and M. Haiman, Schubert polynomials for the classical groups, *J. Amer. Math. Soc.*, 8, no. 2: 443–482, 1995.

[7] T. Dowling and R. Wilson, The slimmest geometric lattices, *Trans. Amer. Math. Soc.*, 196: 203–215, 1974.

[8] K. Nyman and E. Swartz, Inequalities for the h- and flag h-vectors of geometric lattices, preprint, 2003.

[9] K. Nyman, The peak algebra of the symmetric group, *J. Alg. Comb.*, 17: 309–322, 2003.

[10] K. Nyman, Enumeration in geometric lattices and the symmetric group. Ph.D. thesis, Cornell University, Ithaca, NY, 2001.

[11] L. Solomon, A Mackey formula in the group ring of a Coxeter group, *J. Algebra*, 41: no. 2, 255–268, 1976.

[12] R.P. Stanley, *Combinatorics and Commutative Algebra* 2nd edition, Progress in Mathematics, Vol. 41, Birkhäuser, Boston, 1996.

[13] R.P. Stanley, The number of faces of a simplicial convex polytope, *Adv. in Math.*, 35: 236–238, 1980.

[14] T. Zaslavsky, The slimmest arrangements of hyperplanes, *Geometriae Dedicata*, 14: 243–259, 1983.