1. (10 pts) Use the graph of \( f \) at right to find the indicated functions (i)–(iv) among the graphs below (a)–(f).

(i) \( y = f(x) + 2 \)  
(ii) \( y = f(-3(x - 2)) \)  
(iii) \( y = -f(x + 2) + 2 \)  
(iv) \( y = f(-3x + 2) \)

(a)  
(b)  
(c)  
(d)  
(e)  
(f)

2. (10 pts) A population increases from 5.2 million at an annual rate of 3.1%. Find the continuous growth rate.

We have \( P(t) = 5.2(1.031)^t \) (scale in millions). We want \( P(t) = 5.2e^{rt} \). So solve for \( r \):

\[
5.2e^{rt} = 5.2(1.031)^t
\]

\[
e^{rt} = 1.031
\]

\[
r = \ln(1.031) = 0.0305
\]
3. (10 pts)

- (T/F) If $a$ and $b$ are positive, then $\log\left(\frac{a}{b}\right) = \frac{\log a}{\log b}$

  False: the log of a quotient is the difference of the logs.

- (T/F) For any value of $x$, $\ln(e^{2x}) = 2x$

  True: the exponential (with base $e$) and natural log are inverse functions.

- (T/F) For any positive value of $x$, $e^{\ln2x} = 2x$

  True: see above (the phrase “positive $x$” is important... logs don’t accept negative numbers).

- (T/F) If a population doubles every 20 years, its annual continuous growth rate is 20%.

  False: $P(t) = P_0 2^{t/20}$. Set this equal to $P_0 e^{rt}$ and solve for $r$. (Or use common sense; if growth rate is 20%, then it won’t take more than 5 years to double your money.)

- (T/F) If $7.32 = e^t$, then $t = \frac{7.32}{e}$.

  False: the way to pull $t$ out is to take the natural log of each side.

4. (10 pts) If $f(n) = n^2 - 4n + 5$ and $g(n) = -f(2n - 3)$, express $g(n)$ in the form $an^2 + bn + c$.

First, write $g(n)$ in terms of $n$:

$$-f(2n - 3) = -(2n - 3)^2 - 4(2n - 3) + 5.$$

Next expand, collect terms and express as “$an^2 + bn + c$”

$$g(n) = -4n^2 + 6n - 9 + 8n - 12 - 5 = -4n^2 + 14n - 26.$$

5. (10 pts) A ballet dancer is thrown in the air by her partner. The height $h(t)$ above ground, in feet, of the dancer at time $t$ can be described by

$$h(t) = -16t^2 + 32t - 14.$$

How high does she thrown and when does she reach this height? Show your work for credit

First complete the square to put in form $a(t - b)^2 + k$. Then, the point $(b, k)$ is the highest point on the graph. (So at time $t = b$, she’s at height $k$.)

$a = -16$, so factor this out: $h(t) = -16(t^2 - 2t + 14/16)$.

$b = 1$ because $(t - 1)^2 = t^2 - 2t$. Now add and subtract $b^2$ to write

$$h(t) = -16((t^2 - 2t + 1) - 1 + 14/16) = -16(t - 1)^2 + 2.$$
6. (10 pts)

- (T / F) If \( g(t) = f(t - 2) \), then the graph of \( g(t) \) can be obtained by shifting the graph of \( f \) two units to the left.
  False: two units to the right.

- (T / F) Vertical and horizontal shifts are called translations. True.

- (T / F) If \( g(x) = x^2 + 4 \), then \( g(-x) = -g(-x) \).
  False: Well \( g \) is even, so \( g(-x) = g(x) \). Using this fact, we can rewrite the above as \( g(x) = -g(x) \), which makes no sense whatsoever (unless the function is constantly zero \( 0 = -0 \)), but this is not the case.

- (T / F) Multiplying the argument \( x \) of a function \( f(x) \) by a constant \( k \), with \( k < 1 \), horizontally compresses its graph.
  False: it will yield a horizontal stretch: \( f(3x) \) happens three times as fast as \( f \); likewise, \( f((1/3)x) \) happens \( 1/3 \) as fast as \( f \) . . . a horizontal stretch.

- (T / F) If a parabola is concave up, its vertex is a maximum point.
  False. Concave up means it looks like “∪” not like “∩.”

7. (10 pts) A state judge imposes the following fine on Oak Park for failure to maintain its streets: $0.01 the morning of August 2, 2010, and the fine increases by $2 each morning thereafter. If Oak Park defies the judge’s order until the afternoon of August 28, 2010, what does it owe the state? Justify your answer.

This is not exponential growth, but linear growth. (For exponential growth, it might read “find doubles each morning thereafter.”) Here is what the first few days look like:

<table>
<thead>
<tr>
<th>date</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>additional fine</td>
<td>0.01</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>total fine</td>
<td>0.01</td>
<td>2.01</td>
<td>4.01</td>
<td>6.01</td>
</tr>
</tbody>
</table>

This linear growth is described by the equation \( F(t) = 0.01 + 2t \). Since \( 28 - 2 = 26 \), 26 days pass by that carry a fine of $2. So the answer is $52.01.

8. (10 pts) Suppose $300 was deposited into one of three bank accounts and \( t \) is time in years. for each verbal description (i)–(iii), state which formula (a)–(c) could represent it.

(i) \( 300(1.03)^{2t} \) (a) Effective annual yield of investment was less than 3%.
(ii) \( 300(1.03)^{4t} \) (b) Investment earned 6% annually, compounded semi-annually.
(iii) \( 300(1.06)^{t/2} \) (c) Investment earned more than 20% over two years.
9. (10 pts)

- (T / F) If we are given two data points, we can find a linear function and an exponential function that go through these points. **True.**

- (T / F) A decreasing exponential function always becomes smaller than any decreasing linear function in the long run. **False. Consider** $E(t) = 2^{-t}$ versus $L(t) = 2 - 2t$ on the interval $0 \leq t \leq 5$.

- (T / F) In the formula $Q = ab^t$, if $a > 1$, the graph always rises as we read from left to right. **False. We also need $b > 1$ for this to be true. Consider** $E(t) = (1/4)^t$.

- (T / F) A population that has 200 members and decreases at 10% per year can be modeled as $P = 200(0.10)^t$. **False. Each year has 90% of the previous year, so you should multiply by 0.90 (or 1 - 0.1) to build the formula: $P(t) = 200(0.90)^t$.**

- (T / F) If $f(x) \rightarrow \infty$ as $x \rightarrow k$, we say that the line $x = k$ is a horizontal asymptote. **False. This scenario describes a vertical asymptote. Horizontal asymptotes have the feature that $f(x) \rightarrow k$ as $x \rightarrow \infty$.**

10. (10 pts) Solve $x$ exactly if $\ln(3x - 3) - \ln(x) = \ln(x - 1)$. Show your work. **Hint: there is only one valid solution.**

Try to eliminate the logs to get an expression involving only $x$’s.

\[
\ln(3x - 3) - \ln(x) = \ln(x - 1)
\]

\[
\ln\left(\frac{3x - 3}{x}\right) = \ln(x - 1)
\]

\[
\frac{3x - 3}{x} = x - 1
\]

Now, clear denominator and solve for $x$: $3x - 3 = x^2 - x$, or $x^2 - 4x + 3 = 0$, or $x = 1, 3$.

But wait! Looking back at the original problem, $x = 1$ is not a valid solution because $\ln([1] - 1) = \ln(0)$ is undefined. So the answer is $x = 3$.

11. (10 pts) Find the coordinates (to three decimal places) of the point $P$ on a circle of radius 3 centered at the origin if the angle it makes with respect to the positive $x$-axis is $235^\circ$.

Rectangular coordinates $(x, y)$ are equal to $(r \cos \theta, r \sin \theta)$ when $\theta$ is measured with respect to the positive $x$-axis, so the answer is $(3 \cos 235^\circ, 3 \sin 235^\circ) = (-1.7207, -2.45)$.

Warning: if your calculator is set to radians, first use the conversion $235^\circ \times \frac{\pi \text{ rad}}{180} = X \text{ rad}.$
12. (10 pts) Find a formula for the function $f$ pictured at right.

This looks like a sinusoidal graph, so we’ll try to determine $a, b, h, k$ in $f(t) = a \sin(b(t - h)) + k$.

The midline seems to be at 2.5, so $k = 2.5$.
The amplitude seems to be 2 (as it ranges from 0.5 to 4.5), so $a = 2$.
The function satisfies $f(0) = 2.5$, i.e., it starts at its midline, which is what $\sin(t)$ does too, so there is no horizontal translation. Thus $h = 0$.
The period seems to be 4, so $b = (\text{usual period})/(\text{new period}) = 2\pi/4 = \pi/2$. To summarize,

$$f(t) = 2 \sin\left(\frac{\pi}{2}(t - 0)\right) + 2.5.$$  

13. (10 pts) Give exact (precise) coordinates for two points on the curve within the indicated interval.

$y = \tan x \quad -\pi/2 \leq x \leq \pi/2$.

Since $\tan \theta = (\sin \theta)/(\cos \theta)$, we simply recall the coordinates on the unit circle for some of our special reference angles. $0 : (1, 0) \quad \pi/6 : (\sqrt{3}/2, 1/2) \quad \pi/4 : (\sqrt{2}/2, \sqrt{2}/2) \quad -\pi/4 : (\sqrt{2}/2, -\sqrt{2}/2)$. So any of the points $(\theta, \tan \theta)$ above will work (as well as many others). Here are my choices:

$$(\theta, \tan \theta) : (0, 0), (\pi/4, 1), (-\pi/4, -1)$$

14. (10 pts)

• ( T / F ) $\sin(-x) = \sin(x)$
  False. Sine is an odd function, so $\sin(-x) = -\sin(x)$.

• ( T / F ) $\cos(x - \pi/2) = \sin x$
  True. Shifting the cosine graph to the right by 1/4 of a period produces the graph of sine.

• ( T / F ) The cosine of $30^\circ$ is the same as the sine of $\pi/6$.
  False. The cosine of $30^\circ$ is the same as the sine of $60^\circ$, or $\pi/3$.

• ( T / F ) An angle of one radian is about equal to an angle of one degree.
  False. One radian is approximately $\pi/3$ radians, or $60^\circ$.

• ( T / F ) The function $f(x) = \cos 3x$ has period three times as large as the function $g(x) = \cos x$.
  False. It runs three times as fast, which means it’s period is 1/3 the period of cosine.