• Your final exam is scheduled for **Saturday, December 18**, at 9:00 a.m. in our usual room.

• You may bring an $8\frac{1}{2} \times 11$ inch, one-sided, sheet of paper with formulas and definitions to the final exam. No other books or notes will be allowed.

• Calculators will be allowed.

• The exam will last two hours.

• There will be **TWENTY (20)** questions; many of them will be similar to (and/or simpler than) the problems appearing here.

• In addition to these problems, it would be a good idea to look over the “Check your understanding” problems from Chapters 6 through 9 and the past quizzes and exams.

• Solutions to the past quizzes and exams will be posted on our course website by the end of the week.

• There will be some True/False questions and some Matching questions on the exam, but otherwise all the problems will be “work out” questions (no multiple choice).

• If you work some problem on your calculator, then **write down** some of your intermediate steps!!! … Partial credit will be awarded for wrong answers, but **only** if you show your work.

• **Submit** solutions to the last ten problems on **Thursday (12/09)** for your final **quiz grade**.
1. (a) How can you use the notion of average rate of change to define / characterize lines among all types of functions?

Lines have constant rate of change.

(b) Which sentence best characterizes which type of function?

- **periodic functions**
  - average rate of change decreases (but is always positive) on same-size intervals, e.g., [1,2], [2,3], [3,4], ...

- **exponential functions**
  - average rate of change goes to zero on larger and larger intervals

- **logarithmic functions**
  - average rate of change increases on same-size intervals, e.g., [1,2], [2,3], [3,4], ...

2. Fill in the missing points.

   First figure: \((-1, e^{-1})\) and \((2, e^2)\)

   Second figure: the perpendicular line is \(y = -2x + 8\). Find where the two lines meet. Unknown points: \((4, 0)\) and \((12/5, 16/5)\).

3. You have $4 to buy refreshments for the party and want to split it between soda (selling for $0.5 per can) and chips (selling for $0.8 per bag).

   (a) What are your options if you want to bring home less than $0.5 in change?

   (b) What is the least amount of change you can bring home?

   (c) If \(y\) represents the number of cans you can buy and \(x\) represents the number of bags you can buy, express \(y\) as a function of \(x\). (Never mind that this assumes you can buy fractional cans of soda.)

   (d) Plot your function in (c) and your data in (a) on the same graph.

4. Match the data to the type of function it could possibly define

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   a line

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   an exponential function

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   a logarithmic function

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   a periodic function
5. Which sets may describe functions. For those that may be, state which is the independent variable and which is the dependent variable. Also, give possible domains and ranges.

(a) • calendar date
   • closing value of the S&P-500

(b) A soda vending machine

(c) • average daily high temperature
   • months (January, . . . , December)

(d) • average daily high temperature
   • months in the calendar year 2011

(e) • average daily temperature
   • days since January 1, 2000.

(f) • max temp. of your shower this morning
   • number of showers that others in your dorm/apartment took before you.

(g) 2 3 2 3 1 4 7 8 −9

(h) 2 3 5 6 1 4 7 8 −9

(i) 2 3 5 6 1 4 7 2 −9

(a) - yes: domain = all dates; range = positive rational numbers
(b) - yes: domain = ($0.50,button); range = soda cans
(c) - no: depends on which January you look at (not unique output for each input)
(d) - yes: now there is only one output for each input
(e) - yes: domain is calendar date since January 1, 2000; range is all real numbers
(f) - no: depends on who took the showers too (some people steal all the hot water!)
(g) - yes: use second line of table as domain
(h) - yes: use first line of table as domain
(i) - no: neither line can be domain (doesn’t pass vertical line test)

6. If \( f(x) = \frac{\sin((\pi/2)x + \pi/2)}{e^{x-2}} \), determine \( f(2) \). If \( g(x) = \sqrt{\log_{\ln 2}(x^9)} + 16 \), determine \( g(\ln 2) \).

\[ f(2) = -1/1 \quad g(\ln 2) = \sqrt{9 + 16} = 5. \]

7. • If \( f(2) = 7 \) and \( f^{-1}(3) = 4 \), simplify the expression \( 3f^{-1}(7) - 2f(4) \).
   • Assume further that \( f \) is a line. What is \( f(7) \)?

(a) Using properties of inverse functions, note that \( f(4) = 3 \) and \( f^{-1}(7) - 2 \). The rest is arithmetic
(b) Use the two points (4, 3) and (2, 7) to build a line; \( f(7) = -3 \).

8. You plan to compete in a triathlon which consists of 3 miles swimming, followed by 10 miles running, followed by 60 miles cycling.
(a) If you average 0.75 mph swimming and 5 mph running, how fast must you cycle to finish the race in 7 hours? (Is it feasible?)

(b) Graph a piecewise function describing your distance traveled as a function of time (assume an average of 30 mph cycling).

(c) Develop a (piecewise) formula for this function.

\[
(a) \frac{3}{.75} + \frac{10}{5} + \frac{60}{x} = 7, \text{ so } x = 60. \text{ (Not feasible)}
\]

(b) Should be three straight lines. First one goes through (0, 0) and \((3/.75, 3)\); second one goes through \((3/.75, 3)\) and \((3/.75 + 10/5, 3 + 10)\); third goes through the last point computed and \((3/.75 + 10/5 + 60/30, 3 + 10 + 60)\).

9. (a) (T / F) If \(f\) is positive and concave up for \(x > 0\), then the average rate of change is increasing over the intervals \([1,2], [2,3], [3,4], \ldots\)

(b) (T / F) If \(f\) is positive and concave up for \(x\) in the interval \([1,4]\), then \(f\) is increasing over this interval.

T: by definition of concave up
F: could be decreasing concave up (e.g., \(e^{-x}\))

10. Let \(f(x) = 3x + 1\) and \(g(x) = \frac{1}{x + x^2}\). Answer the following.

(a) Calculate \(f \circ g, g \circ f, f \circ f\) and \(g \circ g\), simplifying as much as possible.

(b) Let \(h(x)\) be the first function you calculated in (a). Verify that \(\frac{x + 3}{x}\) and \(x + x^2\) are two other functions (aside from \(f\) and \(g\)) that decompose \(h(x)\).

(a) I’ll just do two: \((f \circ g)(x) = \frac{3}{x + x^2} + 1 = \frac{3 + x^2}{x + x^2}; (f \circ f)(x) = 3[3x + 1] + 1 = 9x + 4.\)

(b) If \(\tilde{f}(x) = \frac{x + 3}{x}\) and \(\tilde{g}(x) = x + x^2\), then it is easy to see that \((\tilde{f} \circ \tilde{g})(x) = h(x)\).

11. (a) Develop a piecewise function (formula) that describes the balance in my savings account after \(t\) years if I begin with $100 and the following happens.

\- If \(t \leq 4\), then account grows at nominal rate of 2%, compounded quarterly.
\- If \(t > 4\), then account grows at 0.4%, compounded continuously.

(b) How much is in my account after 4 years? after 6 years?

Here is the piecewise function: \(f(x) = \begin{cases} 100(1 + .02/4)^{4t} & t \leq 4 \\ \left[100(1 + .02/4)^{16}\right] e^{0.004(t-4)} & t > 4 \end{cases}\)

12. (a) A reservoir is losing water exponentially. If it had 432,000 ft\(^3\) on 1/1/2001 and had 321,000 ft\(^3\) on 1/1/2010, when will it have 210,000 ft\(^3\)?

(b) A population of bacteria doubles every 4 hours. If there were 10 cells to start with, how many will there be two days later?

I like to use the model \(P(t) = P_0 e^{rt}\) for these types of problems, though any base (say 2) would work. Using the model \(P(t) = P_0(1 + r)^t\) also works, but it requires a bit more arithmetic.
Now, you have two unknowns \( (P_0 \text{ and } r) \) and two data points. It is always easiest to treat \( t = 0 \) to be the time of the first data point. So in (a) we have the two data points: \((0, 432000)\) and \((9, 321000)\). This makes \( P_0 = 432000 \) since \( P_0 = P(0) \) always. Use logarithms and the other data point to solve for \( r \).

13. Find the domain and range for each function.

\[
\begin{align*}
\ln x & \quad \ln(x+3) & \quad \ln(x^2+3) & \quad \ln(x^2) \\
D : \{x > 0\} & \quad D : \{x > -3\} & \quad D : \{\text{all real } x\} & \quad D : \{x \neq 0\} \\
R : \{\text{all real } y\} & \quad R : \{\text{all real } y\} & \quad R : \{\text{all real } y \geq \ln 3\} & \quad R : \{\text{all real } y\}
\end{align*}
\]

14. If \( 500(1.03)^{2t} \) represents my savings account balance after 2 years, then . . .

(a) How many times per year does it compound?

(b) What is the (nominal) annual rate?

(c) What is the equivalent annual continuously compounding rate?

(d) Which is a better deal? The account above or one growing by the rule \( 500e^{0.0607t} \)?

TYPO!! (I meant to write \( (1.03)^{2t^2} \) in the statement of the problem.)

(a) If two years have passed, then \( 2 \cdot 2 \) indicates that it has compounded 4 times, or 2 times per year.

(b) This means \( 1.03 = 1 + \frac{r}{2} \), or \( r = 6\% \).

(c) Set \( (1.03)^{2t} = e^{kt} \) and solve for \( k \). (Sufficient to put \( t = 1 \), then take logs.)

(d) I got \( k \approx 6\% \) above, so \( 500e^{0.0607t} \) is the better deal (has a larger \( k \)).

15. Given the facts below, determine which is bigger. (Your answer may be \( x \), \( y \), or “not enough information.”)

(a) \( \log_3(x/y) < 0 \quad y \)

(b) \( \log_6(x) > \log_6(y) \quad x \)

(c) \( \log_6(1/x) > \log_3(1/y) \quad \text{Put } b = \log_3(y) \text{ and } c = \log_6(x). \text{ After quite a bit of work, we get } b > c \text{ and } y = e^b \text{ and } x = 3^{1.63c}. \text{ We don’t know how much bigger } b \text{ is than } c, \text{ so we don’t know if } y \text{ is bigger than } x. \)

(d) \( \log_6(1/x) > \log_3(1/y) > 1 \quad \text{Same problem as above. Adding the extra fact that they are both bigger than 1 doesn’t help at all.} \)

16. (a) If \( \log_a(b) = 2 \), determine \( \log_a(b^3) - \log_a(b^2) \).

(b) If \( \log_a(b) = 2 \), determine \( \log_{\sqrt{2}}(b^3) \).
(c) Solve for $x$ if $e^{3x}2^{2x} = e^5$.

(a) Use logarithm rules to get 2.

(b) TYPO!! (I meant to write $\sqrt{a}$, not $\sqrt{2}$.) First step is to write $\log_{\sqrt{a}}(b^3) = L$ then exponentiate both sides and manipulate. You get $L/2 = 6$, or $L = 12$.

(c) $\ln(e^{3x}2^{2x}) = 5$, or $3x(1) + 2x\ln(2) = 5$, or $x = 5/(3 + 3\ln(2))$.

17. Done in class.

18. Done in class.

19. Two radians is approximately how many degrees? $2 \cdot \frac{180^\circ}{\pi} \approx 114.59$

20. Suppose $f(x) = 3x^2 − 18x + 34$. Complete the square, then use that form to verify that $x = −1$ and $x = 5$ are solutions to the equation $f(x) = 19$.

\[
f(x) = 3 \left( [x^2 − 6x + \frac{9}{3}] + \frac{34}{3} \right)
\]
\[
f(x) = 3 \left( [x^2 − 6x + 9] − 9 + \frac{34}{3} \right)
\]
\[
f(x) = 3[x − 3]^2 + (34 − 27) = 3(x − 3)^2 + 7
\]

Now, plug in $−1$ and $5$ or you easily see that you get $19$ each time. Alternatively, you could solve for $x$ directly: $3(x − 3)^2 + 7 = 19 \implies (x − 3) = \pm(19 − 7)/3 = \pm 4$. Hence $x = −1, 5$.

21. True or false? $\log_4 x = \frac{\ln x}{\ln 4}$. True. And useful, for when you want to compute using a calculator!

22. (a) Solve for $x$ if $\log(x/2) = −1/2$.

(b) Verify that $x = 1$ would be the only solution to $\log(x(x − 3)) − \log(2x − 6) = −1/2$. Explain why this cannot actually be a solution to this equation. (*Hint: domain!*)

(a) Exponentiate (with $10$) to get: $\frac{x}{2} = 10^{-1/2} \implies x = \frac{2}{\sqrt{10}}$.

(b) TYPO!! $(x = \frac{2}{\sqrt{10}}.)$ Combine the logs on the left-hand side and simplify to get the equation in part (a). Now $2 \cdot \frac{2}{\sqrt{10}} − 6$ is negative, so the second log in the problem is undefined there. Hence, *no solution*.

23. Solve for the unknowns $\phi$ and $x$.

$\sin^{-1}(.7346) = 47.275^\circ$. Now subtract this from $180^\circ$ to get the true answer.

Law of Cosines.

24. If $\sin \theta < 0$ and $\cos \theta = −.7$, determine $\theta$.

If sine and cosine are both negative, then $\theta$ is in the third quadrant. $\cos^{-1}(−.7) = 2.3462$, so the true answer is $2\pi − 2.3462$.\n
25. Sketch a graph of \( \cos^{-1} x \). What is its domain and range? Name (precisely) 3 points on the curve.  
Draw \( \cos x \) between 0 and \( \pi \) and then flip across the line \( y = x \).

26. Express \( 3 \cos(2t - 1) - 2 \sin(2t - 3) \) as a single sinusoidal function \( A \sin(B(t - h)) \).  
Key-Idea: Put \( \phi = -Bh \) and rewrite \( A \sin(Bt + \phi) \) using the angle-addition identity, then match up with the original sum. Since the original sum also has phase shifts, you also need to use angle-addition identities on these. Result of the latter: \( 3 \cos(1) - 2 \sin(3) \cos(2t) + (3 \sin(1) + 2 \cos(3)) \sin(2t) = A \cos \phi \sin(2t) + A \sin \phi \cos(2t) \). (See page 322 to complete the problem.)

27. Match the graphs to the functions. (Two graphs will be used twice.)

TYPO!! (I want (b) to read \( 3 + \cos(\frac{\pi}{4} t) \)).

(a) row 3-2  (b) row 3-1  (c) row 2-2  (d) row 2-2  
(e) row 2-1  (f) row 1-2  (g) row 1-1  (h) row 1-1

28. Find all solutions (solve exactly) for \( 0 \leq \alpha < 2\pi \): \( \cos(2\alpha) = -\sin(\alpha) \).  
First, replace \( \cos(2\alpha) \) with \( 1 - 2 \sin^2(\alpha) \). Then see the idea for solutions on quiz #6.

29. Suppose \( \cos(\theta) = -0.07073 \) and \( \sin(2\theta) = -0.14112 \). Determine \( \cos(3\theta) \) to four decimal places.  
This problem got quite complicated. I could solve it for you, but you wouldn’t be any happier than by my just telling you to ignore it. The skills I wanted to test are: (i) do you know which quadrant \( \alpha \) belongs to? (ii) Do you know when to use trig identities. These two skills are better tested in Problems (29) and (24).

30. Given the triangle below, express \( \cos(2\theta) \) in terms of \( x \).  
TYPO!! (The bottom-left angle should be labeled \( \theta \).)

Here, you should use \( \cos(2\theta) = 1 - 2 \sin^2(\theta) = 1 - 2 \left( \frac{\overline{x}}{\sqrt{1 + x^2}} \right)^2 = 1 - \frac{2x}{1 + x^2} \).

31. Use an angle addition formula to compute \( \sin\left(\frac{5\pi}{12}\right) \). (Hint: \( 5\pi = 3\pi + 2\pi \).)  
\( \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \). Since Now use angle-addition formula for sine. Since you know (exactly) the sine and cosine values of \( \pi/4 \) and \( \pi/6 \), you are “done.”

32. Plot the indicated polar curves for \( 0 \leq \theta < 2\pi \): \( r(\theta) = 3 + 2 \cos \theta \quad r(\theta) = \sin \theta \quad r(\theta) = \tan \theta \)  
It’s okay to use your graphing calculator to plot these. (Ask a friend if you don’t know how.) However, you need to be able to identify why the picture looks the way it does, by being able to graph a few points.

33. Done in class.

34. (a) Plot the points \( (r, \theta) = (3, 3) \) and \( (r, \theta) = (3, 7) \) in the Cartesian plane.  
(b) Find the polar coordinate form for the points \( (x, y) = (-2, -3) \) and \( (x, y) = (2, -3) \).  
(a) These polar coordinates correspond to the \( (x, y) \) coordinates \( (-2.9100, 0.4234) \) and \( (2.2617, 1.9710) \), respectively.
(b) These rectangular coordinates correspond to the \((r, \theta)\) coordinates \((\sqrt{13}, 4.1244)\) and \((\sqrt{13}, -0.9827)\), respectively.

35. Let \(p(x)\) and \(q(x)\) be polynomials.

(a) Explain the three possibilities for the long-term behavior of the rational function \(p(x)/q(x)\) and give an example of each.

(b) What must happen at \(x = 3\) for \(p(x)/q(x)\) to have a root there? to have a vertical asymptote there? (Hint: your answer should be in terms of \(p(3)\) and/or \(q(3)\).)

(a) The limit \(\lim_{x \to \infty} \frac{p(x)}{q(x)}\) depends on the degrees of \(p\) and \(q\).
If \(\deg(p) = \deg(q)\), then it goes to some finite number \(k\) (the ratio of leading coefficients).
If \(\deg(p) < \deg(q)\), then it goes to 0.

(b) \(p(3) = 0; q(3) = 0\).

36. Let \(f(x) = \frac{1}{x+3}\) and \(g(x) = \ln x\). Find the domain and range of \(f \circ g\) and \(g \circ f\).

(a) domain: \(x > 0\) but not equal to \(1/e^3\). range: \(y \neq 0\)
(b) domain: \(x > -3\). range: all real numbers \(y\)

37. Show (either algebraically or graphically) that no inverse function exists for the function \(f(x) = x^2\).
Show that the function \(g\) with domain \([0, \infty)\) and rule \(g(x) = x^2\) does have an inverse.
Which is it? \(f^{-1}(x) = \sqrt{x}\) or \(f^{-1}(x) = -\sqrt{x}\)\? Don’t know which to take. (Alternatively, \(f\) doesn’t pass the horizontal line test, so \(f^{-1}\) can’t exist.)

38. Let \(f(x) = \frac{x}{x+3}\). Find the inverse and verify that it is one by simplifying \(f(f^{-1}(x))\) completely. Also, state the range of \(f\) and the domain of \(f^{-1}\).
\[f^{-1}(x) = 3x/(1 - x)\] domain(\(f\)) = range(\(f^{-1}\)) = \(\{x \neq -3\}\).

39. Suppose \(m, n\) are two even numbers satisfying \(m > n > 1\). Match the function to the graph.

(a) \(x^{-m}\) (b) \(x^{-n}\) (c) \(-x^m\) (d) \(-x^n\)

(a) is the solid in the first picture; (b) is the dotted in the first picture; (c) is the solid in the second picture; (d) is the dotted in the first picture.

40. (a) (T / F) A polynomial of degree \(n\) always has \(n\) (possibly repeated) real roots.
(b) (T / F) A poly. of odd degree \(n\) always has at least one real root and at most \(n\) real roots.
(c) (T / F) A poly. of degree \(n\) always has \(n\) (possibly repeated) complex roots.
(d) (T / F) A quadratic polynomial can have 0, 1, or 2 distinct real roots.

F (at most \(n\))
T (because complex roots come in pairs!)
T (every polynomial can be factored completely over the complex numbers)
T (consider: \(x^2 + 2, (x - 1)^2, \) and \((x - 1)(x + 1))\)