Midterm Exam #1-A
Applied Calculus – Math 131.009 – Spring 2012

Date: 09/27/2012 Name: Solutions

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- No books or notes of any kind are allowed. Approved calculators are allowed.
- There are eight (8) questions.
- The value of each question is displayed above and on the respective problem pages.
- Show the details of your work.
- Partial credit will be awarded, but ONLY if you show your work.
- The exam lasts 75 minutes.

Bonne Chance!!
1. (10 pts) The supply and demand "curves" for a product are given, respectively, by

\[ q_S(p) = 3p - 50 \quad \text{and} \quad q_D(p) = 100 - 2p. \]

(a) Give a rough sketch of the supply and demand curves below according to the economists' custom, labeling both axes and both curves.

(b) What is the equilibrium price and quantity for this market?

\[ 3p - 50 = 100 - 2p \]

\[ 5p = 150 \]

\[ p^* = 30 \]

\[ q^* = q_D(p^*) = 100 - 2(30) = 40 \]

\[ (p^*, q^*) = (30, 40). \]
2. (14 pts) The algae Cladophora is native to Lake Michigan and blooms (to the point of nuisance) in the presence of high concentration of phosphorous in the water. There were severe blooms between the 1950s and 1970s, until fertilizer regulations were put in place. But the blooms are back! (Scientists believe they are getting their phosphorous from the invasive zebra mussels.) On May 1st, a team of Loyolans began to monitor the problem by recording phosphorous concentrations. Here are their findings. (Below, \( t \) is measured in days after May 1st, so \( t = 0 \) means May 1st, and \( P \) is in \( \mu g/L \))

<table>
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<tr>
<th>( t )</th>
<th>0</th>
<th>31</th>
<th>61</th>
<th>92</th>
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<td>( P(t) )</td>
<td>5.1</td>
<td>8.1</td>
<td>9.1</td>
<td>10.2</td>
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(a) (4 pts) During what month (May, Jun., Jul., Aug.) was the phosphorous concentration greatest?

August: both 10.2 and 14.3

(b) (5 pts) During what month (May, Jun., Jul., Aug.) was the average monthly rate of change in phosphorous concentration the greatest? Estimate it, including units in your answer.

May: \[
\frac{8.1 - 5.1}{31} = .097
\]

July: \[
\frac{10.2 - 9.1}{92 - 61} = .035
\]

June: \[
\frac{9.1 - 8.1}{61 - 31} = .033
\]

Aug: \[
\frac{14.3 - 10.2}{122 - 92} = .137
\]

August: \( .137 \mu g/L \) per day

(c) (5 pts) During what month (May, Jun., Jul., Aug.) was the average monthly relative rate of change in phosphorous concentration the greatest? Estimate it, including units in your answer.

May: \[
\frac{.097}{5.1}
\]

July: \[
\frac{.035}{9.1}
\]

June: \[
\frac{.033}{8.1}
\]

Aug: \[
\frac{.137}{10.2}
\]

May: \( 0.019/\text{day} \)

or \( 1.9\%/\text{day} \)
3. (8 pts) A chocolate bar manufacturer has fixed costs of $500,000 per month. It sells chocolate bars for a price of $2 each. The variable cost of producing each individual chocolate bar is $0.50 per bar.

(a) Write the total monthly cost, $C(q)$, and revenue, $R(q)$, functions

$$C(q) = 500,000 + 0.50q$$

$$R(q) = 2q$$

(b) How many chocolate bars must the manufacturer sell in order to make a positive profit?

Need $R > C$:

$$2q = 500,000 + 0.5q$$

$$1.5q = 500,000$$

$$q = 333,333.33$$

$$\Rightarrow \text{ Must sell 333,334 chocolate bars per month to turn a profit}$$
4. (16 pts) A frothy golden "pollutant" begins flowing into the Chicago River at the Goose Island Brewery when a pipe ruptures. Suppose the total spill volume is given by

\[ V(t) = 20(1 - 2^t 3^{-t}) , \]

with \( V \) measured in thousands of gallons and \( t \) in hours.

(a) Estimate \( V'(1) \) and explain what it means in practical terms. (Include units in your answer.)

\[
V'(1) \approx \frac{V(1.001) - V(1)}{1.001 - 1} = 20\left(1 - 2 \cdot 3^{-1}\right) - 20\left(1 - 2^t \cdot 3^{-t}\right)
\]

\[
= 5.41 \text{ thousand gallons per hour}
\]

This means, one hour after spill started, the total spill volume was increasing at rate of 5.41 gallons per hour. (i.e. 1 hour later, there would likely be an extra 5 gallons spilled)

(b) If this model holds for all future time, when will 10,000 gallons have spilled?

(Give a precise answer—no decimal numbers.)

Solve for \( t \):

\[ 10 = 20\left(1 - 2^t 3^{-t}\right) \]

\[ \frac{1}{2} = 1 - 2^t 3^{-t} \]

\[ 2^t 3^{-t} = \frac{1}{2} \]

\[ (\frac{2}{3})^t = \frac{1}{2} \]

\[ t \cdot \ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{2}\right) \]

\[ t = \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)} \]
5. **(10 pts)** Circle T if the statement is true or circle F if it is false.

( T / F ) The domain of a function is the set of its outputs.

( T / F ) If \( C(n) \) is the total cost, in dollars, to feed \( n \) students in the campus cafeteria, then the average rate of change of \( C(n) \) has units of students per dollar.

( T / F ) The instantaneous rate of change is constant for exponential functions.

( T / F ) An equation of a line with slope \(-1\) that passes through the point \((2, 5)\) is given by \( y = -x + 7 \).

( T / F ) If the total-cost function \( C(q) \) is linear, it is likely to be a decreasing function of \( q \).
6. (12 pts) Some transformation questions.

The deer mice and Fennec fox populations are known to be cyclical. (Large mice population means plenty of food for the foxes, which leads to more foxes. Too many foxes means a suddenly scarce food supply, which leads to fewer foxes and a renewed abundance of mice.) Suppose that the fox population in Illinois is modeled by \( F(t) = 10 + 6 \sin \left( \frac{\pi}{6} t \right) \), where \( t \) is measured in months (since January 2012), and \( F \) is measured in hundreds of foxes.

(a) Find the period \( P \) for the population (including units).

\[
\frac{\pi}{6} t = 2\pi \quad \text{solve for } t
\]

\[
t = 12 \text{ months}
\]

(b) Determine the first time \( T \) when the fox population is smallest (including units).

\[
\sin t \text{ is lowest at } \frac{3\pi}{2}.
\]

\[
\text{solve for } t: \quad \frac{\pi}{6} t = \frac{3\pi}{2}, \quad t = 9 \text{ months}
\]
7. (20 pts) Below is the graph of a function $f$ defined on $[-1.5, 5]$.

(a) Estimate $f''(a)$ at each point $a$:

- $a = 0$
  \[ f''(0) \approx \frac{4.5 - z}{5 - 0} \]
- $a = 3.5$
  \[ \approx \]
- $a = 4.5$
  \[ \frac{10 - (-12.5)}{5 - 4} = 22.5 \]

(c) Use calculus and your work above to give an estimate for $f(5)$.

\[
f(5) \approx f(4.5) + f'(4.5)(5 - 4.5)
\]

\[
= 0 + 22.5(0.5) = \underline{11.25}
\]

(d) On what intervals is $f'$ positive? negative?

\( (+ ) : \left[ -1.5, 1 \right] \cup (3.5, 5) \)

\( (- ) : (1, 3) \cup (3, 5) \)

(e) On what intervals is $f''$ increasing? decreasing?

\( (\nearrow ) : (3, 5) \)

\( (\searrow ) : (1, 3) \)
8. (10 pts) A function $f$ is defined on $[-1, 25]$. Below is the graph of its derivative, $f'$.

Suppose $f(0) = -3$.

(a) (4 pts) What is $f(1)$?

$f'$ is constant at 4 on $(0, 1)$. So $[\text{distance} = \text{rate} \cdot \text{time}]$ gives

$$f(1) = -3 + 4 = 1.$$

(b) (6 pts) Estimate the sign of each of the following, giving a brief justification for each:

- $f(4.75)$ (+)
  
  $f'$ is positive for all of $(0, 4.75)$, so $f$ increasing throughout the entire time. And it is already positive at $x = 1$.

- $f(20)$ (-)
  
  $f'$ is negative and becomes very negative for the range $(4.75, 20)$, so $f$ is decreasing (rapidly) on that range. So $f(20)$ is likely to be negative—even though $f(4.75)$ is quite positive—
BONUS: (10 pts) Given: any nice (smooth, continuous) function $f$.

For each transformation below, use the definition of derivative to explain the formulas.

- Define $g$ by $g(x) = f(x) + 2$. Then $g'(x) = f'(x)$.

\[
g'(x) \sim \frac{g(b) - g(x)}{b - x} = \frac{(f(b) + 2) - (f(x) + 2)}{b - x} = \frac{f(b) - f(x)}{b - x} \sim f'(x),
\]

- Define $g$ by $g(x) = 2f(x)$. Then $g'(x) = 2f'(x)$.

\[
g'(x) \sim \frac{g(b) - g(x)}{b - x} = \frac{(2f(b)) - (2f(x))}{b - x} = 2 \frac{f(b) - f(x)}{b - x} \sim 2f'(x).
\]