Midterm Exam #2-A
Applied Calculus – Math 131.009 – Spring 2012

Date: 11/01/2012  Name: Solutions

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- No books or notes of any kind are allowed. Approved calculators are allowed.
- There are NINE(9) questions.
- The value of each question is displayed above and on the respective problem pages.
- *Show the details of your work.*
- Partial credit will be awarded, but **ONLY** if you show your work.
- The exam lasts 75 minutes.

*Bonne Chance!!*
1. **(18 pts)** A charter bus company’s per-person costs and revenue both depend on the number of people carried. Let us help them find the optimum number of people to transport on any given trip. Suppose their average cost and revenue, per person, are given by \( \frac{40}{10+n} \) and \( \frac{20}{n} - \frac{10}{n^2} \), respectively. Here, \( n \) is the number of people, measured in tens of people, and prices are in hundreds of dollars.

(a) Give formulas for the cost \( C \), revenue \( R \), and profit \( P \) in terms of the number of people \( n \) carried.

\[
\frac{40}{10+n} = \frac{C(n)}{n} \quad \text{and} \quad \frac{20}{n} - \frac{10}{n^2} = \frac{R(n)}{n} \quad \Rightarrow \\

C(n) = \frac{40n}{10+n} \quad R(n) = 20 - \frac{10}{n} \\

P(n) = R - C = 20 - \frac{10}{n} - \frac{40n}{10+n}
\]

(b) What number maximizes profit?

Find \( P'(n) = 0 \).

\[ P'(n) = 0 - (-10n^{-2}) - \frac{[40](10+n)-(40n)[1]}{(10+n)^2} \]

\[ = \frac{10}{n^2} - \frac{400}{(10+n)^2} \]

Now \( 0 = \frac{10}{n^2} - \frac{400}{(10+n)^2} \Rightarrow \frac{400}{(10+n)^2} = \frac{10}{n^2} \Rightarrow 400n^2 = (10+n)^2 = 100 + 20n + n^2 \]

\[ \Rightarrow 39n^2 - 20n - 100 = 0 \Rightarrow n = \frac{-20 \pm \sqrt{(-20)^2 - 4(39)(-100)}}{2 \cdot 39} \approx 1.878 \text{ or } -1.365 \]

**Ans:** 1.9, or 19 people

(c) What is that profit?

\[ P(19) = 20 - \frac{10}{1.9} - \frac{40(1.9)}{10+1.9} \approx 8.35, \text{ or } \$835 \]
2. (10 pts) Here is a graph of $f'$, together with four candidate graphs of $f$. Choose the best candidate, and justify your answer using calculus (e.g., matching inflection points, critical points, and local extrema of $f$ on the graphs of $f$ and $f'$).
3. (20 pts) Compute the indicated derivatives

(a) \( \frac{d}{dx} \left( x - (4x - 1)^2 \right)^3 \)
\[ = 3 \left( x - (4x - 1)^2 \right)^2 \cdot \left( 1 - 2(4x - 1)^3 \right) (4) \]

(b) \( \frac{d}{dx} \sin x \cos 2x \)
\[ = \left[ \cos x \right] (\cos 2x) + \sin x \left[ -2 \sin 2x \right] \]
\[ = \cos x \cos 2x - 2 \sin x \sin 2x \]

(c) \( f''(\ln 2), \text{ if } f(x) = e^{2x} \) (simplify completely)
\( f'(x) = 2e^{2x} \)
\( f''(x) = 4e^{2x} \)
\( f''(\ln 2) = 4 e^{2 \ln 2} = 4 e^{\ln 4} = 4 (2^2) = 16 \)

(d) \( \frac{d}{dx} x \ln 2x \)
\[ = \left[ 1 \right] \ln 2x + x \left[ \frac{2}{2x} \right] = \ln 2x + 1 \]
4. (15 pts) Compute the indicated derivatives

(a) \[ \frac{d}{dx} e^{2x} \left( 1 + x \right) \]
\[ = \frac{\left[ 2 e^{2x} (1 + x) - \left( e^{2x} \right) [1] \right]}{(1 + x)^2} \]

(b) \[ \frac{d}{dx} e^{2x + \ln x} \]
\[ = \left( e^{2x + \ln x} \right) \left( \frac{2x}{x} + \frac{1}{x} \right) \]

(c) \[ \frac{d}{dx} e^{x + \ln(2x^2)} \]
\[ = \left( e^{x + \ln(2x^2)} \right) \left( 1 + \frac{4x}{2x^2} \right) \]
5. (12 pts) Circle T if the statement is true or circle F if it is false.

(T) Marginal cost equals marginal profit at critical points of marginal revenue.

(F) The parameter \( k \) in the logistic function \( P = \frac{L}{1 + Ce^{-kt}} \) affects the rate at which the curve approaches the asymptote \( L \).

(T) The function \( f(x) = x^3 - 3x^2 \) restricted to the domain \([0, 3]\) achieves its global maximum within the interval (i.e., away from the endpoints).

\[
\begin{align*}
\frac{df}{dx} &= 3x^2 - 6x = 3x(x - 2) \\
\frac{d^2f}{dx^2} &= 6x - 6, \quad \frac{d^2f}{dx^2}(2) > 0
\end{align*}
\]

The function achieves a local maximum at \( x = 2 \).

(F) The derivative of a product of two functions is the product of their derivatives.

(T) If \( f'(x) = e^{-2x}(1 - x) \), then \( f \) increases up to \( x = 1 \), and decreases thereafter.

\[
\begin{align*}
f'(x) &= e^{-2x}(1 - x) \\
f'(0) &= 1 > 0 \\
f''(x) &= -2e^{-2x}(1 - x) < 0
\end{align*}
\]

(F) If \( f(-2) = 12, f'(-2) = -12, f(3) = -7, f'(3) = 7, g(0) = -2 \), and \( g'(0) = 3 \), then \( \frac{d}{dx} f(g(x)) \) at \( x = 0 \) is \(-14\).

\[
\begin{align*}
\frac{d}{dx} f(g(x)) &= f'(g(x)) \cdot g'(x) \\
&= f'(-2) \cdot 3 = (-12)(3) = -36
\end{align*}
\]
6. (5 pts) The graph of $f'$ is given below. (Note that $f$ has two critical points. How many inflection points?) Use the information about $f'$ and $f''$ that you can gather from this graph to sketch the graph of $f$ in the region provided. (Assume $f(0) = 0$.)

\[ f': \begin{array}{cccccc}
& (-) & (-) & (+) & (+) & (+) \\
\end{array} \]

\[ f'': \begin{array}{cccccc}
& (+) & (+) & (-) & (-) & (+) \\
\end{array} \]
7. (6 pts) The following are three snapshots of revenue (solid) and cost (dotted) curves for a tricycle manufacturer, as a function of quantity produced.

For each scenario, indicate in the space provided which of (a), (b) or (c) is most appropriate, and also which of (i), (ii) or (iii) is most appropriate.

(a) the company should increase production  
(b) the company should decrease production  
(c) the company should maintain production  

(i) the company is making a profit  
(ii) the company is losing money  
(iii) the company is breaking even.

\[ \text{answers: } \boxed{a} \quad \boxed{ii} \]

\[ \text{answers: } \boxed{a} \quad \boxed{i} \]

\[ \text{answers: } \boxed{b} \quad \boxed{i} \]
8. (6 pts) If \( C(q) = q^3 - 4q + 8 \), find the quantity \( q \) that minimizes the average cost of the items produced.

\[
a(q) = \frac{C(q)}{q} = \frac{q^2}{4} - \frac{8}{q}
\]

\[
a'(q) = 2q - \frac{8}{q^2}.
\]

\[
a'(q) = 0 \implies 2q = \frac{8}{q^2} \implies 2q^3 = 8 \implies q = \sqrt[3]{4}
\]

This is the only critical point, so likely a min... Check: \( a''(q) = \frac{24q^2}{q^4} \). Yeah, positive inside. @ \( \frac{\sqrt[3]{4}}{4} \).

9. (8 pts) (Elasticity of Demand)

(a) Name a commodity that you expect to: (i) have elastic demand; (ii) have inelastic demand.

(i) Cable provider (too big any sizeable increase and you will switch providers)

(ii) Potatoes (if you're reduced to buying potatoes at the grocery, you won't stray, so demand much less, if price increase)

(b) Determine which of (i) or (ii) is most likely to be represented by \( q(p) = (p - 2)^2 \) at \( p = 1 \).

Justify your answer with some mathematics.

\[
E = \left| \frac{p}{(p-2)^2} \cdot 2(y-2) \right| = \left| \frac{2p(p-2)}{(p-2)^2} \right| = \left| \frac{2p}{p-2} \right|.
\]

\[
E \text{ at } p = 1 : \left| \frac{2p}{p-2} \right| = 2. \text{ So elastic so (i).}
\]

(c) Should the company increase or decrease production? decrease, for greater revenue.
BONUS: (5 pts) A rectangle in the first quadrant has one side on the \(x\)-axis, one side on the \(y\)-axis, one vertex at the origin, and the opposite vertex on the curve \(y = -\ln(2x)\). Find the critical points for the function \(A(x)\) giving the total area of the rectangle.

\[
y = -\ln(2x).
\]

\[
A(x) = x \cdot y = -x \ln(2x).
\]

\[
A'(x) = -\ln(2x) + \frac{2x}{2x}
\]

\[
0 = -\ln(2x) + 1 \Rightarrow \ln(2x) = 1
\]

\[
\Rightarrow 2x = e^1
\]

\[
\Rightarrow x = \frac{1}{2} e^2
\]

Geometric reasoning alone cannot determine whether or not your answer above corresponds to a maximum of \(A(x)\). (Why?) Use the second derivative test to verify that this is indeed a maximum.

really small \(x\) means really big \(y\) \(\Rightarrow\) don't know whether these products would be bigger or smaller than \(A'(\frac{e^2}{2})\).

\[
A''(x) = \frac{-2x}{2x} = -\frac{1}{x}.
\]

So at \(x = \frac{e^2}{2}\), we get \(A''(\frac{e^2}{2}) < 0\) so corresponds to a max
BONUS: (5 pts) What effect does a change in the parameter $C$ have in the logistic curve

$$P(t) = \frac{10}{1 + Ce^{-0.5t}}?$$

Initial population.

At what time does $P$ reach one-half of its carrying capacity?

$$\frac{10}{2} = \frac{10}{1 + Ce^{-0.5t}}$$

$$2 = 1 + Ce^{-0.5t}$$

$$1 = \frac{C}{e^{0.5t}}$$

$$e^{0.5t} = C$$

$$0.5t = \ln C$$

$$t = \frac{\ln C}{0.5} = \sqrt{2 \ln C}$$