Midterm Exam #1-A
Applied Calculus – Math 132.003 – Fall 2011

Date: 10/06/2011    Name: ____________________________

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- No books or notes of any kind are allowed. Calculators are allowed.

- There are SEVEN(7) questions.

- The value of each question is displayed above and on the respective problem pages.

- Show the details of your work.

- Partial credit will be awarded, but only if you show your work.

- The exam lasts 75 minutes.

*Bonne Chance!!*
1. (15 pts) The algae Cladophora is native to the Great Lakes and blooms (to the point of nuisance) in the presence of high concentration of phosphorous in the water. There were increasingly severe blooms from the 1950s to the 1970s, until regulations on fertilizers were put in place. But the blooms are back! (It is believed that they are getting their phosphorous from the invasive zebra mussels.) On May 1st, a team of Loyolaans began to monitor the problem by recording phosphorous concentrations. Here are their findings. (Below, $t$ is in days after May 1st, so $t = 0$ means May 1st, and $P$ is in $\mu g/L$.)

<table>
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<tr>
<th>$t$</th>
<th>0</th>
<th>31</th>
<th>61</th>
<th>92</th>
<th>122</th>
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<td>$P(t)$</td>
<td>5.1</td>
<td>8.1</td>
<td>9.1</td>
<td>10.2</td>
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(a) During what month was the average rate of change in phosphorous the highest?

Look for largest \( \frac{\Delta P}{\Delta t} \).\[ \frac{14.3 - 10.2}{122 - 92} \]

seems largest, so **August**.

(b) During what month was the average relative rate of change in phosphorous the highest?

Look for largest \( \frac{1}{P} \frac{\Delta P}{\Delta t} \).

\[ \frac{8.1 - 5.1}{8.1 \cdot 31 - 0} \]

seems largest, e.g. larger than \( \frac{1}{10.2} \frac{14.3 - 10.2}{122 - 92} \)

So **May**.

(c) Estimate the bioavailability of phosphorous for the algae over the duration of the study. (You need only compute the left Riemann sum; be sure you get the units correct.)

\[
\text{bioavailability} = \int_0^T P(t) \, dt \approx \frac{5.1}{31} + \frac{8.1}{30} + \frac{9.1}{31} + \frac{10.2}{30}
\]

units are $\mu g \cdot \text{days}$.
2. (15 pts) A frothy golden "pollutant" begins flowing into the Chicago River at the Goose Island Brewery when a pipe ruptures. Suppose 100 gallons have spilled by the time the rupture is detected. As the workers repair the damage, the rate at which pollution enters is given by

\[ r(t) = (t + 1)^{-1} \ln(t + 1) \text{ gallons per hour (with } t \text{ in hours).} \]

(a) If the pipe is repaired in three hours, how much pollution is spilled during the repair? (You may use your calculator to get the answer, but indicate what you enter.)

\[
\text{ans} = \int_0^3 r(t) \, dt = \frac{1}{2} \left( \ln (t+1) \right)^2 \bigg|_0^3 \text{ gallons} = \frac{1}{2} (\ln 4)^2 - \frac{1}{2} (\ln 1)^2 = \frac{1}{2} (\ln 4)^2 \text{ gallons}
\]

(b) In your estimation, would a finite or infinite amount of pollutant enter the river if the repairmen worked from now until eternity?

- A picture of \( r(t) \) suggests that \( \int_0^\infty r(t) \, dt = \infty \)
- \( \ln \ln (r(t), t, 0, 10,000) \) is a very large number, suggests \( \infty \) amount
- \( \frac{1}{2} (\ln (T+1))^2 = \int_0^T r(t) \, dt \) Letting \( T \to \infty \) in the graph of \( [\ln(T+1)]^2 \) shows that it also goes to \( \infty \), proves \( \infty \) amount

(c) Determine the value of \( b > 0 \) so that if the leak is repaired after \( b \) hours, then the total pollutant from the spill is 102 gallons.

\[
\text{Total} = \text{Initial} + \int_0^b \text{Rate of change} = 100 + \int_0^b r(t) \, dt = 102 \text{ gallons.}
\]

Solve for \( b \):

\[
\int_0^b \frac{\ln(t+1)}{t+1} \, dt = 2.
\]

(you may use your calculator)

\[
\frac{1}{2} (\ln(b+1))^2 = 2 \Rightarrow [\ln(b+1)]^2 = 4
\]
\[
\Rightarrow \ln(b+1) = 2
\]
\[
\Rightarrow b + 1 = e^2 - 1
\]
3. (10 pts) Circle T if the statement is true or circle F if it is false.

(T/F) A consequence of the fundamental theorem of calculus is that if \( F''(t) = f(t) \) then
\[
F(x) = F(a) + \int_a^x f(t) \, dt.
\]
\[
F(x) - F(a) = \text{total change} = \int_a^x F'(t) \, dt
\]

(T/F) If \( \int_0^2 |f(x)| \, dx = \int_0^2 f(x) \, dx \) then \( f(x) \geq 0 \) for all \( 0 \leq x \leq 2 \). Should be \( \geq \), but okay.

(T/F) If \( f(t) \) is a increasing function, then \( \int_a^b f(t) \, dt \) is over-estimated by a left Riemann sum.

(T/F) Suppose electricity costs \( f(t) \) dollars per day. A Riemann sum with 20 subintervals that estimates the total cost of electricity during June has \( \Delta t = 3/2 \).
\[
\Delta t = \frac{b - a}{n} = \frac{30}{20} = \frac{3}{2}
\]

(T/F) The substitution \( w = 2q^3 - 6q \) converts
\[
\int (2q^3 - 6q)^{10} \sin(6q^2 - 6) \, dq
\]
into
\[
\int w^{10} \cos(w) \, dw.
\]
4. (24 pts) Evaluate the following integrals. Show all steps and give exact answers. Approximations and incomplete solutions will not receive full credit.

(a) \[ \int_1^2 x^{1/3} - \frac{1}{2x} \, dx \]
\[ = \left[ \frac{3}{4} x^{4/3} - \frac{1}{2} \ln|x| \right]_1^2 = \left( \frac{3}{4} \cdot \sqrt[3]{16} - \frac{1}{2} \ln 2 \right) - \left( \frac{3}{4} \cdot 1 - \frac{1}{2} \cdot 0 \right) \]

(b) \[ \int x e^x \, dx \quad w = x^2 \quad dw = 2x \, dx \]
\[ \rightarrow \quad \frac{1}{2} \int e^w \, dw \quad \rightarrow \quad \frac{1}{2} e^w + C \]

(c) \[ \int_0^1 \frac{2t}{\sqrt{9 - 5t^2}} \, dt \quad w = 9 - 5t^2 \quad dw = -10t \, dt = -5(2t \, dt) \]
\[ \rightarrow \quad \frac{1}{5} \int_0^3 \frac{dw}{\sqrt{w}} = \frac{1}{5} \cdot 2 \sqrt{w} \bigg|_0^3 = -\frac{2}{5} \left( 9 - 5t^2 \right)^{1/2} \bigg|_0^1 \]
\[ = \left( -\frac{2}{5} \sqrt{4} \right) - \left( -\frac{2}{5} \sqrt{9} \right) = \frac{6}{5} - \frac{4}{5} = \frac{2}{5} \]

(d) \[ \int_1^2 x \cos(2x) \, dx \quad u = x \quad dv = \cos(2x) \, dx \]
\[ \Rightarrow \quad du = dx \quad v = \frac{1}{2} \sin(2x) \]
\[ \int_1^2 uv \, dx = \frac{1}{2} \int_1^2 x \sin(2x) \, dx - \frac{1}{2} \left[ \sin(2x) \right]_1^2 \]
\[ = \frac{1}{2} \sin(2x) \bigg|_1^2 + \frac{1}{4} \cos(2x) \bigg|_1^2 \]
\[ = \sin(4) - \frac{1}{2} \sin(2) + \frac{1}{4} \cos(4) - \frac{1}{4} \cos(2) \]
5. (10 pts) Your house will need a new roof in 10 years, so you decide to start a savings account specifically for that purpose. If it will cost $60,000, is it sufficient to invest at the continuous annual rate of $S(t) = 4500 + 45t$ dollars per year? (Assume an annual continuously compounding interest rate of 5%).

$$FV = \int_0^{10} (4500 + 45t) e^{0.05(10-t)} \, dt$$

$\text{INT} \rightarrow 61,061$, so yes.

Detailed answer requires integration by parts:

$$\int 45t \, e^{0.05(10-t)} \, dt \quad u = 45t \quad dv = e^{0.05(10-t)} \, dt$$

$$du = 45 \, dt \quad v = \frac{1}{0.05} e^{0.05(10-t)}$$

6. (16 pts) Below is the graph of a function $f$ defined on $[0, \infty)$. Let $F'(x) = f(x)$.

(a) What are the critical points of $F$?

Places where $F' = 0$, so

$x = 6$ and $x = 12$.

(b) On what intervals is/are $F$ increasing? decreasing?

inc: $(6, 12)$

dec: $(0, 6) \cup (12, \infty)$

(c) If $F(3) = 0$, sketch a graph of $F$, taking care to get the concavity correct. (Be sure it agrees with your answers to (a) and (b).)
7. (10 pts) Circle T if the statement is true or circle F if it is false.
(T/F) The integral of the relative growth rate of a population \( P \) gives the total change in \( P \).

(T/F) If the relative growth rate of a population \( P \) is given by \( (1/4)t \) for \( 0 \leq t \leq 2 \), then \( P(2) \) is roughly 65% greater than \( P(0) \).

\[
\frac{P'}{P} = \frac{1}{4}t \quad \Rightarrow \quad \ln P \bigg|_0^2 = \int_0^2 \frac{1}{4}t \, dt = \frac{1}{8} \bigg|_0^2 = \frac{1}{2},
\]

\[
\ln P(2) - \ln P(0) = \ln \left( \frac{P(2)}{P(0)} \right) = \frac{1}{2} \quad \Rightarrow \quad \frac{P(2)}{P(0)} = e^{\frac{1}{2}} \approx 1.65
\]

So, \( P(2) \approx 1.65 \cdot P(0) \).

(T/F) \( \int_1^2 3x^2 \, dx = \frac{9}{2} - \frac{3}{2} \).

(T/F) Integration by parts is a useful method to use in evaluating \( \int 10x^4\sqrt{2x^5 + 1} \, dx \).

With substitution \( w = 2x^5 + 1 \)

\[
dw = 10x^4 \, dx
\]

(T/F) Assuming a positive interest rate, the present value of an income stream is always less than its future value.

If \( FV \) in \( T \) year is \( 100 \), then \( PV = 100 \cdot \frac{e^{-T}}{1} \),

\(< 1\)
Extra work space for Problem ☐.