Midterm Exam #2-B
Applied Calculus – Math 132.003 – Fall 2011

Date: 11/15/2011    Name: ____________________________

<table>
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<th>Ques.</th>
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- No books or notes of any kind are allowed. Calculators are allowed.
- There are eight (8) questions, worth a varying amount of points, as listed above. A perfect score is 100 points, but you have 110 chances to get it.
- *Show the details of your work.*
- Partial credit will be awarded, but only if you show your work.
- The exam lasts 75 minutes.

*Bonne Chance!!*
1. (16 pts) The algae Cladophora, native to Lake Michigan, blooms in the presence of high concentration of phosphorous. Last April 30th, a team of Loyolans endeavored to measure concentration versus time elapsed and size of zebra mussel population by maintaining small populations on the lakebed near Loyola Beach and measuring the concentration throughout the summer. Here are their findings. (Below, $t$ is in days after April 30th, $z$ is number of zebra mussels, and phosphorous $P$ is in $\mu g/L$.)

<table>
<thead>
<tr>
<th>$z$</th>
<th>$t$ (days)</th>
<th>0</th>
<th>30</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>150</th>
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<tr>
<td>10</td>
<td>2.2</td>
<td>3.1</td>
<td>8.2</td>
<td>9.9</td>
<td>14.2</td>
<td>11.3</td>
<td></td>
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<td>20</td>
<td>2.1</td>
<td>3.3</td>
<td>8.1</td>
<td>9.1</td>
<td>12.3</td>
<td>10.8</td>
<td></td>
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<tr>
<td>30</td>
<td>2.2</td>
<td>3.8</td>
<td>9.1</td>
<td>11.4</td>
<td>14.6</td>
<td>13.7</td>
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<tr>
<td>40</td>
<td>2.1</td>
<td>2.8</td>
<td>8.9</td>
<td>10.9</td>
<td>10.5</td>
<td>9.9</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>2.1</td>
<td>3.1</td>
<td>9.0</td>
<td>-</td>
<td>11.2</td>
<td>7.0</td>
<td></td>
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</tbody>
</table>

(a) Based on the data, would you say: (a) zebra mussels have an effect on phosphorous concentration? (b) How about water temperature? Justify your answers.

(a) Not really, for different dates, it's just as likely to decrease as increase, moving down the columns.

(b) Yes, if we correlate distance into the summer with water temp.

(b) Estimate $\partial P/\partial z$ and $P_t$ at $(z, t) = (40, 60)$. Include units.

$$\frac{\partial P}{\partial z} \approx \frac{9.0 - 8.9}{50 - 40} = \frac{0.1}{10} = 0.01 \frac{\mu g}{L \cdot mussel}$$

$$P_t \approx \frac{10.9 - 8.9}{90 - 60} = 0.0667 \frac{\mu g}{L \cdot day}$$

(c) Use the computations in (b) to explain in real terms the effect an additional day and an additional mussel has on phosphorous concentration. Include units.

$$\Delta P \approx P_z \Delta z + P_t \Delta t = (-0.01)(4) + (0.0667)(4) = 0.0567 \frac{\mu g}{L}$$

(adding a mussel decreases P a bit; adding a day increases P a bit more.)

(d) The student responsible for measuring the phosphorous data at $(z, t) = (50, 90)$ skipped school that day. Use the computations in (b) to estimate $P(z = 50, t = 90)$. Include units.

$$\Delta P \approx (-0.01)(10) + (0.0667)(30) = 1.901$$

$$P(50, 90) \approx P(40, 60) + 1.901 = 10.801 \frac{\mu g}{L}$$
2. (16 pts) Suppose \( p(t) = \begin{cases} 0.25 - 0.03125t, & 0 \leq t < 8 \\ 0, & 8 \leq t \leq 9 \end{cases} \) is the probability density function for shelf-life of a Chiquita banana. (\( t \) is in days.)

(a) Find the cumulative distribution function \( P(t) \).

\[
P(t) = \begin{cases} 
\int_0^t p(x) \, dx & 0 \leq t < 8 \\
1 & t \geq 8
\end{cases}
\]

\[
P(t) = \left[ 0.25x - 0.03125x^2 \right]_0^t = 0.25t - 0.01563t^2
\]

(b) Use \( P(t) \) to determine the median shelf-life. (You may use your calculator to estimate, but you must indicate your steps.)

Find \( T \) so that \( P(T) = \frac{1}{2} \):

\[
\frac{1}{2} = 0.25T - 0.01563T^2 \quad \text{Solve for } T.
\]

\[
T = \frac{-0.25 \pm \sqrt{(0.25)^2 - 4(-0.01563)(\frac{1}{2})}}{2(-0.01563)} = 2.34
\]

(c) Determine the mean shelf-life.

\[
\mu = \int_0^9 x \cdot p(x) \, dx = \int_0^8 (0.25x - 0.03125x^2) \, dx \quad \text{In Int} \rightarrow 2.667
\]
3. (10 pts) Only one of the following could possibly represent the concentration $C(d, t)$ of a drug in the blood as a function of the dose $d$ administered and time $t$ elapsed. Underline that one. Explain one or two features of the functions/graphics that cause you to pick or not pick each one.

(a) $C(d, t) = t^2 - td + 5$
(b) $C(d, t) = 3de^{-0.5t}$
(c) $C(d, t) = \frac{2.3td}{1 + t^2}$

(a) For small doses, concentration increases over time; counter-intuitive.
(b) Concentration increases with dose and decreases with time.
(c) [I can see why some said this is valid, but I was thinking of starting the timer only after dose was fully absorbed] Concentration increases, then decreases with time; counter-intuitive.

4. (12 pts) The length travelled $\ell$ by a football, kicked for a 45 yard field goal, is assumed to be normally distributed. Measurements taken during week 10 of the NFL season seem to corroborate this. Suppose they lead to the following cumulative distribution function $P$ for true-length:

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>43</th>
<th>46</th>
<th>48</th>
<th>49</th>
<th>50</th>
<th>51</th>
<th>52</th>
<th>54</th>
<th>56</th>
<th>60</th>
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</thead>
<tbody>
<tr>
<td>$P(\ell)$</td>
<td>0.0002</td>
<td>0.0225</td>
<td>0.1574</td>
<td>0.3068</td>
<td>0.4980</td>
<td>0.6897</td>
<td>0.8401</td>
<td>0.9770</td>
<td>0.9986</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

(a) Estimate the mean.

\[ \mu \approx 50, \quad \text{since} \quad P(50) = 0.4980 \]

(b) Estimate the standard deviation. Include units.

\[ P(\mu + \sigma) - P(\mu - \sigma) \approx 0.64 \]

\[ \text{Guess } \sigma = 2: \quad P(52) - P(50) = 0.8401 - 0.4980 = 0.3421 \]

\[ \text{good guess! } \sigma \approx 2 \text{ yards} \]
5. **(15 pts)** A certain widget is made in two different factories. Cost (in $1000) of making \( x \) units at one and \( y \) units at another is given by

\[
C(x, y) = 10 + (2x - 4)(y - 4),
\]

(where \( x \) and \( y \) are in thousands of widgets). If you would like to make 8,000 widgets, determine how much should be made at each factory to minimize your cost.

(a) What is the objective function? The constraint equation?

\[
\begin{align*}
\n & \downarrow \\
C(x, y) \equiv & \downarrow \\
 & \Rightarrow \\
& g(x, y) = x + y = 8
\end{align*}
\]

(b) Solve using the method of Lagrange multipliers.

\[
\begin{align*}
C_x = \lambda \quad & \rightarrow \quad 2y - 8 = \lambda \\
C_y = \lambda \quad & \rightarrow \quad 2x - 4 = \lambda \\
g = 8 & \rightarrow \quad x + y = 8 & \rightarrow & \quad x + (x + 2) = 8 & \Rightarrow & \quad 2x = 6 & \Rightarrow & \quad x = 3
\end{align*}
\]

Finally, \( 2(3) - 4 = \lambda \), or \( \lambda = 2 \)

To minimize cost, choose \( (x, y) = (3, 5) \).

(c) Approximately what extra cost is incurred if you wish to increase your output to 10,000 widgets?

\[
\Delta C \approx \lambda \Delta q \quad \text{so} \quad \Delta C \approx 2(2)
\]

\[
\text{extra cost} \approx \$4,000
\]
6. **(15 pts)** Compute the indicated quantities:

(a) \( \frac{\partial}{\partial u} [u^2 - uv + v + 3v^2 - 10] \)

\[ 2u - v + 0 + 0 - 0 = 2u - v \]

(b) \( \frac{\partial}{\partial v} [u^2 - uv + v + 3v^2 - 10] \)

\[ 0 - u + 1 + 6v - 0 = -u + 6v + 1 \]

(c) \( \frac{\partial}{\partial x} \frac{\partial}{\partial y} [\sin(xy)] \)

\[ \frac{\partial}{\partial y} = \cos(xy) \quad \frac{\partial}{\partial x} = [\sin(xy) + x[-\sin(xy) \cdot y]] \]

\[ = \cos(xy) - xy \sin(xy) \]

(d) \( P_{ss}(3, -1) \) if \( P(r, s) = 2r^2 s^2 \)

\[ P_s = 2r \cdot 2s = 4r \cdot s \]

\[ P_{ss} = 4r^2 \quad \text{at} \ (3, -1) = 4(3)^2 = 36 \]

(e) the third partial derivative of \( P \) with respect to \( s \) if \( P(r, s) = 2r^2 s^2 \)

\[ P_{ss} = 4r^2 \quad \Longrightarrow \quad P_{ssss} = 0 \]
7. (10 pts) Circle T if the statement is true or circle F if it is false.

(T/F) In constrained optimization, the method of Lagrange multipliers returns \((p,q)\) when the level curve for the constraint equation is perpendicular to the level curves of the objective function at \((p,q)\).

parallel

(T/F) The second derivative test cannot be used to determine if \(f(x,y) = x^4 + y^3\) has a local max/min at the origin.

\[
D = f_{xx} f_{yy} - (f_{xy})^2 = (4.3x^2)(3.2y) - (0)^2
\]

at the origin, \(D = 0\), which is the only value we don't know how to handle.

(T/F) Both \(\frac{0.154 - 0.022}{48 - 48}\) and \(\frac{0.247 - 0.154}{49 - 48}\) estimate \(p(48)\), if \(p(\ell)\) is the probability density function for length travelled by a football. (See Problem 4.)

There were some typos. Everyone gets full credit.

\[
p(48) = p'(48) = \frac{\Delta P}{\Delta x} = \frac{0.3068 - 0.1574}{49 - 48}, \text{ for one estimate.}
\]

(T/F) If \(f_x(2,3) = f_y(2,3) = 0\), \(D(2,3) < 0\), and \(f_{xx}(2,3) > 0\), then \(f\) assumes a local minimum at \((x,y) = (2,3)\).

D < 0, so saddle point!

(T/F) You suddenly find yourself with an extra hour to study for the exam. You have already solved the constrained optimization problem for points won \(P\) given hours studied \(H\) as a function of days \(d\) devoted to it and other commitments \(c\). If you found that \(\Delta P/\Delta H < 0\), then you should just go to bed.

\[
\Delta P \approx \lambda \Delta H. \text{ If } \frac{\Delta P}{\Delta H} < 0, \text{ then an extra hour of studying will make } \Delta P \text{ negative, or you lose points.}
\]

Go to bed.
8. (16 pts) If \( C(u, v) = u^2 - uv + v + 3v^2 - 10 \), then use the second derivative test to determine all local maxima/minima and saddle points of \( C \).

\[
C_u = 2u - v \quad \text{(from #6)} = 0 \quad \Rightarrow \quad v = 2u
\]

\[
C_v = -u + 6v + 1 \quad \text{(from #6)} = 0 \quad \Rightarrow \quad -u + 6(2u) + 1 = 0
\]

\[
\Rightarrow \quad 11u = -1 \quad \text{or} \quad u = -\frac{1}{11}
\]

\[
v = \frac{-2}{11}
\]

One single critical point \( (u, v) = \left( -\frac{1}{11}, -\frac{2}{11} \right) \)

\[
D = \det(C_{uu}C_v^2 - C_{uv}C_{uv}) = 11 > 0.
\]

\[
\Rightarrow \text{the critical point is a local minimum}
\]

\[
C_{uu} = 2 > 0
\]