1. (10 pts) Which might describe a function? Which cannot describe a function? Justify your answer.

(a) This exam. (Input: you and your classmates. Output: your grades.)

(b) Tabulation of temperature and precipitation data in Chicago:

<table>
<thead>
<tr>
<th>temp (in °F)</th>
<th>25</th>
<th>48</th>
<th>67</th>
<th>82</th>
<th>74</th>
<th>48</th>
<th>91</th>
</tr>
</thead>
<tbody>
<tr>
<td>precip (in inches)</td>
<td>.5</td>
<td>.72</td>
<td>1.3</td>
<td>.8</td>
<td>.72</td>
<td>.33</td>
<td>.74</td>
</tr>
</tbody>
</table>

First: sure, unique output for each input. Second: no, because no matter which row we call the independent variable, there are not unique outputs for each input.

2. (10 pts) In the figure below, determine which is $f$, which is $f'$, and which is $f''$. Justify your answer.

Look near $x = 1$, the solid curve seems to have a horizontal tangent line there, while the dotted curve passes through zero there. (The dashed curve does not, so cannot be the derivative of the solid line.) Also, the dashed curve is zero near $x = 3$, while the dotted curve appears to have a horizontal tangent line there. (The solid line is nonzero there, so cannot be the derivative of the dotted line.) Conclude: solid curve is $f$, dotted curve is $f'$, and dashed curve is $f''$.

More evidence: solid line is concave down from near $x = 0$ to near $x = 1.5$ and the dashed line is negative there.
3. (10 pts) Find the domain and range of \( \frac{1}{\sqrt{4 - x^2}} \). Also, indicate any special properties the function may have on its domain (e.g., even, odd, invertible, differentiable, continuous, asymptotes, etc.).

Domain: note that \(|x| \leq 2\), since square roots of negative numbers are undefined (over the real numbers). Also, \( x \neq \pm 2 \) because the function \( u \mapsto 1/u \) cannot accept \( u = 0 \) as input. Conclude: \( \mathcal{D} = \{x : -2 < x < 2\} \).

Range: note that the numerator and denominator of the function are always positive. Also, the largest the denominator gets is when \( x = 0 \), giving an output of \( 1/2 \). As \( x \) approaches \( \pm 2 \), the denominator goes to zero, so the entire function approaches infinity.

The function is even (\( f(-x) = f(x) \)), not invertible (because it is even!), and differentiable on its domain (in particular, continuous on its domain). There are no horizontal asymptotes; there are vertical asymptotes at \( x = \pm 2 \).

4. (10 pts) If \( \lim_{x \to -1} f(x) = 4 \) and \( \lim_{x \to 4} g(x) = 1 \), then compute \( \lim_{x \to -1} \left( g(f(x)) - f(x) \right)^2 \).

Assume \( g \) is continuous. The hypotheses tell us that \( \lim_{x \to -1} g(f(x)) = g(\lim_{x \to -1} f(x)) = g(4) = 1 \). Thus the entire expression goes to \((1 - 4)^2 = 9\).

5. (10 pts) Use the sandwich/squeeze theorem to prove that \( \lim_{x \to \infty} \frac{\cos(x - 1)}{x} = 0 \).

We need to find functions \( g \) and \( h \) with \( g(x) \leq f(x) \leq h(x) \) as \( x \to \infty \). Since \( \cos(\theta) \) is always between \(-1\) and \(1\), we can use \( g(x) = -1/x \) and \( h(x) = 1/x \). Since \( \lim_{x \to \infty} g(x) = \lim_{x \to \infty} h(x) = 0 \), conclude that the same holds for the values of \( f \).

6. (10 pts)

(a) (5 pts) State the \( \epsilon-\delta \) definition of the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \).

(b) (5 pts) State the (limit) definition of the derivative of a function \( f \).

(a) \( \lim_{x \to a} f(x) = L \) if, for any given \( \epsilon > 0 \), one can find a \( \delta > 0 \) so that \[ 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon \].

(b) \( f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \), if this limit exists (and undefined otherwise).

7. (10 pts) Find \( a \) and \( b \) so that \( f \) is differentiable at \( x = 2 \) if

\[
f(x) = \begin{cases} 
afx^2 + b, & \text{if } x < 2 \\
e^{2x-4}, & \text{if } x \geq 2
\end{cases}
\]

We need (i) \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \) and also (ii) \( \lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x) \)

(i) gives us \( a(2)^2 + b = e^{2(2)-4} = 1 \), while (ii) gives us \( 2a(2) = 2e^{2(2)-4} = 2 \). Conclude that \( a = 1/2 \) and \( b = -1 \).
8. (10 pts) A diver on a 10m platform makes his dive. Assume his jump gives him an initial upward velocity of 2.45m/s. If gravity acts downward at a rate of 9.8m/s², answer the following.

If it takes 0.7s for him to straighten out in preparation for splashdown, how long does he have to execute the never-before-attempted-in-competition quintuple shimmy-shake luxor?

Projectile motion is described as follows (where \((s, v, a) = (\text{position}, \text{velocity}, \text{acceleration})\) are measured in meters and \(t\) is measured in seconds). Since \(a(t) = -9.8\), we have \(v(t) = -9.8t + v_0\), where \(v_0\) is the initial velocity. We are told that this is 2.45m/s, so \(v(t) = s'(t)\), so \(s(t) = -4.9t^2 + 2.45t + s_0\), where \(s_0\) is the initial displacement. We are told this is 10m.

We are asked to find the total time in the air, minus 0.7s for him to prepare for hitting the water. So we consider \(s(t) = 0\) and solve for \(t\). The answer is roughly 1.7s (I have discarded the solution \(t\) that is negative), so the final answer is \(\approx 1\) second.

9. (10 pts) Solve one of the problems below. (Indicate which you want graded.)

(a) Use the \(\epsilon-\delta\) definition of limit to verify that \(\lim_{x \to 2}(3x - 2) = 4\). (Hint: \(\delta = \epsilon/3\) may work.)

(b) Use the limit definition of derivative to verify that \(f'(-2) = -1/4\) when \(f(x) = 1/x\).

(c) A circular plate is sitting on the counter near a hot oven. Use the mean value theorem to prove that there are two antipodal points on the plate’s rim with exactly the same temperature.

(a) \(|(3x - 2) - (4)| < \epsilon\) if and only if \(|3x - 6| < \epsilon\), if and only if \(3|x - 2| < \epsilon\), or \(|x - 2| < \epsilon/3\). I am allowed to control \(|x - 2|\) with \(\delta\), so putting \(\delta = \epsilon/3\) guarantees that \(|f(x) - L| < \epsilon\).

(b) \(\lim_{h \to 0} \frac{1/(-2+h) - 1/(-2)}{h} = \lim_{h \to 0} \frac{h}{h(2)(h-2)} = \lim_{h \to 0} \frac{1}{2(h-2)} = -1/4\).

(c) Let \(N\) represent the point on the rim of the plate nearest the oven and \(F\) the point farthest from the oven. Imagine placing a pencil at the center of the plate, facing \(F\), facing away from \(N\), then rotating that pencil clockwise through an angle \(\theta\). Let \(\text{diff}\) be the function that takes the difference of the temperatures of the plate’s rim at the point-end of the pencil and the butt-end of the pencil. Clearly, \(\text{diff}(0) < 0\) while \(\text{diff}(\pi) > 0\). Since \(\text{diff}\) is a continuous function, there is some angle \(0 < c < \pi\) so that \(\text{diff}(c) = 0\).
10. (10 pts) Compute the following derivatives:

(a) (4 pts) \( \frac{d}{dx} \left( x^3 + \frac{x}{\cos(x)} + 8 \right) \)

(b) (3 pts) \( \frac{d}{dx} \left( \sqrt{x + e^{\sqrt{x+1}}} \right) \)

(c) (2 pts) \( \frac{d^2}{d\theta^2} \left( \sec 2\theta \right) \)

(d) (1 pt) \( \frac{d^{2010}}{dt^{2010}} \left( e^{2t+1} \right) \)

(a) \( 3x + \frac{[1](\cos x) - (x)[- \sin x]}{\cos^2(x)} + 0 \)

(b) \( (1/2) \left[ x + e^{\sqrt{x+1}} \right]^{-1/2} \cdot \left( 1 + e^{\sqrt{x+1}} \cdot \{(1/2)(x + 1)^{-1/2}(1)\} \right) \)

(c) \( = \frac{d}{d\theta} \left( 2 \sec(2\theta) \tan(2\theta) \right) = 2 \left[ 2 \sec(2\theta) \tan(2\theta) \right] \tan(2\theta) + 2 \sec(2\theta) \left[ 2 \sec^2(2\theta) \right] \)

(d) Each derivative introduces another power of 2, starting with 1 after the 1st derivative, so the answer is \( 2^{2010}e^{2t+1} \).

11. (10 pts) Lake Mead is losing water at an exponential rate due to the exponential population growth in the Las Vegas area. If the water level at its deepest point was 40m in 2005 and 38m in 2010, when will it fall below 30m?

We know that \( H(t) = H_0 e^{kt} \), where \( H_0 \) is the initial height and \( k \) is the continuous exponential rate of decay. (So \( k < 0 \).) Using the data points (2005, 40) and (2010, 38), we can determine \( H_0 \) and \( k \). In fact, lets count from the year 2005, so \( t = 0 \) at that point. Then we have:

\[
40 = H_0 e^{k(0)} \Rightarrow H_0 = 40 \\
38 = 40e^{k(5)} \Rightarrow k = (1/5) \ln(38/40) \approx -0.0103
\]

Finally, we are asked to find \( t \) to that \( 30 = 40e^{-0.0103t} \). We get \( t \approx 28.04 \), so the answer is early in the year of 2034.

12. (10 pts) Recall that \( \sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B) \). Compute the following limit:

\[
\lim_{\theta \to 0} \frac{\sin(2\theta)}{\theta^2}.
\]

The above equals \( \lim_{\theta \to 0} \frac{2\sin(\theta) \cos(\theta)}{\theta^2} = 2 \left( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \right) \left( \lim_{\theta \to 0} \frac{\cos(\theta)}{\theta} \right) \). The first terms go to 2 \cdot 1. Let’s look at the last term. Since the numerator \( \cos \theta \) is positive for \( \theta \) near zero, and the denominator is negative (since \( \theta \) approaches zero from the left), the fraction looks like 1/0, but passing through negative numbers (i.e, the limit is \(-\infty\)). Multiply this by 2 and by 1 and we get \(-\infty\).