Final Exam Review (and Quiz #5)

Calculus I – Math 161.005 – Fall 2010

- Your final exam is scheduled for Thursday, December 16, at 9:00 a.m. in our usual room.

- No books or notes will be allowed. (If you think you need a trig identity to solve a problem, I may be willing to offer a hint.)

- Calculators will be allowed.

- The exam will last two hours.

- There will be approximately fifteen (15) questions; many of them will be similar to (and/or simpler than) the problems appearing here.

- In addition to these problems, it would be a good idea to look over the past quizzes and exams. (Solutions to these will be posted on our course website by the end of the week.)

- There may be some True/False questions or Matching questions on the exam, but otherwise all the problems will be “work out” questions (no multiple choice).

- If you work some problem on your calculator, then write down some of your intermediate steps!! 

  ... Partial credit will be awarded for wrong answers, but only if you show your work.

- Submit solutions to problems 14–22 on Thursday (12/09) for your final quiz grade. (Note: my records indicate that we’ll only have had five quizzes. I will record your highest quiz twice to get to the promised six quizzes.)
1. Which sets may describe functions. For those that may be, state which is the independent variable and which is the dependent variable. Also, give possible domains and ranges.

(a) • calendar date
   • closing value of the S&P-500

(b) A soda vending machine

\[
\begin{array}{cccccccc}
2 & 3 & 2 & 3 & 1 & 4 & 7 & 8 & -9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
2 & 3 & 5 & 6 & 1 & 4 & 7 & 8 & -9 \\
1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
2 & 3 & 5 & 6 & 1 & 4 & 7 & 2 & -9 \\
1 & 2 & 3 & 4 & 5 & 5 & 7 & 8 & 9 \\
\end{array}
\]

(a) - yes: domain = all dates; range = positive rational numbers
(b) - yes: domain = ($0.50, button); range = soda cans
(c) - yes: use second line of table as domain
(d) - yes: use first line of table as domain
(e) - no: neither line can be domain (doesn’t pass vertical line test)

2. (a) If \( f(2) = 7 \) and \( f^{-1}(3) = 4 \), simplify the expression \( 3f^{-1}(7) - 2f(4) \).
   
   (b) Assume further that \( f \) is a line. What is \( f(7) \)?
   
   \( 3 \cdot 2 - 2 \cdot 3 = 0 \)

3. (a) A reservoir is losing water exponentially. If it had 432,000 ft³ on 1/1/2001 and had 321,000 ft³ on 1/1/2010, when will it have 210,000 ft³?
   
   (b) A population of bacteria doubles every 4 hours. If there were 10 cells to start with, how many will there be two days later?

4. If \( \sin \theta < 0 \) and \( \cos \theta = -0.7 \), determine \( \theta \).
   
   Angle must be in third quadrant. Get approx. \( 2\pi - 2.346 \).

5. Find all solutions (solve exactly) for \( -\pi \leq \alpha < \pi \):
   
   (a) \( \cos(2\alpha) = -\sin(\alpha) \).
   
   (b) \( \cos(3\alpha) = -0.5 \)

   (a) \( \cos(2\alpha) = 1 - 2\sin^2 \alpha \). Now solve a quadratic equation \( aX^2 + bX + c = 0 \), where \( X = \sin \alpha \). Get \( X = 1, -1/2, \) so \( \alpha = \pi/2, -\pi/6, -5\pi/6 \).
   
   (b) \( 3\alpha = -8\pi/3, -4\pi/3, -2\pi/3, 2\pi/3, 4\pi/3, 8\pi/3 \)
6. Given the triangle below, express \( \cos(2\theta) \) in terms of \( x \). (Hint: use a double angle identity.)

![Triangle Diagram](image)

\[
1 - 2/x^2
\]

7. Let \( p(x) \) and \( q(x) \) be polynomials.

(a) Explain three possibilities for the long-term behavior of the rational function \( p(x)/q(x) \) and give an example of each.

(b) What must happen at \( x = 3 \) for \( p(x)/q(x) \) to have a root there? to have a vertical asymptote there? (Hint: your answer should be in terms of \( p(3) \) and/or \( q(3) \).)

(a) The limit \( \lim_{x \to \infty} \frac{p(x)}{q(x)} \) depends on the degrees of \( p \) and \( q \).
If \( \deg(p) = \deg(q) \), then it goes to some finite number \( k \) (the ratio of leading coefficients).
If \( \deg(p) < \deg(q) \), then it goes to 0.

(b) \( p(3) = 0; \ q(3) = 0. \)

8. Let \( f(x) = \frac{1}{x+3} \) and \( g(x) = \ln x \). Find the domain and range of \( f \circ g \) and \( g \circ f \).

(a) domain: \( x > 0 \) but not equal to \( 1/e^3 \). range: \( y \neq 0 \)
(b) domain: \( x > -3 \). range: all real numbers \( y \)

9. Sketch a graph of \( \cos^{-1} x \). What is its domain and range? Name (precisely) 5 points on the curve.

Draw \( \cos x \) between 0 and \( \pi \) and then flip across the line \( y = x \).
10. Fill in the blanks

\[ f(x) = x^3 - 2x^2 \]

\[ g_1(x) = -2f(x) \]

\[ g_2(x) = \]

\[ g(x) = g_2(x - 1) = \]

(re-express as a transformation of \( f \))
11. Match the graphs to the functions. (Two graphs will be used twice.)

(a) \( \sin(\pi t) + 2 \)  
(b) \( 2 \cos(\frac{\pi}{4} t) + 3 \)  
(c) \( 3 - 2 \sin(-\frac{\pi}{4} t + \frac{\pi}{4}) \)  
(d) \( 2 \sin(\frac{\pi}{4} (t - 1)) + 3 \)  
(e) \( 1 - 2 \sin(\pi t) \)  
(f) \( 1 + \sin(\pi t - \pi) \)  
(g) \( 3 + 2 \cos(t/2) \)  
(h) \( 2 \sin(-\frac{1}{2} - \frac{\pi}{2}) + 3 \)  

Typo above: (b) needs to be \( 3 + \cos(\frac{\pi}{4} t) \).

(a) row 3-2  
(b) row 3-1  
(c) row 2-2  
(d) row 2-2  
(e) row 2-1  
(f) row 1-2  
(g) row 1-1  
(h) row 1-1

12. Show (either algebraically or graphically) that no inverse function exists for the function \( f(x) = x^2 \). Show that the function \( g \) with domain \([0, \infty)\) and rule \( g(x) = x^2 \) does have an inverse. Which is it? \( f^{-1}(x) = \sqrt{x} \) or \( f^{-1}(x) = -\sqrt{x} \)? Don’t know which to take. (Alternatively, \( f \) doesn’t pass the horizontal line test, so \( f^{-1} \) can’t exist.)

13. Let \( f(x) = \frac{x}{x+3} \). Find the inverse and verify that it is one by simplifying \( f(f^{-1}(x)) \) completely. Also, state the range of \( f \) and the domain of \( f^{-1} \).

\[ f^{-1}(x) = \frac{3x}{1 - x} \]

14. Compute the area bounded by the curves \( y = 0 \), \( y = 2 - x \), and \( y = 20 - 2(x - 3)^2 \).

\[ 20 + \frac{2}{3} \]

15. Find the area bounded by the curves \( y = \sqrt{x} \), \( y = 2 \), and \( y = 2 - x \) by using integration with respect to \( y \) (“\( dy \)” instead of “\( dx \)”).

16. Compute the following.

(a) \( \int_{2}^{16} \frac{dx}{2x\sqrt{\ln x}} \)
(b) \[\int_{\pi/4}^{\pi/2} \cot t \, dt\]

(c) \[\int_1^4 \frac{dy}{2\sqrt{y}(1 + \sqrt{y})^2}\]

(d) \[\int_3^\pi \frac{t}{t + 1} \quad (\text{Hint: } t = (t + 1) - (1).)\]

(e) \[\int_0^\pi t(t^2 + 1)^{1/3} \, dt\]

(f) \[\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta\]

(g) \[\int_0^\pi |\sin x| \, dx\]

(h) \[\int_{-1}^1 |x^2 - 1| \, dx\]

(i) \[\int_0^{\pi/4} \cos^2(4t - \pi/4) \, dt\]

(j) \[\int_1^8 \frac{\log_4 \theta}{\theta} \, d\theta\]

(k) \[\frac{d}{dx} \int_0^{x^2 + 1} \cos(t) \, dt\]

17. Verify that \[\int \ln x \, dx = x \ln x - x + C.\]

18. Let \(f(x) = |x|\) and let \(F\) and \(G\) be defined as follows,

\[F(x) = \int_0^x f(x) \, dx \quad \quad G(x) = \begin{cases} \frac{1}{2}x^2 + 2 & x \geq 0, \\ \frac{-1}{2}x^2 - 2 & x < 0. \end{cases}\]

(a) What are the domains and ranges of \(F\) and \(G\)? (b) Find a formula for \(F\), then graph \(F\) and \(G\) together. (c) Compute \(F'(x)\) for all \(x\) in the domain of \(F\) and do the same for \(G\). (This will require evaluating left- and right-hand limits for at least one of these... both if you’re not being clever.) (d) Graph the results from (c) together. Explain why this does not violate of Corollary 2 of the MVT.

19. (a) Evaluate by interpreting as a definite integral:

\[\lim_{n \to \infty} \frac{7}{n} \sum_{k=1}^n \sin \left(\pi \left(\frac{1}{2} + \frac{7}{n}k\right)\right)\]

(b) Evaluate using Riemann sums, limits, and formulas for \(\sum k\), \(\sum k^2\), etc.: the (signed) area under the curve \(y = -3(x - 1)^2 + 2\) between 0 and 2.
20. Find the average value of $xe^{x^2}$ on the interval $[-1, 1]$.

\[
\frac{e^{-1/e^2}}{2}
\]

21. Here is the graph of the velocity of a particle moving along the $x$-axis. Determine when the particle first returns to the origin.

Find $b$ so that $\int_{14/3}^b ((-3/2)x + y) \, dx = -(12 + 1/3 - \pi)$.

22. Let $a_k = \cos((\pi/3)k)$ for all $k \geq 0$. Compute \( \sum_{k=2}^{4} \frac{1}{3a_k + k^2} \).

23. You are a day trader waiting to find the right time to buy KRAFT stock and sell NESTLE stock. Suppose that

\[
K(1:00 \text{ p.m.}) = 45.10 \quad N(1:00 \text{ p.m.}) = 44.30
\]

and further, suppose you have observed that $\Delta K \approx -0.2$ over the past few hours; with $\Delta N \approx +0.3$ over the same time.

Use Newton’s method to estimate when you will be able to sell NESTLE, buy KRAFT, and make a profit doing so.

Several problems with this problem. I wanted to find the zeros of the function $F(t) = N(t) - K(t)$, but I didn’t even tell you what units $\Delta t$ was being measured in (much less how to calculate $F'(t)$ for points besides $t = 1:00$).


25. Work book exercise 4.6.46. Additionally, determine if/when one weight passes the other while going in the same direction (for $0 \leq t \leq 2\pi$).

26. Let $r(x)$, $c(x)$ and $p(x) = r(x) - c(x)$ be the revenue, cost, and profit functions for some company (as a function of selling $x$ items, say). Show that profit is maximized when marginal cost equals marginal revenue. Verify this when $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$.

27. Work these book exercises in Section 4.5: 20, 22, 26, 29, 54, 62, 66.
28. Interpret as a derivative and compute using differentiation rules:

\[
\begin{align*}
\lim_{x \to 0} \frac{e^x - 1}{x} & \quad \lim_{x \to \pi/2} \frac{\sin x - 1}{x - \pi/2} & \quad \lim_{x \to e} \frac{\ln x^{1/x} - 1}{x - e}
\end{align*}
\]

Let \( f(x) \) be the function. Then (a) looks for \( f'(0) \) when \( f(x) = e^x \); (b) looks for \( f'\left(\frac{\pi}{2}\right) \) when \( f(x) = \sin x \); (c) looks for \( f'(e) \) when \( f(x) = (\ln x)^{1/x} \). The answers are 1, 0, and \((1/e^2)(1/e)\).

29. The following is a graph of \( f' \) and \( f'' \). Which is which? Use this information to sketch a graph of \( f \) if \( f(0) = 1 \).

![Graph of f' and f''](image)

Estimate the slope of each near \( x = 0 \) to determine that the dotted line is the derivative of the dashed line.

Since \( f(0) = 1 \) and it’s increasing concave up for \( x > 0 \) and increasing concave down for \( x < 0 \), the graph effectively looks like \( x^3 + 1 \). (In fact, it is \( \int e^{|x|} \, dx \).)


31. (a) \( (\text{T} / \text{F}) \) If \( f \) is concave up for \( x \geq 0 \), then

\[
\frac{1}{3} \int_1^4 f(x) \, dx < \frac{1}{3} \int_5^8 f(x) \, dx.
\]

(b) \( (\text{T} / \text{F}) \) If \( f \) is concave up for \( x \geq 0 \), then \( \int_1^4 f'(x) \, dx > 0 \).

Both are false, because concave up does not imply increasing. The first is a comparison of two average values; the second is \( f(4) - f(1) \).

32. Use curve-sketching techniques to sketch the following:

\[
\frac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2} \quad \frac{x - 1}{x^2(x - 2)}
\]

(Include horizontal, vertical, and oblique (slant) asymptotes.)
33. You drove past mile marker 10 at 1:00 p.m. and past mile marker 100 at 2:15 p.m. What speed can the copy who pulls you over safely claim you attained at least once during your ill-fated trip?

34. If HANS stock is selling for $45 and in the last half-hour its price changed from 44.7, use linearization to guess how much it will be worth in the next half-hour.

35. Use the “they have the same derivative” argument (§4.2) to prove the identity

\[ \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \]


37. PJ and John spy a creepy urn with dead flowers sitting at the bottom of a drained pool. It’s too deep to climb down and get it, so they decide to fill the pool with water in the hopes that the urn will float as the water rises to ground-level. If the pool is cylindrical with radius 3 feet and depth 15 feet, and if they can carry about 3 ft\(^3\) of water per minute from a nearby stream, how long before they can get their hands on the urn?

I get 45\(\pi\) minutes. This comes from implicit differentiation (with respect to time) of the volume equation \(V = \pi r^2 h\) (with \(dr/dt = 0\) and \(dV/dt = 3\) and \(dh/dt\) the unknown we need to solve for... then divide 15 ft by this quantity.

38. Understand Example 6 from §3.10.

39. Look back at the soccer problem from the second exam. Prove that the formula given for the viewing angle is the correct one. (See also book exercise 3.9.43.)

40. Use the triangle below to prove that \(\frac{d}{dx} \csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}\) (for \(x > 1\)).

\[ x \]
\[ \sqrt{x^2-1} \]
\[ 1 \]

41. Compute the second derivative of \(f(x) = \ln \cos x\). If a function \(g\) satisfies \(g''(x) = -\sec^2 x\), determine the most general form for \(g\).

\(g(x) = \ln \cos x + Cx + D\)

42. Show that the ellipse \(2x^2 + 3y^2 = 5\) and the cissoid \(y^2 = x^3\) cross perpendicularly at the points \((1, 1)\) and \((1, -1)\).

43. Determine \(y''\) at the point \((0, -1)\) if \(xy + y^2 = 1\).

44. Compute the following.

(a) \(\frac{d^2}{dx^2} (5 - 2x)^{-3} + \frac{1}{8} (\frac{2}{x} + 1)^4\)

(b) \(\frac{d}{dx} xe^{x\ln x}\)
(c) \( \frac{d}{dx} \left( \sin(x + \sqrt{e^{2x} + 1}) \right) \)

(d) \( \frac{d}{dx} 2^x \)

(e) \( \frac{d}{dx} \log_2 x^2 \)

45. Compute \( \frac{d}{d\theta} \sin(\cos^{-1}(\sin \theta)) \). Then use the triangle below to compute \( \frac{d}{d\theta} \sqrt{1 - x^2} \). You should get the same answer!

![Triangle Diagram]

You should get \(-x\) in both cases.

46. If \( f(2) = -3, f'(2) = 3, g(1) = 1/2, g'(1) = 1, h(1) = 2, \) and \( h'(1) = -1, \) then determine

\[ \frac{d}{dx} \left[ f(h(x))^2 \cdot g(x) - \frac{h(x)}{g(x)} \right] \]

at \( x = 1. \)

47. Work book exercises 3.4.16 and 3.4.18.

48. State the formal definition of limit. Use it to show that \( \lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} \).

49. State formal definition of derivative, then use it to compute \( f'(0) \) if \( f(x) = \sqrt{x + 1}. \)

50. Work these book exercises in Section 2.6: 25, 26, 29, 56, 57.

51. State the formal definition of continuity, then work book exercise 2.5.48. Also, is there any choice of constants that allow \( g \) to be differentiable everywhere?

52. Determine the line \( y = mx + b \) with negative slope and positive \( y \)-intercept that passes through \( (3, 2) \) and minimizes the area of the region in the first quadrant enclosed by the line and the coordinate axes.