Practice Exam 2 (Chapter 3)
Math 161

1. A continuous function is defined for all values of x and has the following properties:

   - f is always increasing.
   - $f^{-1}(4) = 3$
   - f is always concave down.
   - $f'(3) = \frac{2}{3}$

   Answer the following questions:

   (a) Sketch a possible graph of $f(x)$.

   (b) How many zeros can $f(x)$ have?

   (c) In what intertvals (of x-values) can the zeros be located? Be as accurate as you can be.

   (d) Compute $\lim_{x\to+\infty} f(x)$.

   (e) Explain why or why not $f'(5) = \frac{1}{3}$ is possible.

   (f) Explain why or why not $f'(5) = 2$ is possible.

2. Determine whether the following statements are true or false. If true briefly explain why, if false give a counterexample.

   (a) Every continuous function is differentiable.

   (b) If $f''(x) > 0$, then for $a < b$, the average rate of change of f on the interval (a, b) is less than the instantaneous rate of change of f at $x = a$.

   (c) Every differentiable function is continuous.

3. The number of gallons of water in a tank t minutes after the tank begins to drain is $Q(t) = 100\sqrt{20 - t}$.

   (a) How fast is water draining out of the tank 4 minutes after the tank has started to drain?

   (b) What is the average rate of change at which the water flows out of the tank during the first 4 minutes?

4. A company manufactures and sells widgets. The cost of manufacturing x widgets is $C(x) = 300 + 1.1x$, while the revenue from selling x widgets is given by $R(x) = 5x - 0.003x^2$

(a) What is the marginal cost of producing 600 widgets? Explain what this means in practical terms.

(b) What is the marginal revenue from selling 600 wdigets? Explain in practical terms.

(c) Based on your answers above, explain why you think this company should produce more or less than 600 widgets.

(d) How many widgets should be manufatured in order to maximize their profits?

5. Find the following derivatives:

(a) $f''(x)$, where $f(x) = 3x \cdot 2^{5x}$

(b) $\frac{dy}{dt}$, where $y = (t + 2^t)\, e^{t^2}$

(c) $g'(\theta)$, where $g(\theta) = \frac{\theta \cos(\sin\theta)}{\tan\left(e^{\theta}\right)}$

(d) $h'(r)$, where $h(r) = \sqrt{3r} + 3\sqrt{r} - \sqrt{\frac{3}{r}} + \sqrt{3}$

(e) $\frac{dz}{dx}$, where $z = \sqrt{\arctan(x^3)}$

(f) $\frac{du}{dt}$, where $u = \frac{t^3}{\ln(t^2)}$

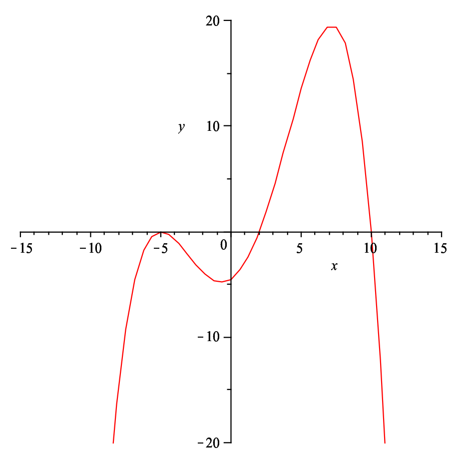
(g) $\frac{df}{dx}$, where $f(x) = \ln\left[\left(\frac{1-\cos x}{1+\cos x}\right)^3\right]$

(h) $h'(z)$, where $h(z) = b\cos(az^2)a^{b\tan z}$ (everything is a constant but z !!)

(i) $\frac{dy}{dx}$, for $e^{\cos y} = x^3 \text{arctan} y$

(j) $\frac{du}{dv}$, for $e^{v^2} + \ln u = 4$

6. Consider the graph of $f(x)$ below:



(a) Where are the turning points?

(b) Over what interval(s) is $f(x)$ increasing? decreasing?

(c) Approximately where are the inflection points?

(d) Over what interval(s) is the function concave up? concave down?

(e) Based on your answers above, sketch possible graphs for both $f'(x)$ and $f''(x)$. Your graphs should reflect the properties and intervals stated in the previous questions.

7. Use calculus to find the set of points where the function $f(x) = 3x^5 - 20x^3$ is both decreasing and concave up (find EXACT intervals).

8. A stone is intially thrown upward from a cliff, and its height above ground (in feet) t seconds after being thrown is is given by the formula $h(t) = 320 + 128t - 16t^2$.

   (a) What is the intial velocity and acceleration of the stone?

   (b) When does the stone hit the ground?

   (c) What is the velocity and acceleration of the stone when it hits the ground?

   (d) What is the maximum height of the stone, and at what time does the stone reach this height?

9. Given $x^3 + y^3 - xy^2 = 8$,

   (a) Find the linear approximation to the curve at the point $(x, y) = (2, 2)$.

   (b) Based on your answer in (a), if $x = 2.2$, find an approximation for the value of y.

10. A particle moves along the curve defined by the parametric equation

$$x(t) = 2\cos(3t) \quad , \quad y(t) = -2\sin(3t) \quad \text{for} \quad \frac{\pi}{6} \leq t \leq \frac{\pi}{2}.$$

   (a) Sketch the parametric curve, clearly indicating the start and end points, and the direction in which you move along the curve.

   (b) What is the speed of the particle when $t = \frac{\pi}{4}$?

11. The radius of an oil slick (which is shaped like a circle) is increasing at a rate of 5 feet per minute. At what rate is the area of the slick increasing when the radius is 100 feet?

12. Find the equation of the tangent line to $f(t) = t \ln t$ at $t = 1$.

13. Consider the family of functions $f(t) = Ae^t + Bte^t$. Find the values for A and B such that $f(0) = 2$ and $f'(0) = 1$.

14. Given $r(2) = 4$, $s(2) = 1$, $s(1) = 4$, $r'(2) = -1$, $s'(2) = 3$, and $s'(4) = 3$. Compute the following derivatives:

   (a) $H'(2)$, for $H(x) = r(x)s(x)$.

   (b) $F'(2)$, for $F(x) = s(r(x))$.

   (c) $G'(2)$, for $G(x) = \ln\left[(r(x))^2\right]$

   (d) $K'(1)$, for $K(x) = s^{-1}(x)$ (the inverse of the function $s(x)$)

15. *(This is a difficult problem, so proceed with this in mind)*
   Sand falls from a hopper at a rate of 0.1 cubic meters per hour and forms a conical pile beneath. If the side of the cone makes a constant angle of $\frac{\pi}{6}$ radians with the vertical of the cone, find the rate at which the height of the cone increases when the height is 6 meters. At what rate does the radius of the base grow when the height is 6 meters?

   HINT: The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.

16. Let $P(x) = 200xe^{-x/400}$ be the profit of a manufacturer who sells x items.

   (a) Find the differential dP.

   (b) Find dP at $x = 145$ if $dx = -5$.

   (c) Explain the significance of your answer to part b.

17. The diameter of a tree was 10 inches. During the following year, the circumfrence increased 2 inches.

   (a) Find a formula for the diameter x as a function of the circumfrence C.

   (b) Find dx

   (c) About how much did the tree's diameter increase?

   (d) Find a formula for the tree's area A as function of its circumfrence C.

   (e) About how much did the tree's cross-sectional area increase?