

# Math Bite: Finding $e$ in Pascal's Triangle

HARLAN J. BROTHERS

Brothers Technology, LLC

Branford, CT 06405

harlan@brotherstechnology.com

Mathematicians have long been familiar with the tidy way in which the  $n$ th row of Pascal's triangle sums to  $2^n$  (the top row conventionally labeled as  $n = 0$ ). It is less obvious how the rows behave when we multiply their items.

			1					.....1		
			1	1				.....1		
			1	2	1			.....2		
			1	3	3	1		.....9		
			1	4	6	4	1	.....96		
			1	5	10	10	5	1	..... 2500	
			1	6	15	20	15	6	1	.....162000

Let  $s_n$  be the product for row  $n$ ; that is,  $s_n = \prod_{k=0}^n \binom{n}{k}$ . On the right-hand side of the figure above, we see the sequence  $\{s_n\}$  grows very quickly. To get a sense of its rate of growth, we can look at the ratios of successive terms,  $r_n = s_n/s_{n-1}$ . The sequence  $\{r_n\}$  itself grows rapidly. Examining the ratios of ratios, a familiar pattern emerges:

$n$	$s_n$	$r_n = s_n/s_{n-1} \approx$	$r_n/r_{n-1} \approx$
1	1	1	
2	2	2	2
3	9	4.5	2.25
4	96	10.667	2.370
5	2500	26.042	2.441
6	162000	64.800	2.488
⋮	⋮	⋮	⋮
1000	$\approx 1.68 \cdot 10^{215681}$	$2.49 \cdot 10^{432}$	2.71692

**THEOREM.**  $\lim_{n \rightarrow \infty} \frac{s_{n+1}/s_n}{s_n/s_{n-1}} = e.$

*Proof.* By direct calculation we get

$$s_n = (n!)^{n+1} \prod_{k=0}^n (k!)^{-2}, \quad n \geq 0$$

$$s_n/s_{n-1} = \frac{n^n}{n!}, \quad n \geq 1, \quad \text{and}$$

$$\frac{s_{n+1}/s_n}{s_n/s_{n-1}} = \left(1 + \frac{1}{n}\right)^n.$$

Given that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ , the result follows. ■

**Summary** If  $s_n$  is the product of the entries in row  $n$  of Pascal's triangle then  $(s_{n+1}/s_n)/(s_n/s_{n-1}) = (1 + 1/n)^n$ , which has the limiting value  $e$ .