These questions are based on a course in 2005 from a different textbook, so they may not be all that representative. But hopefully they give you some idea of what type of questions to expect. A better study tool for the final exam might be to go over the two midterms.

In these questions, column vectors are written using capital letters, just like matrices. Also, column vectors are sometimes written as a transpose of a row vector, in order to save on space; e.g., $B = [1, 2, 3]^\mathsf{T}$ is the column vector with entries 1, 2, and 3.

1. (2 points each) TRUE-FALSE QUESTIONS. No reason is necessary.

   (a) A linear system $AX = B$ is solvable if and only if B belongs to the column space of A.

   (b) If a 4×4 matrix A is nonsingular then the linear system $AX = B$ has a unique solution vector for every column vector B in $\mathbb{R}^4$.

   (c) It is possible for a 4×6 matrix to have 5 independent columns.

   (d) A square matrix is invertible if and only if its determinant is zero.

   (e) A square matrix is nonsingular if and only if its rows are independent.

   (f) The columns of a matrix A are linearly independent if and only if the rank of A is the same as the number of columns.

2. Let $A = \begin{bmatrix} 0 & 0 & 3 & 6 \\ 1 & 2 & 0 & 2 \\ 2 & 4 & 1 & 6 \\ 3 & 6 & 0 & 6 \end{bmatrix}$.

   (a) (4 points) Carefully find the reduced row echelon form of A. Show your steps.

   (b) (3 points) What is the rank of A?

   (c) (3 points) What is the determinant of A?

   (d) (3 points) Is A invertible?

   (e) (3 points) Find a basis for the column space of A.

   (f) (3 points) Find a basis for the nullspace of A.

   (g) (3 points) Given that the column vector $X = [1, 1, 1, 0]^\mathsf{T}$ is a particular solution to $AX = B$ where $B = [3, 3, 7, 9]^\mathsf{T}$, write out the complete solution to $AX = B$.

   (h) (3 points) What is the dimension of the range of the linear map $f(X) = AX$?

3. Let $\mathcal{P}_3$ be the vector space of all polynomials of degree ≤ 3. A natural basis for $\mathcal{P}_3$ is the independent spanning set $B = \{1, x, x^2, x^3\}$. Let $h : \mathcal{P}_3 \to \mathcal{P}_3$ be the operator defined by the rule $h(p(x)) = x\frac{dp}{dx} - p(x)$, for any polynomial $p = p(x) \in \mathcal{P}_3$.

   (a) (4 points) Show that h is linear.

   (b) (6 points) Find the matrix $\mathrm{Rep}_{B,B}(h)$ of h with respect to B. Show your work.

   (c) (3 points) Is h an isomorphism? Explain why or why not.

4. (6 points) Let $V = \mathcal{C}(\mathbb{R})$ be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let W be the subset of V consisting of all differentiable functions $y = f(x)$ such that $x\frac{df}{dx} = f(x)$. Show that W is a subspace of V. [You must show that W is non-empty and is closed under sums and scalar multiples, or that W is non-empty and closed under linear combinations.]

5. Let $A = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

   (a) (3 points) What is the determinant of A?

   (b) (3 points) What are the eigenvalues of A?

   (c) (6 points) Find as many independent eigenvectors of A as possible. Show your work.

   (d) (4 points) Is A diagonalizable? Why or why not?

   (e) (5 points) If possible, find an invertible matrix Q and a diagonal matrix D such that $AQ = QD$.

6. (a) (5 points) Let $\mathcal{M}_{2\times 2}$ be the vector space of all 2×2 matrices. Find a basis B for the space $\mathcal{M}_{2\times 2}$ and give reasons why it is a basis.

   (b) (5 points) Consider the linear map f from $\mathcal{M}_{2\times 2}$ to $\mathcal{M}_{2\times 2}$ defined by the rule $f(A) = A^{\mathsf{T}}$, for any matrix $A \in \mathcal{M}_{2\times 2}$. Find the matrix $F = \mathrm{Rep}_{B,B}(f)$ of f with respect to your chosen basis B from part (a) above. Show your steps.

7. (4 points) Compute the determinant of a plane rotation by θ radians counterclockwise.

8. Let A be the $n \times n$ matrix (for $n \geq 2$) consisting entirely of 1's. Observe that the column vector $X = [1, 1, \ldots, 1]^{\mathsf{T}}$ consisting entirely of 1's is an obvious eigenvector for A.

   (a) (3 points) What is the eigenvalue corresponding to the eigenvector X as above?

   (b) (3 points) Observe that $\lambda = 0$ is another eigenvalue. Explain why it must be an eigenvalue.

   (c) (3 points) Compute the dimension of the eigenspace for $\lambda = 0$.

   (d) (5 points) [EXTRA CREDIT] Is A diagonalizable? Explain why or why not.

   (e) (6 points) [EXTRA CREDIT] Describe, if possible, an invertible matrix Q and a diagonal matrix D such that $AQ = QD$.