Linear Algebra and Applications
Practice Exam #1
(First Exam: Monday, July 11)

1 Theory

1.1 True, False, Neither

1. If $A$ is a $3 \times 4$ real matrix, then the equation $A\mathbf{x} = \mathbf{b}$ always has a nonzero solution $\mathbf{x} \in \mathbb{R}^4$ for any given $\mathbf{b} \in \mathbb{R}^3$.

Solution: See OS1.

2. If $A$ is any matrix with the property $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{y}^T A = \mathbf{0}$ for some $\mathbf{y}^T \neq \mathbf{0}$.

Solution: If $A$ is a square matrix, then this is true, and clear from either the FTLA or from the fact that $\det A = \det A^T$. However, it is entirely possible for the row vectors to be linearly independent while the column vectors are not... examples?

3. If $A$ is a $3 \times 3$ matrix which is diagonalizable, and $B$ is a matrix satisfying $AB = BA$, then $A$ and $B$ share a complete set (3) eigenvectors.

Solution: If $B$ is also diagonalizable, then $AB = BA$ says that $A$ and $B$ are simultaneously diagonalizable. Since the columns of $P$ (the diagonalizing matrix) are the eigenvectors for $A$ (and for $B$) we have made the statement into a true statement. See problem 2.4 for an example of what can happen when $B$ is not also diagonalizable.

4. If $A$ is an $n \times n$ symmetric matrix, then the nullspace $\mathcal{N}(A)$ of $A$ is equal to the nullspace $\mathcal{N}(A^2)$ of $A^2$, and the column spaces $\mathcal{R}(A)$ and $\mathcal{R}(A^2)$ are also equal.

Solution: See OS1.

1.2 Fix it

Find the flaw in the following reasoning (and give an associated true statement): If $A$ is an $n \times n$ matrix satisfying $A^T = -A$, then $\det A = -\det A$, i.e. $\det A = 0$.

Solution: $\det A^T = \det(-A) = (-1)^n \det A$ when $A$ is an $n \times n$ matrix. So if $n$ is even, we needn’t conclude that $\det A = -\det A$.

1.3 Prove it

If $A = [a_1|a_2|\cdots|a_n]$ is any $m \times n$ matrix satisfying $A \cdot A^T = I$, then the mapping $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ given by

$$
\mathbf{b} \mapsto (a_1^T \mathbf{b})a_1 + (a_2^T \mathbf{b})a_2 + \cdots + (a_n^T \mathbf{b})a_n
$$

is a linear transformation, and indeed is the identity linear transformation. Prove these statements, and compare the sizes of $m$ and $n$.

Solution: Need to check that $P(\mathbf{b} + \alpha \mathbf{c}) = P(\mathbf{b}) + \alpha P(\mathbf{c})$. That’s straightforward. Next, need to notice that the image of $P$ may be rewritten as $P\mathbf{b} = AA^T\mathbf{b} = \mathbf{b}$. If $A$ is right-invertible, then $m < n$... it’s easiest to argue with a picture here.

2 Computation

2.1 The FTLA

Let $A$ be the matrix

$$
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 2 & -2 & 0 \\
1 & 2 & -2 & 1
\end{bmatrix}.
$$

---

1 Prof. B. Osofsky’s midterm, October 27, 2004
For each of the four fundamental subspaces of $A$, find a basis.

Solution: See OS1.

2.2 Alternative Inverses

You are given matrices

$$U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

Without doing any matrix multiplications (in particular without multiplying $L$ by $U$), and without finding the inverse of the matrix $U$, find the inverse of the matrix $B = LU$. Then check your work doing appropriate matrix multiplications. Finally, solve $Bx = c$ for

$$c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution: See OS1.

2.3 Extend a Basis

Let $C$ be the matrix

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}.$$

1. Find an orthogonal $3 \times 3$ matrix $Q$ and a matrix $R$ in echelon form such that $C = QR$.

2. Find the least squares best solution to the inconsistent system of equations

$$C \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: See OS1.

2.4 AB=BA

Find all eigenvalues and eigenvectors of the matrices $A$ and $B$ below. (Afterward, you may want to revisit your answer to Problem 1.c.i.)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Solution: Straightforward. Eigenvectors/values for $A$ are $\{2 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ and $\{3 : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$.

Eigenvectors/values for $B$ are $\{1 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$ and $\{-2 : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$.

Notice that $AB = BA$, but $B$ doesn’t have a full set of eigenvectors.