1 Theory

1.1 True, False, Neither

Indicate whether the following statements are always true, never true, or sometimes true. Justify your answer by giving a brief proof if the statement is always true or always false, or examples illustrating that it is sometimes true and sometimes false if that is the case.

1. A $3 \times 3$ matrix $A$ has eigenvalues 1, 2, and -3. Another $3 \times 3$ matrix $B$ has eigenvalues -4, 5, and 6. $A$ and $B$ must be congruent to each other. (Recall: two matrices $A$ and $B$ are congruent to each other if there is a nonsingular matrix $P$ such that $P^T AP = B$.)

Solution. If $A$ and $B$ are symmetric/Hermitian, then there are orthog/Hermitian matrices $Q_A, Q_B$ such that $A = Q_A \cdot \text{diag}(1, 2, -3) \cdot Q_A^T$ and $B = Q_B \cdot \text{diag}(-4, 5, 6) \cdot Q_B^T$. Getting an affirmative answer from here is not too hard. On the other hand, if $A$ is symmetric and $B$ is not, it clearly can’t be possible.

2. If $A$ is an $m \times c$ matrix and $x$ is a $c \times 1$ eigenvector for $A^H$, then $Ax$ is an eigenvector for $AA^H$.

Solution. I think this problem meant to read “if $x$ is an eigenvector for $A^H A$, then $Ax$ is an eigenvector for $AA^H$.” It doesn’t make sense as written because $c \times m$ matrices don’t have eigenvectors. Finally, we must consider the case $c = m$. If the hypotheses imply the conclusion here, and if $A$ is invertible, then $x$ is also an eigenvector for $A$. It is easy to find $2 \times 2$ matrices where $A^H \neq A$, $A$ is invertible, but $A^H$ has an eigenvector $x$ which is not also an eigenvector for $A$.

3. If $A$ is an $n \times n$ complex matrix, and $x$ is an eigenvector for $A$ with real eigenvalue $\lambda$, then $\lambda$ is also an eigenvalue for $A^H$.

Solution. Use Schur’s unitary triangularization to write $A = UTU^H$. Then the eigenvalues of $A$ are the diagonal entries of $T$ and the eigenvalues of $A^H$ are the diagonal entries of $T^H$.

4. For $A$ an $m \times n$ matrix with complex number entries, and $x$ an $n \times 1$ with complex number entries, $A^H Ax = 0$ is true if and only if $Ax = 0$.

Solution. True. $Ax = 0 \Rightarrow A^H Ax = 0$. On the other hand, $A^H Ax = 0 \Rightarrow x^H A^H Ax = \|Ax\|^2 = 0$, or $Ax = 0$.

5. Let $P$ be a nonsingular $n \times n$ matrix and $D$ a diagonal $n \times n$ matrix. Set $A = PDP^{-1}$. Then $A$ has some set of $n$ eigenvectors which span $\mathbb{R}^n$.

Solution. Assuming the entries of $P$ are real, yes. Write as $AP = PD$, or $[Ap_1 | Ap_2 | \cdots | Ap_n] = [d_1 p_1 | \cdots | d_n p_n]$. 


1.2 Real vs. Complex

As a transformation on $\mathbb{R}^n$, if $A$ is orthogonally diagonalizable, then $A$ is symmetric. Find what breaks down over $\mathbb{C}^n$, that is: if $A$ is unitarily diagonalizable, it is not necessarily the case that $A$ is Hermitian. *Hint: the answer is almost in 1.1.3 above; start by trying to prove the first statement (orthog. implies symm.).*

Solution. $A = Q\Lambda Q^T$ means $A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T = A$, i.e., $A$ is symmetric. Now, doing the same thing with conjugate transpose doesn’t reach the same conclusion, but rather $A^H = Q\Lambda^H Q^H$, which is equal to $A$ only if $\Lambda^H = \Lambda$, i.e., . . . ?

1.3 Characterization of “Symmetric”

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix}$. Without computing any eigenvalues or eigenvectors of $A$, determine whether or not there is some set consisting of 3 pairwise orthogonal eigenvectors of $A$ (which must then form a basis of $\mathbb{R}^3$).

Solution. If so, then $A$ is symmetric (see above). $A$ is clearly not symmetric, so there cannot be such a set of vectors.

1.4 Diagonalization

Let $P$ be a nonsingular $n \times n$ matrix and $D$ a diagonal $n \times n$ matrix. Set $A = PDP^{-1}$. Find (that is, describe) some set of $n$ eigenvectors of $A$ which span $\mathbb{R}^n$.

Solution. The eigenvectors are the columns of $P$.

1.5 Definitions

Let $u$ be a vector in $\mathbb{C}^n$. Show that $A = I - \frac{2}{u^H u} uu^H$ is . . .

1. Hermitian

Solution. $(I + \frac{2}{u^H u} uu^H)^H = I + \frac{2}{u^H u} u^H u^H = A$.

2. Unitary

Solution. Similarly easy, straight from the definition.
1.6 Givens’ Method

Suppose $A$ is a $4 \times 4$ symmetric matrix whose $LU$ form looks like

$$A = \begin{bmatrix} 1 & * & 1 & * \\ * & 1 & * & 1 \\ * & * & 1 & * \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} -3 & * & * & * \\ * & 2 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{bmatrix}$$

and moreover, $A + 12I$ and $A + 11I$ factor, respectively, as

$$\begin{bmatrix} 1 & * & 1 & * \\ * & 1 & * & 1 \\ * & * & 1 & * \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} 8 & * & * & * \\ 10 & * & * & * \\ -2 & * & 1 & * \\ 11 & * & * & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ -1 & * & * & * \\ * & 7 & * & * \\ * & * & 12 & * \end{bmatrix}$$

1. What is the rank of $A$?

**Solution.** The rank of $A$ is three. The eigenvalue breakdown is $(-, 1, 0, +)$. 

2. Give an estimate on the size of the greatest negative eigenvalue of $A$.

**Solution.** $A$ has an eigenvalue between $-12$ and $-11$ because $A + 12I$ has one negative pivot, while $A + 11I$ has two negative pivots. 

1.7 “Unsymmetric” Pos.Def.

Suppose $A = QJQ^T$ for an orthogonal matrix $Q$ and a Jordan block $J = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ with $a > 1/\sqrt{2}$.

1. Rewrite $ay_1^2 + (y_1 + ay_2)y_2 + (y_2 + ay_3)y_3$ in such a way to demonstrate that it is always positive (no matter what $[y_1, y_2, y_3]$ is).

**Solution.** $a(y_1 + y_2/2a)^2 + a(y_3 + y_2/2a)^2 + (a - 2/(4a)^2)y_2^2$. 

2. Prove that $A$ is positive definite from the definition ($x^T A x > 0$ for all $x \neq 0$).

**Solution.** Put $x = Qy$, or $Q^T x = y$. 

1.8 Rayleigh’s Quotient

Show that the smallest eigenvalue $\lambda_1$ of the generalized eigenvalue problem $A \mathbf{x} = \lambda M \mathbf{x}$ is not larger than the ratio $a_{11}/m_{11}$ of the corner entries.

**Solution.** On the one hand, $A \mathbf{x} = \lambda M \mathbf{x}$ is the same as $C^T A C \mathbf{y} = \lambda \mathbf{y}$ (writing $M = R^T R$ for $C = R^{-1}$, and putting $R \mathbf{x} = \mathbf{y}$). Then $\mathbf{y}^T B \mathbf{y}/\mathbf{y}^T \mathbf{y}$ has its minimum value at $\lambda_1$ ($B = C^T AC$), the least eigenvalue for the generalized eigenvector problem. On the other hand, this quotient is equal to $x^T A x/x^T M x$, which sometimes equals $a_{11}/m_{11}$, e.g., when $x$ equals the standard unit vector $e_1$. 
