Math 315

1. Consider Alice, who chooses where in class to sit each day: by the door, in the middle, or by the window. Alice chooses somewhat arbitrarily. If she sat by the window the previous day, she flips a coin to determine if she’ll sit by the door or in the middle. If she sat in the middle the previous day, she rolls a fair three-sided die to determine where to sit next. If she sat by the door the previous day, she always sits by the window the next day.

(a) Give both the matrix and graph representation for this Markov process (Ordering: door, middle, window).

(b) If Alice sits in the middle today, what is the probability she will sit in the middle in two days (work out by hand, using recursion)? in 32 days (you may use Mathematica for help)?

2. Let

\[
A = \begin{bmatrix}
1 & 3 & 2 \\
1 & 1 & 4 \\
1 & 3 & 4 \\
1 & 1 & 2
\end{bmatrix}, \quad
b = \begin{bmatrix}
0 \\
4 \\
2 \\
2
\end{bmatrix}
\text{ and } c = \begin{bmatrix}
1 \\
4 \\
2
\end{bmatrix}
\]

(a) Solve the system \(Ax = b\) using the method of LU factorization.

(b) Solve the system \(Az = c\) using a QR factorization and the method of least-squares.

3. Suppose \(T \in \mathcal{L}(V)\) satisfies \(T^2 = -T\) for some finite dimensional vector space \(V\) over \(F\). Show that \(V\) has a basis of eigenvectors for \(T\). (Hint: first show that \(V = \text{null}T \oplus \text{range}T\))

Math 488

Submit all of the above, plus the following.

4. Suppose \(T \in \mathcal{L}(V)\) for some \(n\) dimensional vector space \(V\) over \(F\). Show that there is a basis for \(V\) such that \(M(T)\) is a diagonal matrix if and only if \(V\) contains \(n\) linearly independent eigenvectors for \(T\).

5. Given \(T \in \mathcal{L}(V)\) for some (not necessarily finite dimensional) vector space \(V\), a \(T\)-cyclic subspace of \(V\) is any space of the form \(\text{span}(v, Tv, T^2v, \ldots)\).

(a) Fix \(v \in V\) and let \(W\) be the \(T\)-cyclic subspace of \(V\) generated by \(v\). Given any \(w \in V\), show that \(w \in W\) if and only if there exists some polynomial \(g(t) \in \mathcal{P}(F)\) satisfying \(w = g(T)v\).

(b) Now suppose \(V\) is finite dimensional and suppose \(V\) is a \(T\)-cyclic subspace of itself. Given \(U \in \mathcal{L}(V)\), prove that \(UT = TU\) if and only if \(U = f(T)\) for some polynomial \(f(t) \in \mathcal{P}(F)\).