Math 315

1. Use the Gershgorin disc theorem to approximate the eigenvalues of \( A = \begin{bmatrix} 3 & .1 & .1i \\ -.1 & 0 & .1 \\ -.1i & .1 & 2 \end{bmatrix} \).

2. Let \( T \) be the transformation with \( \mathcal{M}(T) = \begin{bmatrix} 20 & -6 & -2 \\ 46 & -8 & -8 \\ 74 & -16 & -11 \end{bmatrix} \).

Find a single eigenpair \((\mathbf{w}, \lambda)\) for \( T \) by first forming the \( T \)-cyclic spaces \( \langle \mathbf{v} \rangle_T \) indicated below, then proceeding in the Axler manner. (Sometimes you’re lucky, sometimes you’re not.)

(a) \( \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}^T \)

(b) \( \mathbf{v} = \begin{bmatrix} 46 \\ 100 \\ 2 \end{bmatrix}^T \)

(c) \( \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \)

3. Here we follow the inductive procedure laid out by Axler. In Part 1(a) above, you should have found 1 as an eigenvalue. Now, à la Axler, set \( U = \text{range}(T - 1 \cdot I) \), a two-dimensional space and view \( T - I \) as a map in \( \mathcal{L}(U) \). Repeat the cyclic subspace game as in Problem 1(a), with any vector of your choosing belonging to \( U \). This should enable you to find two complex eigenpairs. (Hopefully a linear combination of what you found in Part 1(b) above.)

Math 488

Submit all of the above, plus the following.

4. A map \( T \in \mathcal{L}(V) \) is \textit{diagonalizable} (for \( V \) dimension \( n \)) if there is an invertible operator \( S \in \mathcal{L}(V) \) such that \( \mathcal{M}(S^{-1}TS) \) is a diagonal matrix. Two operators \( T, T' \) are \textit{simultaneously diagonalizable} if the same \( S \) works for both.

Suppose both \( T \) and \( T' \) have \( n \) distinct eigenvalues. In particular, they are diagonalizable. Prove that \( T \) and \( T' \) are simultaneously diagonalizable if and only if \( TT'(v) = T'T(v) \) for all \( v \in V \). That is, \( T \) and \( T' \) \textit{commute} as operators.

5. Given vectors \( v, w \in V \) (not necessarily finite dimensional) over \( \mathbb{F} \) (not necessarily algebraically closed). Suppose \( T \in \mathcal{L}(V) \) and the \( T \)-cyclic spaces corresponding to \( v, w \) are all finite dimensional, and yield the polynomials \( g_v(z) \) and \( g_w(z) \) à la Axler’s procedure for finding eigenpairs. (a) Prove that \( h = \text{lcm}(g_v, g_w) \) has the property that \( h(T)v = 0 \) and \( h(T)w = 0 \). (b) If \( V \) has dimension \( n \), prove that there is a polynomial \( h \) of degree at most \( n \) with the property that \( h(T)v = 0 \) for all \( v \in V \).