Written Assignment #5

1/ Find centers and radii of discs:
\[ C_1 = (3, 0) \quad r_1 = \sqrt{1^2 + 1^2} = 2 \]
\[ C_2 = (0, 0) \quad r_2 = \sqrt{1^2 + 1^2} = 2 \]
\[ C_3 = (2, 0) \quad r_3 = \sqrt{1^2 + 1^2} = 2 \]

Since discs are disjoint, may guess the center points at the possible eigenvalues: \( \lambda = 3, 0, 2 \)

2/ (a) \( \mathbf{v} = [2 \, 4 \, 7]^T, \quad \mathbf{Tv} = [24 \, 7]^T. \) Ah! So \( \mathbf{v} \) is an eigenvector with eigenvalue \( \lambda = 4 \). (The cyclic game stops right away.)

(b) \( \mathbf{v} = \left[ \begin{array}{c} 23 \\ 50 \\ 82 \end{array} \right]^T, \quad \mathbf{Tv} = \left[ \begin{array}{c} -4 \\ 2 \\ 0 \end{array} \right] \) and \( \langle \mathbf{v}, \mathbf{Tv} \rangle \) lin. indep.

\[ T^2 \mathbf{v} = \left[ -92 \quad -200 \quad 328 \right] \quad \text{and} \quad \langle \mathbf{v}, \mathbf{Tv}, T^2 \mathbf{v} \rangle \text{ lin. dependent!} \]

\[ T^2 \mathbf{v} = -4 \mathbf{v} + \mathbf{0} \quad \text{so} \quad g(2) = (2 + 2i)(2 - 2i) \]

(c) Here, \( \langle \mathbf{v}, \mathbf{Tv}, T^2 \mathbf{v} \rangle \) are lin. indep. Can't possibly have \( \langle \mathbf{v}, \ldots, T^3 \mathbf{v} \rangle \) lin. indep. since we are in \( \mathbb{R}^3 \).

\[ T^3 \mathbf{v} = 4 \mathbf{v} - 4T \mathbf{v} + T^2 \mathbf{v} \quad \text{so} \quad g(2) = (2 - 1)(3^2 + 4) \]

\[ g(T) \mathbf{v} = (T - 1)(T^2 + 4) \mathbf{v} = (T - 1) \left[ \begin{array}{c} 23 \\ 50 \\ 82 \end{array} \right] \]

\[ \mathbf{v} = \left[ \begin{array}{c} 23 \\ 50 \\ 82 \end{array} \right] \] has eigenvalue \( \lambda = 1 \), but we already knew this, since it's a multiple of the \( \mathbf{v} \) in part (a).
Let us find a basis for $\text{range}(T-I)$. Pick two vectors "at random":

$$V_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  
Then $\hat{v}_2 = \begin{bmatrix} 19 \\ 74 \end{bmatrix}$, $\hat{v}_3 = \begin{bmatrix} -9 \\ -16 \end{bmatrix}$.  
... look indep. to me!

Since $T$ has distinct eigenvalues, each eigenspace is one dimensional.  
Hence $\dim(\text{null}(T)) = 1$; hence $\dim(\text{range}(T)) = 2$.

(Aside: why I'm looking for two vectors)

How does $T$ act on these? call them $w_2$ & $w_3$, and $E = \text{basis} (w_2, w_3)$.

$$Tw_2 = \begin{bmatrix} -44 \\ -86 \\ -144 \end{bmatrix} \quad & \quad Tw_3 = \begin{bmatrix} 34 \\ -72 \\ 24 \end{bmatrix}, \quad \text{or} \quad \begin{cases} Tw_2 = \frac{-8}{7} w_2 + \frac{26}{7} w_3 \\ Tw_3 = \frac{-10}{7} w_2 + \frac{8}{7} w_3 \end{cases}$$

Thus $M(T|_E) = \begin{bmatrix} -\frac{8}{7} & -\frac{10}{7} \\ \frac{26}{7} & \frac{8}{7} \end{bmatrix}$.  
On this smaller space, we are supposed to apply induction to find some basis $\hat{E}$ with $M(T|_{\hat{E}})$ upper-triangular.

But let's just keep going.

$$w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A w = \begin{bmatrix} -\frac{8}{7} \\ \frac{26}{7} \end{bmatrix}, \quad A^2 w = \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \quad \text{so} \quad \hat{g}(z) = z^2 + 4 \text{ is important.}$$

$$A^2 w + \hat{g} \cdot A w + 4 \cdot w = 0$$

$$\hat{g}(A) w = (A + 2i)(A - 2i) w = (A^2 - 4i^2) w = \begin{bmatrix} -\frac{8}{7} - 2i \\ \frac{26}{7} \end{bmatrix} = 0 \quad \therefore \quad \begin{bmatrix} -\frac{8}{7} - 2i \\ \frac{26}{7} \end{bmatrix} \text{ is an eigenvector with eigenvalue } -2i.$$  

Similarly, $\hat{g}(A) w = (A - 2i)(A + 2i) w = (A^2 + 4i^2) w = 0$, so this gives final eigenpair:

$$A \cdot \begin{bmatrix} -\frac{8}{7} + 2i \\ \frac{26}{7} \end{bmatrix} = T \cdot \begin{bmatrix} \frac{-8}{7} + 2i \\ \frac{26}{7} \end{bmatrix}$$

Now, $A \cdot \begin{bmatrix} -\frac{8}{7} + 2i \\ \frac{26}{7} \end{bmatrix} \equiv T \cdot \begin{bmatrix} \frac{-8}{7} + 2i \\ \frac{26}{7} \end{bmatrix}$

So we're done.  
Basis $\hat{E} = \begin{bmatrix} \frac{2}{7} \\ \frac{9}{7} \end{bmatrix}$.  
$x_1 = (\frac{-8}{7} + 2i) w_2 + (\frac{26}{7}) w_3 = \begin{bmatrix} -44 + 38i \\ -86 + 92i \end{bmatrix}$

$$x_2 = (\frac{-8}{7} - 2i) w_2 + (\frac{26}{7}) w_3 = \begin{bmatrix} -44 - 38i \\ -86 - 92i \end{bmatrix}$$

$$x_3 = (\frac{-8}{7} - 2i) w_2 + (\frac{26}{7}) w_3 = \begin{bmatrix} -144 + 148i \\ -144 - 148i \end{bmatrix}$$

This basis satisfies $M(T)_{\hat{E}}$ is upper-triangular.
\( A/ (\Rightarrow) \) Put \( S^{-1} T S = D_1 \) and \( S^{-1} T' S = D_2 \) for some diagonal matrices \( D_1, D_2 \).

Easy to see that \( D_1, D_2 = D_2 P_1 \).

Then \( T T' = SD_1 S^{-1} S D_2 S^{-1} = SD_2 S^{-1} S D_1 S^{-1} = SD_2 S^{-1} S D_2 S^{-1} = T' T \) \( \square \)

\( (\Leftarrow) \) Note that \( T' \) preserved the eigenspaces of \( T \):

if \( T v = \lambda v \), then \( T(T'v) = T'(Tv) = T' \lambda v = \lambda (T'v) \).

So \( T'v \) is another eigenvector with eigenvalue \( \lambda \). But each eigenspace is
one-dimensional, since we were given that \( T, T' \) had distinct eigenvalues, so \( T'v \) is a multiple of \( v \).

i.e., \( v \) is also an eigenvector for \( T' \).

Given a basis \( S = \{v_1, v_2, \ldots, v_n\} \) of eigenvectors for \( T \), we have

\[ T \cdot S = S \cdot D_1 \quad \text{for some diagonal matrix } D_1 \text{ of eigenvalues of } T \text{, depending on the diagonal.} \]

and also, by your work,

\[ T' \cdot S = S \cdot D_2 \quad \text{for some other diagonal matrix.} \]

\( B/ (a) \) If \( \mathbf{h} = \mathbf{lcm}(g_1, g_2) \), then \( h(x) = p_1(x) g_1(x) = p_2(x) g_2(x) \) for some \( p_1, p_2 \in \mathbb{P}(\mathbb{F}) \).

Thus \( \begin{cases} h(T)v = p_1(T)g_1(T)v = p_1(T)(0) = 0, \\ h(T)w = p_2(T)g_2(T)w = p_2(T)(0) = 0. \end{cases} \)

(b) First apply the Axler method to find a basis \( \{v_1, v_2, \ldots, v_n\} \) for which \( T \) is given by an upper-triangular matrix.

We may assume that like numbers appear consecutively on the diagonal.

[Note:] we must pass to algebraic closure of \( \mathbb{F} \), to do this!!]

[Note 2: reread the proof to see why... if you haven't found all the \( \lambda \)'s yet, after \( k \) steps, then peel more of those \( \lambda \)'s, before hunting for next value.]

Suppose there are \( r \) distinct eigenvalues, appearing \( \lambda_1, \ldots, \lambda_r \), \( d_i \) times (i.e., \( \lambda_i \)). Relabel the \( \{v_1, v_2, \ldots, v_n\} \) to \( \{v_1, v_2, \ldots, v_{d_1}, \ldots, v_{d_2}, \ldots\} \).

Lemma: \( (T-\lambda_i)^{d_i}v_j = 0 \) \( \forall i \leq j < d_i \) (\( \forall i \)).
Formal proof would go by induction.

Here's "why", though: e.g. \[
\begin{align*}
(T-\lambda_1)^3 V_{11} &= (T-\lambda_1)^2 \left((T-\lambda_1)V_{11}\right) = (T-\lambda_1)^2 (0) = 0 \\
(T-\lambda_1)^3 V_{12} &= (T-\lambda_1)^2 \left(\lambda V_{12} + a_{12,11} V_{11}\right) - (\lambda V_{12}) \\
&= (T-\lambda_1)^2 (a_{12,11} V_{11}) = (T-\lambda_1) (0) = 0 \\
(T-\lambda_1)^3 V_{13} &= (T-\lambda_1)^2 (a_{12,12} V_{11} + a_{12,12} V_{12}) = 0
\end{align*}
\]

Finally, \[ (T-\lambda_1)^d_1 (T-\lambda_2)^d_2 \cdots (T-\lambda_r)^d_r V_{ij} = 0 \quad Vi_j \quad \text{(by simply rearranging in which order you apply these operators.)} \]

\[
 g(z) = \prod_{i=1}^{r} (z-\lambda_i)^{d_i} \quad \text{is a poly. of degree } r \text{ that kills everything. (Nevermind if smaller polys do the same thing.)}
\]