Math 315

1. A *norm* on a real vector space $V$ (i.e., over some $\mathbb{F} \subseteq \mathbb{R}$) is a map $n: V \rightarrow \mathbb{F}$ satisfying, for all $v, w \in V$ and $\alpha \in \mathbb{F}$:

   (i) $n(v) \geq 0$;  
   (ii) $n(\alpha v) = |\alpha| \cdot n(v)$;  
   (iii) $n(v + w) \leq n(v) + n(w)$.

   Show that $n((x_1 \ , x_2)) := |x_1| + |x_2|$ is a norm on $V = \mathbb{F}^2$.

2. Let $\langle \cdot , \cdot \rangle : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ be the inner product given by

   $$\langle f, g \rangle = \int_0^1 \frac{f(x)g(x)}{\sqrt{x}} \, dx.$$ 

   (a) Using this inner product, apply the Gram-Schmidt procedure to the basis $\mathcal{B} = (1, 3x - 1, x^2)$ to produce an orthonormal basis.

   (b) Find a polynomial $g(x)$ so that $f(1/2) = \langle f, g \rangle$ for all $f \in \mathcal{P}_2(\mathbb{R})$.

   (c) Define a map $T \in \mathcal{L}(\mathcal{P}_2, \mathbb{F})$ by $T(f) = f(1/2)$. Find a formula for $T^*(\alpha)$ for all $\alpha \in \mathbb{F}$.

3. Provide reasons why each of the following *is not* an inner product on the indicated vector spaces.

   (a) $\langle (a, b) , (c, d) \rangle = ac - bd$ on $\mathbb{R}^2$.

   (b) $\langle A, B \rangle = \text{trace}(A + B)$ on $M_{2 \times 2}(\mathbb{R})$.

   (c) $\langle f, g \rangle = \int_0^1 f'(t)g(t) \, dt$ on $\mathcal{P}(\mathbb{R})$, where $'$ denotes differentiation.

Math 488

Submit all of the above, plus the following.

4. Prove *Parseval’s Identity*. Let $(v_1, v_2, \ldots, v_n)$ be an orthonormal basis for $V$ over $\mathbb{F} \subseteq \mathbb{C}$. For any $x, y \in V$, prove that

   $$\langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$ 

5. Fix two vectors $y, z$ in an inner product space $V$. Define a map $T: V \rightarrow V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. Prove that $T$ is linear. Also, show that $T^*$ exists and find an explicit expression for it. (Think in terms of Theorem 6.45.)