HOPF STRUCTURES ON THE MULTIPLIHEDRA

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Abstract. We investigate algebraic structures that can be placed on vertices of the multiplihedra, a family of polytopes originating in the study of higher categories and homotopy theory. Most compelling among these are two distinct structures of a Hopf module over the Loday–Ronco Hopf algebra.

INTRODUCTION

The permutahedra $\mathcal{S}$, form a family of highly symmetric polytopes that have been of interest since their introduction by Schoute in 1911 [23]. The associahedra $\mathcal{Y}$, are another family of polytopes that were introduced by Stasheff as cell complexes in 1963 [25], and with the permutahedra were studied from the perspective of monoidal categories and $H$-spaces [17] in the 1960s. Only later were associahedra shown to be polytopes [11, 13, 18]. Interest in these objects was heightened in the 1990s, when Hopf algebra structures were placed on them in work of Malvenuto, Reutenauer, Loday, Ronco, Chapoton, and others [6, 14, 16]. More recently, the associahedra were shown to arise in Lie theory through work of Fomin and Zelevinsky on cluster algebras [7].

We investigate Hopf structures on another family of polyhedra, the multiplihedra, $\mathcal{M}$. Stasheff introduced them in the context of maps preserving higher homotopy associativity [26] and described their 1-skeleta. Boardman and Vogt [5], and then Iwase and Mimura [12] described the multiplihedra as cell complexes, and only recently were they shown to be convex polytopes [8]. These three families of polytopes are closely related. For each integer $n \geq 1$, the permutahedron $\mathcal{S}_n$, multiplihedron $\mathcal{M}_n$, and associahedron $\mathcal{Y}_n$ are polytopes of dimension $n-1$ with natural cellular surjections $\mathcal{S}_n \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \mathcal{Y}_n$, which we illustrate when $n = 4$.

The faces of these polytopes are represented by different flavors of planar trees; permutahedra by ordered trees (set compositions), multiplihedra by bi-leveled trees (Section 2.1), and associahedra by planar trees. The maps between them forget

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the additional structure on the trees. These maps induce surjective maps of graded vector spaces spanned by the vertices, which are binary trees. The span \( G_{\text{Sym}} \) of ordered trees forms the Malvenuto-Reutenauer Hopf algebra \([16]\) and the span \( \mathcal{Y}_{\text{Sym}} \) of planar binary trees forms the Loday-Ronco Hopf algebra \([14]\). The algebraic structures of multiplication and comultiplication on \( G_{\text{Sym}} \) and \( \mathcal{Y}_{\text{Sym}} \) are described in terms of geometric operations on trees and the composed surjection \( \tau: G_{\cdot} \rightarrow \mathcal{Y}_{\cdot} \) gives a surjective morphism \( \tau: G_{\text{Sym}} \rightarrow \mathcal{Y}_{\text{Sym}} \) of Hopf algebras.

We define \( \mathcal{M}_{\text{Sym}} \) to be the vector space spanned by the vertices of all multiplihedra. The factorization of \( \tau \) induced by the maps of polytopes, \( G_{\text{Sym}} \rightarrow \mathcal{M}_{\text{Sym}} \rightarrow \mathcal{Y}_{\text{Sym}} \), does not endow \( \mathcal{M}_{\text{Sym}} \) with the structure of a Hopf algebra. Nevertheless, some algebraic structure does survive the factorization. We show in Section 3 that \( \mathcal{M}_{\text{Sym}} \) is an algebra, which is simultaneously a \( G_{\text{Sym}} \)-module and a \( \mathcal{Y}_{\text{Sym}} \)-Hopf module algebra, and the maps preserve these structures.

We perform a change of basis in \( \mathcal{M}_{\text{Sym}} \) using Möbius inversion that illuminates its comodule structure. Such changes of basis helped to understand the coalgebra structure of \( G_{\text{Sym}} \) \([1]\) and of \( \mathcal{Y}_{\text{Sym}} \) \([2]\). Section 4 discusses a second \( \mathcal{Y}_{\text{Sym}} \) Hopf module structure that may be placed on the positive part \( \mathcal{M}_{\text{Sym}}^+ \) of \( \mathcal{M}_{\text{Sym}} \). This structure also arises from polytope maps between \( G_{\cdot} \) and \( \mathcal{Y}_{\cdot} \), but not directly from the algebra structure of \( G_{\text{Sym}} \). Möbius inversion again reveals an explicit basis of \( \mathcal{Y}_{\text{Sym}} \) coinvariants in this alternate setting.

1. Basic Combinatorial Data

The structures of the Malvenuto-Reutenauer and Loday-Ronco algebras are related to the weak order on ordered trees and the Tamari order on planar trees. There are natural maps between the weak and Tamari orders which induce a morphism of Hopf algebras. We first recall these partial orders and then the basic structure of these Hopf algebras. In Section 1.3 we establish a formula involving the Möbius functions of two posets related by an interval retract. This is a strictly weaker notion than that of a Galois correspondence, which was used to study the structure of the Loday-Ronco Hopf algebra.

1.1. \( G_{\cdot} \) and \( \mathcal{Y}_{\cdot} \). The 1-skeleta of the families of polytopes \( G_{\cdot}, M_{\cdot}, \) and \( \mathcal{Y}_{\cdot} \) are Hasse diagrams of posets whose structures are intertwined with the algebra structures we study. We use the same notation for a polytope and its poset of vertices. Similarly, we use the same notation for a cellular surjection of polytopes and the poset map formed by restricting that surjection to vertices.

For the permutahedron \( G_n \), the corresponding poset is the (left) weak order, which we describe in terms of permutations. A cover in the weak order has the form \( w \prec (k, k+1)w \), where \( k \) preceeds \( k+1 \) among the values of \( w \). Figure 1 displays the weak order on \( G_4 \). We let \( G_0 = \{ \emptyset \} \), where \( \emptyset \) is the empty permutation of \( \emptyset \).

Let \( \mathcal{Y}_n \) be the set of rooted, planar binary trees with \( n \) nodes. The cover relations in the Tamari order on \( \mathcal{Y}_n \) are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus

\[
\begin{align*}
\mathcal{Y}_{\cdot} & \rightarrow \mathcal{Y}_{\cdot} \\
\mathcal{Y}_{\cdot} & \rightarrow \mathcal{Y}_{\cdot} \\
\mathcal{Y}_{\cdot} & \rightarrow \mathcal{Y}_{\cdot} \\
\mathcal{Y}_{\cdot} & \rightarrow \mathcal{Y}_{\cdot}
\end{align*}
\]
is an increasing chain in $Y_3$ (the moving vertices are marked with dots). Figure 1 shows the Tamari order on $Y_4$.

![Figure 1. Weak order on $S_4$ and Tamari order on $Y_4$](image)

The unique tree in $Y_1$ is $Y$. Given trees $t_\ell$ and $t_r$, form the tree $t_\ell \vee t_r$ by grafting the root of $t_\ell$ (respectively of $t_r$) to the left (respectively right) leaf of $Y$. Form the tree $t_\ell \setminus t_r$ by grafting the root of $t_r$ to the rightmost leaf of $t_\ell$. For example,

$$t_\ell \quad t_r \quad t_\ell \vee t_r \quad t_\ell \setminus t_r$$

Decompositions $t = t_1 \setminus t_2$ correspond to pruning $t$ along the right branches from the root. A tree $t$ is indecomposable if it has no nontrivial decomposition $t = t_1 \setminus t_2$ with $t_1, t_2 \neq 1$. Equivalently, if the root node is the rightmost node of $t$. Any tree $t$ is uniquely decomposed $t = t_1 \setminus \cdots \setminus t_m$ into indecomposable trees $t_1, \ldots, t_m$.

We define a poset map $\tau: S_n \rightarrow Y_n$. First, given distinct integers $a_1, \ldots, a_k$, let $\overline{a} \in S_k$ be the unique permutation such that $\overline{a}(i) < \overline{a}(j)$ if and only if $a_i < a_j$. Thus $4726 = 2413$. Since $S_0, Y_0, S_1,$ and $Y_1$ are singletons, we must have

$$\tau : S_0 \rightarrow Y_0 \quad \text{with} \quad \tau : \emptyset \mapsto \downarrow, \quad \text{and} \quad \tau : S_1 \rightarrow Y_1 \quad \text{with} \quad \tau : 1 \mapsto Y.$$

Let $n > 0$ and assume that $\tau$ has been defined on $S_k$ for $k < n$. For $w \in S_n$ suppose that $w(j) = n$, and define

$$\tau(w) := \tau\left(\overline{w(1)}, \ldots, w(j-1)\right) \vee \tau\left(w(j+1), \ldots, w(n)\right).$$

For example,

$$\tau(12) = Y \vee \downarrow = Y, \quad \tau(21) = \downarrow \vee Y = Y, \quad \text{and} \quad \tau(3421) = \tau(3) \vee \tau(21) = \tau(1) \vee \tau(21) = Y \vee Y = Y \vee Y.$$
Loday and Ronco [15] show that the fibers \( \tau^{-1}(t) \) of \( \tau \) are intervals in the weak order. This gives two canonical sections of \( \tau \). For \( t \in \mathcal{Y}_n \),

\[
\min(t) := \min \{ w \mid \tau(w) = t \} \quad \text{and} \quad \max(t) := \max \{ w \mid \tau(w) = t \},
\]

the minimum and maximum in the weak order. Equivalently, \( \min(t) \) is the unique 231-avoiding permutation in \( \tau^{-1}(t) \) and \( \max(t) \) is the unique 132-avoiding permutation. These maps are order-preserving.

The 1-skeleta of \( \mathcal{S}_n \) and \( \mathcal{Y}_n \) form the Hasse diagrams of the weak and Tamari orders, respectively. Since \( \tau \) is an order-preserving surjection, it induces a cellular map between the 1-skeleta of these polytopes. Tonks [27] extended \( \tau \) to the faces of \( \mathcal{S}_n \), giving a cellular surjection.

The nodes and internal edges of a tree are the Hasse diagram of a poset with the root node maximal. Labeling the nodes (equivalently, the gaps between the leaves) of \( \tau(w) \) with the values of the permutation \( w \) gives a linear extension of the node poset of \( \tau(w) \), and all linear extensions of a tree \( t \) arise in this way for a unique permutation in \( \tau^{-1}(t) \). Such a linear extension \( w \) of a tree is an ordered tree and \( \tau(w) \) is the corresponding unordered tree. In this way, \( \mathcal{S}_n \) is identified with the set of ordered trees with \( n \) nodes. Here are some ordered trees,

\[
\begin{align*}
3 & 4 & 2 & 1 \\
1 & 4 & 3 & 2 \\
2 & 3 & 5 & 4 & 1 \\
2 & 5 & 1 & 4 & 3
\end{align*}
\]

Given ordered trees \( u, v \), form the ordered tree \( u \backslash v \) by grafting the root of \( v \) to the rightmost leaf of \( u \), where the nodes of \( u \) are greater than the nodes of \( v \), but the relative orders within \( u \) and \( v \) are maintained. Thus we may decompose an ordered tree \( w = u \backslash v \) whenever \( \tau(w) = r \backslash s \) with \( \tau(u) = r, \tau(v) = s \), and the nodes of \( r \) in \( w \) precede the nodes of \( s \) in \( w \). An ordered tree \( w \) is indecomposable if it has no nontrivial such decompositions. Here are ordered trees \( u, v \) and \( u \backslash v \),

\[
\begin{align*}
1 & 4 & 3 & 2 \\
1 & 3 & 2 \\
4 & 7 & 6 & 5 & 1 & 3 & 2
\end{align*}
\]

We may split an ordered tree \( w \) along a leaf to obtain either an ordered forest (where the nodes in the forest are totally ordered) or a pair of ordered trees,

\[
\begin{align*}
2 & 5 & 1 & 4 & 3 \\
2 & 5 & 1 & 4 & 3 \\
2 & 5 & 1 & 4 & 3
\end{align*}
\]

Write \( w \mapsto (w_0, w_1) \) to indicate that the ordered forest \( (w_0, w_1) \) (or pair of ordered trees) is obtained by splitting \( w \) along some leaf. (Context will determine how to interpret the result.) More generally, we may split an ordered tree \( w \) along a multiset of \( m \geq 0 \) of its leaves to obtain an ordered forest, or tuple of ordered
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trees, written \( w \overset{\gamma}{\rightarrow} (w_0, \ldots, w_m) \). For example,

\[
\begin{align*}
3 & \ 2 \ 7 \ 5 \ 11 \ 6 \ 4 \\
\mp \rightarrow
\end{align*}
\]

\[
\overset{\gamma}{\rightarrow}
\begin{align*}
3 & \ 2 \ 7 \ 5 \ 1 \ 6 \ 4 \\
\end{align*}
\]

Given \( v \in \mathfrak{S}_m \) and an ordered forest \( (w_0, \ldots, w_m) \), let \( (w_0, \ldots, w_m)/v \) be the ordered tree obtained by grafting the root of \( w_i \) to the \( i \)th leaf of \( v \), where the nodes of \( v \) are greater than all nodes of \( w \), but the relative orders within the \( w_i \) and \( v \) are maintained. When \( v \) is the ordered tree corresponding to 1432 and \( w \overset{\gamma}{\rightarrow} (w_0, \ldots, w_m) \) is the splitting (1.1), this grafting is

\[
\begin{align*}
3 & \ 2 \ 8 \ 11 \ 7 \ 5 \ 10 \ 6 \ 9 \ 4 \\
\end{align*}
\]

The notions of splitting and grafting also make sense for the unordered trees \( \mathcal{Y}_n \) and we use the same notation, \( \overset{\gamma}{\rightarrow} \) and \( \overset{\cdot}{\rightarrow} \). (Simply delete the labels in the constructions above.) These operations of splitting and grafting are compatible with the map \( \tau: \mathfrak{S}_* \rightarrow \mathcal{Y}_* \): if \( w \overset{\gamma}{\rightarrow} (w_0, \ldots, w_m) \) then \( \tau(w) \overset{\gamma}{\rightarrow} (\tau(w_0), \ldots, \tau(w_m)) \) and all splittings in \( \mathcal{Y}_* \) are induced in this way from splittings in \( \mathfrak{S}_* \). The same is true for grafting, \( \tau((w_0, \ldots, w_m)/v) = (\tau(w_0), \ldots, \tau(w_m))/\tau(v) \).

1.2. \( \mathfrak{S}Sym \) and \( \mathcal{Y}Sym \). For basics on Hopf algebras, see [19]. Let \( \mathfrak{S}Sym := \bigoplus_{n \geq 0} \mathfrak{S}Sym_n \) be the graded \( \mathbb{Q} \)-vector space whose \( n \)th graded piece has basis \( \{F_w \mid w \in \mathfrak{S}_n\} \). Malvenuto and Reutenauer [16] defined a Hopf algebra structure on \( \mathfrak{S}Sym \). For \( w \in \mathfrak{S}_* \), define the coproduct

\[
\Delta F_w := \sum_{w \overset{\gamma}{\rightarrow} (w_0, w_1)} F_{w_0} \otimes F_{w_1},
\]

where \( (w_0, w_1) \) is a pair of ordered trees. If \( v \in \mathfrak{S}_m \), define the product

\[
F_w \cdot F_v := \sum_{w \overset{\cdot}{\rightarrow} (w_0, \ldots, w_m)/v} F_{(w_0, \ldots, w_m)/v}.
\]

The counit is the projection \( \varepsilon: \mathfrak{S}Sym \rightarrow \mathfrak{S}Sym_0 \) onto the 0th graded piece, which is spanned by the unit, 1 = \( F_{\emptyset} \), for this multiplication.

**Proposition 1.1** ([16]). With these definitions of coproduct, product, counit, and unit, \( \mathfrak{S}Sym \) is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative.
Let $\mathcal{Y}Sym := \bigoplus_{n \geq 0} \mathcal{Y}Sym_n$ be the graded $\mathbb{Q}$–vector space whose $n$th graded piece has basis $\{F_t \mid t \in \mathcal{Y}_n\}$. Loday and Ronco [14] defined a Hopf algebra structure on $\mathcal{Y}Sym$. For $t \in \mathcal{Y}$, define the coproduct
\[
\Delta F_t := \sum_{t \to (t_0, t_1)} F_{t_0} \otimes F_{t_1},
\]
and if $s \in \mathcal{Y}_m$, define the product
\[
F_t \cdot F_s := \sum_{t \to (t_0, \ldots, t_m)/s} F_{(t_0, \ldots, t_m)/s}.
\]
The counit is the projection $\varepsilon : \mathcal{Y}Sym \to \mathcal{Y}Sym_0$ onto the 0th graded piece, which is spanned by the unit, $1 = F_0$, for this multiplication. The map $\tau$ extends to a linear map $\tau : SSym \to \mathcal{Y}Sym$, defined by $\tau(F_w) = F_{\tau(w)}$.

**Proposition 1.2 ([14]).** With these definitions of coproduct, product, counit, and unit, $\mathcal{Y}Sym$ is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative and the map $\tau$ a morphism of Hopf algebras.

Some structures of the Hopf algebras $SSym$ and $\mathcal{Y}Sym$, particularly their primitive elements and coradical filtrations are better understood with respect to a second basis. The M"obius function $\mu$ (or $\mu_P$) of a poset $P$ is defined for pairs $(x, y)$ of elements of $P$ with $\mu(x, y) = 0$ if $x \not< y$, $\mu(x, x) = 1$, and, if $x < y$, then
\[
\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) \quad \text{so that} \quad 0 = \sum_{x \leq z \leq y} \mu(x, z).
\]

For $w \in S$, and $t \in \mathcal{Y}$, set
\[
M_w := \sum_{w \leq v} \mu(w, v)F_v \quad \text{and} \quad M_t := \sum_{t \leq s} \mu(t, s)F_s,
\]
where the first sum is over $v \in S$, the second sum over $s \in \mathcal{Y}$, and $\mu(\cdot, \cdot)$ is the M"obius function in the weak and Tamari orders.

**Proposition 1.3 ([1, 2]).** If $w \in S$, then
\[
\tau(M_w) = \begin{cases} M_{\tau(w)}, & \text{if } w = \max(\tau(w)) \\ 0, & \text{otherwise} \end{cases}
\]
and
\[
\Delta M_w = \sum_{w = u \setminus v} M_u \otimes M_v.
\]
If $t \in \mathcal{Y}$, then
\[
\Delta M_t = \sum_{t = r \setminus s} M_r \otimes M_s.
\]
This implies that the set $\{M_w \mid w \in S$ is indecomposable$\}$ is a basis for the primitive elements of $SSym$ (and the same for $\mathcal{Y}Sym$), thereby explicitly realizing the cofree-ness of $SSym$ and $\mathcal{Y}Sym$. 
1.3. Möbius functions and interval retracts. A pair \( f : P \to Q \) and \( g : Q \to P \) of poset maps is a Galois connection if \( f \) is left adjoint to \( g \) in that

\[
\forall p \in P \text{ and } q \in Q, \quad f(p) \leq_Q q \iff p \leq_P g(q).
\]

When this occurs, Rota [21, Theorem 1] related the Möbius functions of \( P \) and \( Q \):

\[
\forall p \in P \text{ and } q \in Q, \quad \sum_{f(y) = q} \mu_P(p, y) = \sum_{g(x) = q} \mu_Q(x, q).
\]

Rota’s formula was used in [2] to establish the coproduct formulas (1.4) and (1.6), as the maps \( \tau : S_n \to Y_n \) and \( \max : Y_n \to S_n \) form a Galois connection [4, Section 9].

We do not have a Galois connection between \( S_n \) and \( M_n \), and so cannot use Rota’s formula. Nevertheless, there is a useful relation between the Möbius functions of \( S_n \) and \( M_n \) that we establish here in a general form. A surjective poset map \( f : P \to Q \) from a finite lattice \( P \) is an interval retract if the fibers of \( f \) are intervals and if \( f \) admits an order-preserving section \( g : Q \to P \) with \( f \circ g = \text{id} \).

**Theorem 1.4.** Let the poset map \( f : P \to Q \) be an interval retract, then the Möbius functions \( \mu_P \) and \( \mu_Q \) of \( P \) and \( Q \) are related by the formula

\[
(1.7) \quad \mu_Q(x, y) = \sum_{\substack{f(a) = x \\ f(b) = y}} \mu_P(a, b) \quad (\forall x, y \in Q).
\]

In Section 2, we define an interval retract \( \beta : S_n \to M_n \).

We evaluate each side of (1.7) using Hall’s formula, which expresses the Möbius function in terms of chains. A linearly ordered subset \( C = x_0 < \cdots < x_r \) of a poset is a chain of length \( \ell(C) = r \) from \( x_0 \) to \( x_r \). Given a poset \( P \), let \( \mathcal{C}(P) \) be the set of all chains in \( P \). A poset \( P \) is an interval if it has a unique maximum element and a unique minimum element. If \( P = [x, y] \) is an interval, let \( \mathcal{C}(P) \) denote the chains in \( P \) beginning in \( x \) and ending in \( y \). Hall’s formula states that

\[
\mu(x, y) = \sum_{C \in \mathcal{C}(x, y)} (-1)^{\ell(C)}.
\]

Our proof rests on the following two lemmas.

**Lemma 1.5.** If \( P \) is an interval, then \( \sum_{C \in \mathcal{C}(P)} (-1)^{\ell(C)} = 1 \).

**Proof.** Suppose that \( P = [x, y] \) and append new minimum and maximum elements to \( P \) to get \( \hat{P} := P \cup \{\hat{0}, \hat{1}\} \). Then the definition of Möbius function (1.2) gives

\[
\mu(\hat{0}, \hat{1}) = -\sum_{\hat{0} \leq z \leq \hat{1}} \mu(\hat{0}, z),
\]

which is zero by (1.2). By Hall’s formula,

\[ 0 = \mu(\hat{0}, \hat{1}) = \sum_{C \in \mathcal{C}([\hat{0}, \hat{1}])} (-1)^{\ell(C)} = -1 + \sum_{C \in \mathcal{C}(P)} (-1)^{\ell(C)+2}, \]

where the term \(-1\) comes from the chain \( \hat{0} < \hat{1} \). This proves the lemma. \( \square \)
Call a partition $P = K_0 \sqcup \cdots \sqcup K_r$ of $P$ into subposets $K_i$ **monotone** if $x < y$ with $x \in K_i$ and $y \in K_j$ implies that $i \leq j$. Given $\emptyset \subseteq I \subseteq [0, r]$, write $\mathcal{U}_I(P)$ for the subset of chains $C$ in $\mathcal{U}(P)$ such that $C \cap K_i \neq \emptyset$ if and only if $i \in I$.

**Lemma 1.6.** Let $P = K_0 \sqcup \cdots \sqcup K_r$ be a monotonic partition of a poset $P$. If $\bigcup_{i \in I} K_i$ is an interval for all $I \subseteq [0, r]$, then

\begin{equation}
\sum_{C \in \mathcal{U}_{[0, r]}(P)} (-1)^{\ell(C)} = (-1)^r.
\end{equation}

**Proof.** We argue by induction on $r$. Lemma 1.5 is the case $r = 0$ (wherein $K_0 = P$), so we consider the case $r \geq 1$.

Form the poset $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ as in the proof of Lemma 1.5. Since $P$ is an interval, we have $\sum_{C \in \mathcal{U}([0, 1])} (-1)^{\ell(C)} = 0$. As $\mathcal{U}([\hat{0}, \hat{1}]) = \bigcup_I \mathcal{U}_I(P)$ we have,

\begin{equation}
0 = -1 + \sum_{\emptyset \not\subseteq I \subseteq [r]} \left( \sum_{C \in \mathcal{U}_I(P)} (-1)^{\ell(C)} \right) + \sum_{C \in \mathcal{U}_{[0, r]}(P)} (-1)^{\ell(C)},
\end{equation}

where the term $-1$ counts the chain $\hat{0} < \hat{1}$. Applying induction, we have

\begin{align*}
0 &= \sum_{k=0}^r \binom{r+1}{k} (-1)^{k-1} + \sum_{C \in \mathcal{U}_{[0, r]}(P)} (-1)^{\ell(C)}.
\end{align*}

Comparing this to the binomial expansion of $(1 - 1)^{r+1}$ completes the proof. \hfill \box

**Proof of Theorem 1.4.** Fix $x < y$ in $Q$. We use Hall’s formula to rewrite the right-hand side of (1.7) as

\begin{equation}
\sum_{f(x) = a} \sum_{f(y) = b} (-1)^{\ell(C)}.
\end{equation}

Fix a chain $D$: $q_0 \prec \cdots \prec q_r$ in $\mathcal{U}[x,y]$ and let $P|_D$ be the subposet of $P$ consisting of elements that occur in some chain of $P$ that maps to $D$ under $f$. This is nonempty as $f$ has section. Furthermore, the sets $K_i := f^{-1}(q_i) \cap P|_D$, for $i = 0, \ldots, r$, form a monotonic partition of $P|_D$. We claim that $\bigcup_{i \in I} K_i$ is an interval for all $I \subseteq [0, r]$. If so, let us first rewrite (1.9) as a sum over chains $D$ in $Q$,

\begin{equation}
\sum_{D \in \mathcal{U}[x,y]} \sum_{C \in \mathcal{U}_{[0, \ell(D)]}(P|_D)} (-1)^{\ell(C)}.
\end{equation}

By Lemma 1.6, the inner sum becomes $\sum_D (-1)^{\ell(D)}$, which completes the proof.

To prove the claim, suppose that $I = \{i_0 < \cdots < i_s\}$. Each set $K_{i_0}$ is an interval, as it is the intersection of two intervals in the lattice $P$. Thus $K_{i_0}$ and $K_{i_s}$ are intervals with minimum and maximum elements $m$ and $M$, respectively. Any chain in $\bigcup_{i \in I} K_i$ can be extended to a chain beginning with $m$ and ending at $M$, so $\bigcup_{i \in I} K_i$ is an interval. \hfill \box
2. The Multiplihedra $\mathcal{M}$

The map $\tau: \mathcal{G} \to \mathcal{Y}$, forgets the linear ordering of the node poset of an ordered tree, and it induces a morphism of Hopf algebras $\tau: \mathcal{S}\text{Sym} \to \mathcal{Y}\text{Sym}$. In fact, one may take the (ahistorical) view that the Hopf structure on $\mathcal{Y}\text{Sym}$ is induced from that on $\mathcal{S}\text{Sym}$ via the map $\tau$. Forgetting some, but not all, of the structure on a tree in $\mathcal{G}$, factorizes the map $\tau$. Here, we study combinatorial consequences of one such factorization, and later treat its algebraic consequences.

2.1. Bi-leveled trees. A bi-leveled tree $(t; T)$ is a planar binary tree $t \in \mathcal{Y}_n$ together with an (upper) order ideal $T$ of its node poset, where $T$ contains the leftmost node of $t$ as a minimal element. Thus $T$ contains all nodes along the path from the leftmost leaf to the root, and none above the leftmost node. Numbering the gaps between the leaves of $t$ by $1, \ldots, n$ from left to right, $T$ becomes a subset of $\{1, \ldots, n\}$.

Saneblidze and Umble [22] introduced bi-leveled trees to describe a cellular projection from the permutahedra to Stasheff’s multiplihedra $\mathcal{M}_\ast$, with the bi-leveled trees on $n$ nodes indexing the vertices $\mathcal{M}_n$. Stasheff used a different type of tree for the vertices of $\mathcal{M}_\ast$. These alternative trees lead to a different Hopf structure which we explore in a forthcoming paper [9]. We remark that $\mathcal{M}_0 = \{\emptyset\}$.

The partial order on $\mathcal{M}_n$ is defined by $(s; S) \leq (t; T)$ if $s \leq t$ in $\mathcal{Y}_n$ and $S \supseteq T$. The Hasse diagrams of the posets $\mathcal{M}_n$ are 1-skeleta for the multiplihedra. We represent a bi-leveled tree by drawing the underlying tree $t$ and circling the nodes in $T$. The Hasse diagram of $\mathcal{M}_4$ appears in Figure 2.

![Figure 2. The 1-skeleton of the multiplihedron $\mathcal{M}_4$.](image)

2.2. Poset maps. Forgetting the order ideal in a bi-leveled tree, $(t; T) \mapsto t$, is a poset map $\phi: \mathcal{M}_\ast \to \mathcal{Y}_\ast$. We define a map $\beta: \mathcal{G}_\ast \to \mathcal{M}_\ast$ so that $\mathcal{G}_\ast \xrightarrow{\beta} \mathcal{M}_\ast \xrightarrow{\phi} \mathcal{Y}_\ast$ factors the map $\tau: \mathcal{G}_\ast \to \mathcal{Y}_\ast$, and we define a right inverse (section) $\iota$ of $\beta$. 
Let \( w \in \mathcal{S} \) be an ordered tree. Define the set
\[
\mathcal{T}(w) := \{ i \mid w(i) \geq w(1) \}.
\]
Observe that \((\tau(w); \mathcal{T}(w))\) is a bi-leveled tree. Indeed, as \( w \) is a linear extension of \( \tau(w) \), \( \mathcal{T}(w) \) is an upper order ideal which by definition (2.1) contains the leftmost node as a minimal element. Since covers in the weak order can only decrease the subset \( \mathcal{T}(w) \) and \( \tau \) is also a poset map, we see that \( \beta \) is a poset map.

**Theorem 2.1.** The maps \( \beta: \mathcal{S} \rightarrow \mathcal{M} \) and \( \phi: \mathcal{M} \rightarrow \mathcal{Y} \) are surjective poset maps with \( \tau = \phi \circ \beta \).

The fibers of the map \( \beta \) are intervals (indeed, products of intervals); see Figure 3. We prove this using an equivalent representation of a bi-leveled tree and a description of the map \( \beta \) in that representation. If we prune a bi-leveled tree \( b = (t; T) \) above the nodes in \( T \) (but not on the leftmost branch) we obtain a tree \( t'_0 \) (the order ideal) on \( r \) nodes and a planar forest \( t = (t_1, \ldots, t_r) \) of \( r \) trees. If we prune \( t'_0 \) just below its leftmost node, we obtain the tree \( \Upsilon \) (from the pruning) and a tree \( t_0 \), and \( t'_0 \) is obtained by grafting \( \Upsilon \) onto the leftmost leaf of \( t_0 \). We may recover \( b \) from this tree \( t_0 \) on \( r-1 \) nodes and the planar forest \( t = (t_1, \ldots, t_r) \), and so we also write \( b = (t_0, t) \). We illustrate this correspondence in Figure 4.

![Figure 3](image3.png)

**Figure 3.** The preimages of \( \beta \) are intervals.

![Figure 4](image4.png)

**Figure 4.** Two representations of bi-leveled trees.

We describe the map \( \beta \) in terms of this second representation of bi-leveled trees. Given a permutation \( w \) with \( \beta(w) = (t; T) \) and \( |T| = r \), let \( u_1 u_2 \ldots u_r \) be the
restriction of \( w \) to the set \( T \). We may write the values of \( w \) as \( w = u_1 v^1 u_2 \cdots u_r v^r \), where \( v^i \) is the (possibly empty) subword of \( w \) between the numbers \( u_i \) and \( u_{i+1} \) and \( v^r \) is the word after \( u_r \). Call this the bi-leveled factorization of \( w \). For example,

\[
654789132 \mapsto (6, 54, 7, \emptyset, 8, \emptyset, 9, 132).
\]

Note that \( \beta(w) = (\tau(u_2 \ldots u_r), (\tau(v^1), \ldots, \tau(v^r))) \).

**Theorem 2.2.** For any \( b \in \mathcal{M}_n \) the fiber \( \beta^{-1}(b) \subseteq \mathfrak{S}_n \) is a product of intervals.

**Proof.** Let \( b = (t_0, (t_1, \ldots, t_r)) = (t; T) \in \mathcal{M}_n \) be a bi-leveled tree. A permutation \( w \in \beta^{-1}(b) \in \mathfrak{S}_n \) has a bi-leveled factorization \( w = u_1 v^1 u_2 \cdots u_r v^r \) with

\[
(i) \quad w|_T = u_1 u_2 \cdots u_r, \quad u_1 = n+1-r, \quad \tau(u_2 \ldots u_r) = t_0, \quad \text{and}
(ii) \quad \tau(v^i) = t_i, \quad \text{for } i = 1, \ldots, r.
\]

Since \( u_1 < u_2, \ldots, u_r \) are the values of \( w \) in the positions of \( T \), and \( u_1 = n+1-r \) exceeds all the letters in \( v^1, \ldots, v^r \), which are the values of \( w \) in the positions in the complement of \( T \), these two parts of the bi-leveled factorization may be chosen independently to satisfy (2.2), which shows that \( \beta^{-1}(b) \) is a product.

To see that the factors are intervals, and thus \( \beta^{-1}(b) \) is an interval, we examine the conditions (i) and (ii) separately. Those \( u_1 \ldots u_r = w|_T \) for \( w \) in the fiber \( \beta^{-1}(b) \) are exactly the set of \( n+1-r, u_2, \ldots, u_r \) with \( \{u_2, \ldots, u_r\} = \{n+2-r, \ldots, n\} \) and \( \tau(u_2 \ldots u_r) = t_0 \). This is a poset under the restriction of the weak order, and it is in natural bijection with the interval \( \tau^{-1}(t_0) \subseteq \mathfrak{S}_{r-1} \). Its minimal element is \( \min_0(b) = u_1 u_2 \ldots u_r \), where \( u_2 \ldots u_r \) is the unique 231-avoiding word on \( \{n+1-r, \ldots, n\} \) satisfying (i), and its maximal element is \( \max_0(b) = u_1 u_2 \ldots u_r \), where now \( u_2 \ldots u_r \) is the unique 132-avoiding word on \( \{n+1-r, \ldots, n\} \) satisfying (i).

Now consider sequences of words \( v^1, \ldots, v^r \) on distinct letters \( \{1, \ldots, n-r\} \) satisfying (ii). This is also a poset under the restriction of the weak order. It has a minimal element, which is the unique such sequence \( \min(b) \) satisfying (ii) where the letters of \( v^i \) precede those of \( v^j \) whenever \( i < j \), and where each \( v^i \) is 231-avoiding. Its maximal element is the unique sequence \( \max(b) \) satisfying (ii) where the letters of \( v^i \) are greater than those of \( v^j \) when \( i < j \) and \( v^i \) is 132-avoiding. \( \square \)

The fibers of \( \beta \) are intervals so that consistently choosing the minimum or maximum in a fiber gives two set-theoretic sections. These are not order-preserving as may be seen from Figure 5. We have \( \preceq \prec \prec \preceq \) but the maxima in their fibers

\[
\begin{align*}
\text{Figure 5. Fibers of } \beta.
\end{align*}
\]
under $\beta$, 1342 and 2143, are incomparable. Similarly, $\{\, 7, 8, 9, 10 \} \prec \{\, 3, 1 \}$ but the minima in their fibers under $\beta$, 2341 and 3142, are incomparable. This shows that the map $\beta : \mathcal{S}_n \to \mathcal{M}_n$ is not a lattice congruence (unlike the map $\tau : \mathcal{S}_n \to \mathcal{Y}_n$ [20]).

In the notation of the proof, given a bi-leveled tree $b = (t_0, (t_1, \ldots, t_r))$, let $\iota(b)$ be the permutation $w \in \beta^{-1}(b)$ with bi-leveled factorization $w = u_1 v^1 u_2 \ldots u_r v^r$ where $u_1 u_2 \ldots u_r = \min_b(b)$ and $(v^1, \ldots, v^r) = \max_b(b)$. This defines a map $\iota : \mathcal{M}_n \to \mathcal{S}_n$ that is a section of the map $\beta$. For example,

\[
\iota \left( \begin{array}{c}
1 & 2 & 3 \\
& 4 & 5 \\
 & & 6 & 7 \\
& & & 8
\end{array} \right) = \begin{array}{c}
7 & 8 & 6 & 11 & 4 & 5 & 9 & 10 & 2 & 3 & 1 \\
\end{array} = 7861145910231.
\]

**Remark 2.3.** This map $\iota$ may be characterized in terms of pattern avoidance: the permutation $\iota(b)$ is the unique $w \in \beta^{-1}(b)$ avoiding the pinned patterns

\[\{2031, 0231, 3021\},\]

where the underlined letter must be the first letter of a permutation. To see this, note that the first pattern forces the letters in $v^i$ to be larger than those in $v^{i+1}$ for $1 \leq i < r$, the second pattern forces $u_2 \ldots u_r$ to be 231-avoiding, and the last pattern forces each $v^i$ to be 132-avoiding.

**Theorem 2.4.** The map $\iota$ is injective, right-inverse to $\beta$, and order-preserving. That is, $\beta : \mathcal{S}_n \to \mathcal{M}_n$ is an interval retract.

Since $\mathcal{S}_n$ is a lattice [10], the fibers of $\beta$ are intervals, and $\iota$ is a section of $\beta$. That is, we need only verify that $\iota$ is order-preserving. We begin by describing the covers in $\mathcal{M}_n$. Since $\beta$ is a surjective poset map, every cover in $\mathcal{M}_n$ is the image of some cover $w \leq w'$ in $\mathcal{S}_n$.

**Lemma 2.5.** If a cover $w \leq w' \in \mathcal{S}_n$ does not collapse under $\beta$, i.e., $\beta(w) \neq \beta(w')$, then it yields one of three types of covers $\beta(w) \leq \beta(w')$ in $\mathcal{M}_n$.

(i) In exactly one tree $t_i$ in $\beta(w) = (t_0, (t_1, \ldots, t_r))$, a node is moved from left to right across its parent to obtain $\beta(w')$. That is, $t_i < t_i'$.  
(ii) If $\beta(w) = (t; T)$, the leftmost node of $t$ is moved across its parent, which has no other child in the order ideal $T$, and is deleted from $T$ to obtain $\beta(w')$.
(iii) If $T(w) = \{1 = T_1 < \cdots < T_r\}$, then $\tau(w') = \tau(w)$ and $T(w') = T(w) \setminus \{T_j\}$ for some $j > 2$.

**Proof.** Put $w' = (k, k+1)w$, with $k, k+1$ appearing in order in $w$. Let $(t; T)$ and $(t_0, (t_1, \ldots, t_r))$ be the two representations of $\beta(w)$. Write $T = \{T_1 < \cdots < T_r\}$ (with $T_1 = 1$) and $w|_T = u_1 u_2 \ldots u_r$. If $w \leq w'$ and $\beta(w) \leq \beta(w')$, then $k$ appears within $w$ in one of three ways: (i) $u_1 \neq k$, (ii) $u_1 = k$ and $u_2 = k+1$, or (iii) $u_1 = k$ and $u_j = k+1$ for some $j > 2$. These yield the corresponding descriptions in the statement of the lemma. (Note that in type (i), $T(w') = T$, so if we set $\beta(w') = (t_0, (t_1', \ldots, t_r'))$, then $t_i = t'_i$, except for one index $i$, where $t_i < t'_i$.) \[\square\]
Figure 6. Some covers in $\mathcal{M}_7$.

Figure 6 illustrates these three types of covers, labeled by their type.

For $T \subset \{1, \ldots, n\}$ with $1 \in T$, let $\mathcal{G}_n(T) := \{w \in \mathcal{G}_n \mid T(w) = T\}$. Let $\mathcal{M}_n(T)$ be those bi-leveled trees whose order ideal consists of the nodes in $T$. Note that $\beta(\mathcal{G}_n(T)) = \mathcal{M}_n(T)$ and $\beta^{-1}(\mathcal{M}_n(T)) = \mathcal{G}_n(T)$.

**Lemma 2.6.** The map $\iota: \mathcal{M}_n(T) \to \mathcal{G}_n(T)$ is a map of posets.

**Proof.** Let $T = \{1 = T_1 < \cdots < T_r\}$. Setting $T_{r+1} = n + 1$, define $a_i := T_{i+1} - T_i - 1$ for $i = 1, \ldots, r$. Then $b \mapsto (t_0, (t_1, \ldots, t_r))$ gives an isomorphism of posets,

$$\mathcal{M}_n(T) \xrightarrow{\sim} \mathcal{Y}_{a_1} \times \cdots \times \mathcal{Y}_{a_r}.$$

As the maps $\min, \max : \mathcal{Y}_a \to \mathcal{G}_a$ are order-preserving, the proof of Theorem 2.2 gives the desired result. \hfill \square

**Proof of Theorem 2.4.** Let $b < c$ be a cover in $\mathcal{M}_n$. We will show that $\iota(b) \leq \iota(c)$ in $\mathcal{G}_n$. Suppose that $b = (t; T)$, with $T = \{1 = T_1 < \cdots < T_r\}$. Let $\iota(b)$ have bi-leveled factorization $\iota(b) = v^1u_1v^2 \cdots u_r v^r$, and set $k := n + 1 - |T|$.

The result is immediate if the cover $b < c$ is of type (i), for then $b, c \in \mathcal{M}_n(T)$ and $\iota: \mathcal{M}_n(T) \to \mathcal{G}_n$ is order-preserving, as observed in Lemma 2.6.

Now suppose that $b < c$ is a cover of type (ii). Set $w := \iota(b)$. We claim that $w \leq (k, k+1)w$ and $\iota(c) = (k, k+1)w$. Now, $u_1 = k$ labels the leftmost node of $b$, so the first claim is immediate. Note that $u_2$ labels the parent of the node labeled $b$. This parent has no other child in $T$, so we must have $u_2 < u_3$. As $u_2u_3\ldots u_r$ is 231-avoiding and contains $k+1$, we must have $u_2 = k+1$. This shows that

$$\iota(c) = (k, k+1)w = u_2(v^1u_1v^2)u_3\ldots u_r v^r.$$  

Indeed, $u_2$ is minimal among $u_2, \ldots, u_r$ and $u_3\ldots u_r$ is 231-avoiding, thus $\min_0(c) = u_2\ldots u_r$. The bi-leveled factorization of $(k, k+1)w$ gives $(v^1u_1v^2, v^3, \ldots, v^r)$, which we claim is $\max(c)$. As $u_1$ is the largest letter in the sequence, we need only check that $v^1u_1v^2$ is 132-avoiding. But this is true for $v^1$ and $v^2$ and there can be no 132-pattern involving $u_1$ as the letters in $v^1$ are all greater than those in $v^2$.

Finally, suppose that $b < c$ is of type (iii). Then $c = (t; T \setminus \{T_j\})$ for some $j > 2$. We will find a permutation $w' \in \beta^{-1}(b)$ satisfying $(k, k+1)w' \in \beta^{-1}(c)$ and

$$\iota(b) \leq w' \preceq (k, k+1)w' \leq \iota(c).$$  

(2.3)
Let \( w' \in \beta^{-1}(b) \) be the minimal permutation having bi-leveled factorization

\[
w' = u'_1 v^1 u'_2 \ldots u'_r v^r, \quad \text{with} \quad u'_j = k+1.
\]

Here \((v^1, \ldots, v^r) = \max(b)\) is the same sequence as in \(\iota(b)\). The structure of \(\beta^{-1}(b)\) implies that \(\iota(b) \leq w'\). We also have

\[
w' \prec (k, k+1)w' \quad \text{and} \quad \beta((k, k+1)w') = c.
\]

While \(\iota(c)\) and \((k, k+1)w'\) are not necessarily equal, we do have that

\[
(k, k+1)w'|_{\tau \backslash \{T\}} = u'_j u'_2 \ldots u'_{j-1} u'_{j+1} \ldots u'_r
\]

and \(u'_2 \ldots u'_{j-1} u'_{j+1} \ldots u'_{r}\) is 231-avoiding. That is, \((k, k+1)w'|_{\tau \backslash \{T\}} = \iota(c)|_{\tau \backslash \{T\}}\). Otherwise, \(w'\) would not be minimal. The bi-leveled factorization of \((k, k+1)w'\) is

\[
u'_j v^1 u'_2 \ldots u'_{j-1} (v^j-1 u'_i v^j) u'_{j+1} \ldots u'_r v^r,
\]

and we necessarily have \((v^1, \ldots, v^j-1 u'_i v^j, \ldots, v^r) \leq \max(c)\), which implies that \((k, k+1)w' \leq \iota(c)\). We thus have the chain \((2.3)\) in \(\mathcal{S}_n\), completing the proof.

If \(b \prec c\) is the cover of type \((iii)\) in Figure 6, the chain \((2.3)\) from \(\iota(b)\) to \(\iota(c)\) is

\[
4357126 \leq 4367125 \prec 5367124 \leq 5467123.
\]

2.3. Tree enumeration. Let

\[
S(q) := \sum_{n\geq 0} n!q^n = 1 + q + 2q^2 + 6q^3 + 24q^4 + 120q^5 + \cdots
\]

be the enumerating series of permutations, and define \(M(q)\) and \(Y(q)\) similarly

\[
2.4 \quad M(q) := \sum_{n\geq 0} A_n q^n = 1 + q + 2q^2 + 6q^3 + 21q^4 + 80q^5 + \cdots,
\]

\[
Y(q) := \sum_{n\geq 0} C_n q^n = 1 + q + 2q^2 + 5q^3 + 14q^4 + 42q^5 + \cdots,
\]

where \(A_n := |\mathcal{M}_n|\) and \(C_n := |\mathcal{Y}_n|\) are the Catalan numbers \(\frac{1}{n+1}\binom{2n}{n}\), whose enumerating series satisfies

\[
Y(q) = \frac{1 - \sqrt{1 - 4q}}{2q} = \frac{2}{1 + \sqrt{1 - 4q}}.
\]

Bi-leveled trees are Catalan-like [8, Theorem 3.1]: for \(n \geq 1\), \(A_n = C_{n-1} + \sum_{i=1}^{n-1} A_i A_{n-i}\). See also [24, A121988]. Their enumerating series satisfies

\[
M(q) = 1 + qY(q) \cdot Y(qY(q)).
\]

We will also be interested in \(M_+(q) := \sum_{n>0} A_n q^n = qY(q) \cdot Y(qY(q))\).

**Theorem 2.7.** The only nontrivial quotients of the enumerating series \(S(q)\), \(M(q)\), \(M_+(q)\), and \(Y(q)\) whose expansions have nonnegative coefficients are

\[
S(q)/M(q), \quad S(q)/Y(q), \quad M_+(q)/Y(q), \quad \text{and} \quad M(q)/Y(q).
\]
Proof. We prove the positivity of the quotient $S(q)/M(q)$ in Section 4.2. The positivity of $S(q)/Y(q)$ was established after [2, Theorem 7.2], which shows that $\mathcal{G}Sym$ is a smash product over $\mathcal{Y}Sym$.

For the positivity of $M_+(q)/Y(q)$, we use [3, Proposition 3], which computes $Y(qY(q)) = \sum_{n>0} B_n q^n$, where

$$B_1 := C_0 \quad \text{and} \quad B_n := \sum_{k=0}^{n-1} \frac{k}{n-1} \binom{2n-k-3}{n-k-1} C_k \quad \text{for } n > 1.$$  (2.5)

In particular, $B_n \geq 0$ for all $n \geq 0$. Returning to the quotient, we have

$$\frac{M_+(q)}{Y(q)} = \frac{qY(q) \cdot Y(qY(q))}{Y(q)} = qY(qY(q)),$$

so $M_+(q)/Y(q) = \sum_{n>0} B_n q^n$ has nonnegative coefficients.

For $M(q)/Y(q)$, use the identity $1/Y(q) = 1 - qY(q)$ to obtain

$$\frac{M(q)}{Y(q)} = M_+(q) + 1 - qY(q) = 1 + \sum_{n>0} (B_n - C_{n-1}) q^n.$$

Positivity is immediate as $B_n - C_{n-1} \geq 0$ for $n > 0$.

We leave the proof that the remaining quotients have negative coefficients to the reader’s computer. \qed

Remark 2.8. Up to an index shift, the quotient $M_+(q)/Y(q)$ corresponds to the sequence [24, A127632] beginning with $(1, 1, 3, 11, 44, 185, 804)$. We give a new combinatorial interpretation of this sequence in Corollary 4.3.

3. The Algebra $\mathcal{M}Sym$

Let $\mathcal{M}Sym := \bigoplus_{n \geq 0} \mathcal{M}Sym_n$ denote the graded $\mathbb{Q}$-vector space whose $n^{th}$ graded piece has the basis $\{F_b \mid b \in \mathcal{M}_n\}$. The maps $\beta : \mathcal{G} \rightarrow \mathcal{M}$ and $\phi : \mathcal{M} \rightarrow \mathcal{Y}$ of graded sets induce surjective maps of graded vector spaces

$$\mathcal{G}Sym \xrightarrow{\beta} \mathcal{M}Sym \xrightarrow{\phi} \mathcal{Y}Sym \quad F_w \mapsto F_{\beta(w)} \mapsto F_{\phi(\beta(w))},$$

which factor the Hopf algebra map $\tau : \mathcal{G}Sym \rightarrow \mathcal{Y}Sym$, as $\phi(\beta(w)) = \tau(w)$. We will show how the maps $\beta$ and $\tau$ induce on $\mathcal{M}Sym$ the structures of an algebra, of a $\mathcal{G}Sym$-module, and of a $\mathcal{Y}Sym$-comodule so that the composition (3.1) factors the map $\tau$ as maps of algebras, of $\mathcal{G}Sym$-modules, and of $\mathcal{Y}Sym$-comodules.

3.1. Algebra structure on $\mathcal{M}Sym$. For $b, c \in \mathcal{M}$, define

$$F_b \cdot F_c = \beta(F_w \cdot F_v),$$

where $w, v$ are permutations in $\mathcal{G}$, with $b = \beta(w)$ and $c = \beta(v)$.

Theorem 3.1. The operation $F_b \cdot F_c$ defined by (3.2) is independent of choices of $w, v$ with $\beta(w) = b$ and $\beta(v) = c$ and it endows $\mathcal{M}Sym$ with the structure of a graded connected algebra such that the map $\beta : \mathcal{G}Sym \rightarrow \mathcal{M}Sym$ is a surjective map of graded connected algebras.
If the expression $\beta(F_w \cdot F_v)$ is independent of choice of $w \in \beta^{-1}(b)$ and $v \in \beta^{-1}(c)$, then the map $\beta$ is automatically multiplicative. Associative and unital properties for $\mathcal{MSym}$ are then inherited from those for $\mathfrak{SSym}$, and the theorem follows. To prove independence (in Lemma 3.2), we formulate a description of (3.2) in terms of splittings and graftings of bi-leveled trees.

Let $s \xrightarrow{\gamma} (s_0, \ldots, s_m)$ be a splitting on the underlying tree of a bi-leveled tree $b = (s; S) \in \mathcal{M}_n$. Then the nodes of $s$ are distributed among the nodes of the partially ordered forest $(s_0, \ldots, s_m)$ so that the order ideal $S$ gives a sequence of order ideals in the trees $s_i$. Write $b \xrightarrow{\gamma} (b_0, \ldots, b_m)$ for the corresponding splitting of the bi-leveled tree $b$, viewing $b_i$ as $(s_i; S|_n)$. (Note that only $b_0$ is guaranteed to be a bi-leveled tree.) Given $c = (t; T) \in \mathcal{M}_m$ and a splitting $b \xrightarrow{\gamma} (b_0, \ldots, b_m)$ of $b \in \mathcal{M}_n$, form a bi-leveled tree $(b_0, \ldots, b_m)/c$ whose underlying tree is $(s_0, \ldots, s_m)/t$ and whose order ideal is either

\begin{equation}
\begin{aligned}
& (i) \ T, \text{ if } b_0 \in \mathcal{M}_0, \text{ or } \\
& (ii) \ S \cup \{\text{the nodes of } t\}, \text{ if } b_0 \not\in \mathcal{M}_0.
\end{aligned}
\end{equation}

Lemma 3.2. The product (3.2) is independent of choices of $w, v$ with $\beta(w) = b$ and $\beta(v) = c$. For $b \in \mathcal{M}_n$ and $c \in \mathcal{M}_m$, we have

$$F_b \cdot F_c = \sum_{b \xrightarrow{\gamma} (b_0, \ldots, b_m)} F_{(b_0, \ldots, b_m)/c}.$$ 

Proof. Fix any $w \in \beta^{-1}(b)$ and $v \in \beta^{-1}(c)$. The bi-leveled tree $\beta((w_0, \ldots, w_m)/v)$ associated to a splitting $w \xrightarrow{\gamma} (w_0, \ldots, w_m)$ has underlying tree $(s_0, \ldots, s_m)/t$, where $s \xrightarrow{\gamma} (s_0, \ldots, s_m)$ is the induced splitting on the underlying tree $s = \tau(w) = \phi(b)$. Each node of $(w_0, \ldots, w_m)/v$ comes from a node of either $w$ or $v$, with the labels of nodes from $w$ all smaller than the labels of nodes from $v$. Consequently, the leftmost node of $(w_0, \ldots, w_m)/v$ comes from either

\begin{enumerate}
\item[(i)] $v$, and then $T((w_0, \ldots, w_m)/v) = T(v) = T(c)$, or
\item[(ii)] $w$, and then $T((w_0, \ldots, w_m)/v) = T(w) = T(b) \cup \{\text{the nodes of } v\}$.
\end{enumerate}

The first case is when $w_0 \in \mathfrak{S}_0$ and the second case is when $w_0 \not\in \mathfrak{S}_0$. \qed

Here is the product $F_{\gamma} \cdot F_{\gamma}$, together with the corresponding splittings of $\gamma$, $\tau$,

$$F_{\gamma} \cdot F_{\gamma} = F_{\gamma \cdot \gamma} + F_{\gamma \cdot \gamma} + F_{\gamma \cdot \gamma} + F_{\gamma \cdot \gamma} + F_{\gamma \cdot \gamma} + F_{\gamma \cdot \gamma}.$$ 

3.2. $\mathfrak{SSym}$ module structure on $\mathcal{MSym}$. Since $\beta$ is a surjective algebra map, $\mathcal{MSym}$ becomes a $\mathfrak{SSym}$-bimodule with the action

$$F_w \cdot F_b \cdot F_v = F_{\beta(w)} \cdot F_b \cdot F_{\beta(v)}.$$ 

The map $\tau$ likewise induces on $\mathcal{YSym}$ the structure of a $\mathfrak{SSym}$-bimodule, and the maps $\beta, \phi, \tau$ are maps of $\mathfrak{SSym}$-bimodules.
Curiously, we may use the map $\iota: \mathcal{M} \to \mathcal{S}$ to define the structure of a right $\mathcal{S} \text{Sym}$-comodule on $\mathcal{M} \text{Sym}$,

$$F_b \mapsto \sum_{\iota(b) \hookrightarrow (w_0, w_1)} F_{\beta(w_0)} \otimes F_{w_1}.$$  

This induces a right comodule structure, because if $\iota(b) \hookrightarrow (w_0, w_1)$, then $w_0 = \iota(\beta(w_0))$, which may be checked using the characterization of $\iota$ in terms of pattern avoidance, as explained in Remark 2.3.

While $\mathcal{M} \text{Sym}$ is both a right $\mathcal{S} \text{Sym}$-module and right $\mathcal{S} \text{Sym}$-comodule, it is not an $\mathcal{S} \text{Sym}$–Hopf module. For if it were a Hopf module, then the fundamental theorem of Hopf modules (see Remark 4.4) would imply that the series $\mathcal{M}(q)/\mathcal{S}(q)$ has positive coefficients, which contradicts Theorem 2.7.

3.3. $\mathcal{Y} \text{Sym}$-comodule structure on $\mathcal{M} \text{Sym}$. For $b \in \mathcal{M}$, define the linear map $\rho: \mathcal{M} \text{Sym} \to \mathcal{M} \text{Sym} \otimes \mathcal{Y} \text{Sym}$ by

$$\rho(F_b) = \sum_{b \mapsto (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}.$$  

By $\phi(b_1)$, we mean the tree underlying $b_1$.

**Example 3.3.** In the fundamental bases of $\mathcal{M} \text{Sym}$ and $\mathcal{Y} \text{Sym}$, we have

$$\rho(F_{\langle \rangle}) = F_{\langle \rangle} \otimes 1 + F_{\langle \rangle} \otimes F_{\langle \rangle} + F_{\langle \rangle} \otimes F_{\langle \rangle} + F_{\langle \rangle} \otimes F_{\langle \rangle} + 1 \otimes F_{\langle \rangle} \langle \rangle.$$

**Theorem 3.4.** Under $\rho$, $\mathcal{M} \text{Sym}$ is a right $\mathcal{Y} \text{Sym}$-comodule.

**Proof.** This is counital as $(b, 1)$ is a splitting of $b$. Coassociativity is also clear as both $(\rho \otimes 1)\rho$ and $(1 \otimes \Delta)\rho$ applied to $F_b$ for $b \in \mathcal{M}$, are sums of terms $F_{b_0} \otimes F_{\phi(b_1)} \otimes F_{\phi(b_2)}$ over all splittings $b \mapsto (b_0, b_1, b_2)$. □

Careful bookkeeping of the terms in $\rho(F_b \cdot F_c)$ show that it equals $\rho(F_b) \cdot \rho(F_c)$ for all $b, c \in \mathcal{M}$, and thus $\mathcal{M} \text{Sym}$ is a $\mathcal{Y} \text{Sym}$–comodule algebra. Hence, $\phi$ is a map of $\mathcal{Y} \text{Sym}$–comodule algebras, and in fact $\beta$ is also a map of $\mathcal{Y} \text{Sym}$–comodule algebras. We leave this to the reader, and will not pursue it further.

Since $\tau: \mathcal{S} \text{Sym} \to \mathcal{Y} \text{Sym}$ is a map of Hopf algebras, $\mathcal{S} \text{Sym}$ is naturally a right $\mathcal{Y} \text{Sym}$-comodule where the comodule map is the composition

$$\mathcal{S} \text{Sym} \xrightarrow{\Delta} \mathcal{S} \text{Sym} \otimes \mathcal{S} \text{Sym} \xrightarrow{1 \otimes \tau} \mathcal{S} \text{Sym} \otimes \mathcal{Y} \text{Sym}.$$  

With these definitions, the following lemma is immediate.

**Lemma 3.5.** The maps $\tau$ and $\phi$ are maps of right $\mathcal{Y} \text{Sym}$-comodules.

In particular, we have the equality of maps $\mathcal{S} \text{Sym} \to \mathcal{M} \text{Sym} \otimes \mathcal{Y} \text{Sym}$,

$$\rho \circ \beta = (\beta \otimes \tau) \circ \Delta.$$  

3.4. Coaction in the monomial basis. The coalgebra structures of $\mathcal{S} \text{Sym}$ and $\mathcal{Y} \text{Sym}$ were elucidated by considering a second basis related to the fundamental basis via Möbius inversion. For $b \in \mathcal{M}_n$, define

\begin{equation}
M_b := \sum_{b \leq c} \mu(b, c) F_c,
\end{equation}

where $\mu(\cdot, \cdot)$ is the Möbius function on the poset $\mathcal{M}_n$.

Given $b \in \mathcal{M}_m$ and $s \in \mathcal{Y}_q$, write $b \lessdot s$ for the bi-leveled tree with $p + q$ nodes whose underlying tree is formed by grafting the root of $s$ onto the rightmost leaf of $b$, but whose order ideal is that of $b$. Here is an example of $b$, $s$, and $b \lessdot s$.

Observe that we cannot have $b = 1$ in this construction.

The maximum bi-leveled tree with a given underlying tree $t$ is $\beta(\max(t))$, which has order ideal $T$ consisting only of the nodes of $t$ along its leftmost branch. Here are three such trees of the form $\beta(\max(t))$.

\begin{equation}
\beta(\max(t)) = \cdots
\end{equation}

Theorem 3.6. Given $b = (t; T) \in \mathcal{M}_*$, we have

\begin{equation}
\rho(M_b) = \begin{cases} 
\sum_{b = c \lessdot s} M_c \otimes M_s & \text{if } b \neq \beta(\max(t)) \\
\sum_{b = c \lessdot s} M_c \otimes M_s + 1 \otimes M_t & \text{if } b = \beta(\max(t))
\end{cases}
\end{equation}

For example,

\begin{align*}
\rho(M_{\mathcal{T}}) &= M_{\mathcal{T}} \otimes 1 \\
\rho(M_{\mathcal{Y}}) &= M_{\mathcal{Y}} \otimes 1 + M_{\mathcal{Y}} \otimes M_{\mathcal{Y}} \\
\rho(M_{\mathcal{Y} \mathcal{Y}}) &= M_{\mathcal{Y} \mathcal{Y}} \otimes 1 + M_{\mathcal{Y} \mathcal{Y}} \otimes M_{\mathcal{Y}} + M_{\mathcal{Y} \mathcal{Y}} \otimes M_{\mathcal{Y}} + 1 \otimes M_{\mathcal{Y} \mathcal{Y}}.
\end{align*}

Our proof of Theorem 3.6 uses Proposition 1.3 and the following results.

Lemma 3.7. For any bi-leveled tree $b \in \mathcal{M}_*$, we have

\begin{equation}
\beta \left( \sum_{\beta(w) = b} M_w \right) = M_b.
\end{equation}

Proof. Expand the left hand side in terms of the fundamental bases to get

\begin{equation}
\beta \left( \sum_{\beta(w) = b} \sum_{w \leq v} \mu_{\mathcal{S}}(w, v) F_v \right) = \sum_{\beta(w) = b} \sum_{w \leq v} \mu_{\mathcal{S}}(w, v) F_{\beta(v)}.
\end{equation}
As $\beta$ is surjective, we may change the index of summation to $b \leq c$ in $\mathcal{M}$, to obtain

$$
\sum_{b \leq c} \left( \sum_{\beta(w) = b} \mu_{\beta}(w, v) \right) F_c.
$$

By Theorems 1.4 and 2.4, the inner sum is $\mu_{\mathcal{M}}(b, c)$, so this sum is $M_b$. □

Recall that $w = u \setminus v$ only if $\tau(w) = \tau(u) \setminus \tau(v)$ and the values of $w$ in the nodes of $u$ exceed the values in the nodes of $v$. We always have the trivial decomposition $w = (\emptyset, w)$. Suppose that $w = u \setminus v$ with $u \neq \emptyset$ a nontrivial decomposition. If $\beta(w) = b = (t; T)$, then $T$ is a subset of the nodes of $u$ so that $\beta(u) = (\tau(u); S)$ and $b = \beta(u) \setminus \tau(v)$. Moreover, for every decomposition $b = c \setminus s$ and every $u, v$ with $\beta(u) = c$ and $\tau(v) = s$, we have $b = \beta(u \setminus v)$. Thus, for $b \in \mathcal{M}_-$, we have

$$
(3.7) \quad \bigsqcup_{\beta(w) = b} \bigsqcup_{w = u \setminus v, u \neq \emptyset} (u, v) = \bigsqcup_{b = c \setminus t} \bigsqcup_{\beta(u) = c, \tau(v) = t} (u, v). \quad \text{(17)}
$$

**Proof of Theorem 3.6.** Let $b = (t; T)$ with $t \neq 1$. Using Lemma 3.7, we have

$$
\rho(M_b) = \rho\beta \left( \sum_{\beta(w) = b} M_w \right) = \sum_{\beta(w) = b} \rho\beta M_w.
$$

By (3.5), (3.7), and (1.5), this equals

$$
\sum_{\beta(w) = b} \sum_{w = u \setminus v, u \neq \emptyset} \beta(M_u) \otimes \tau(M_v) + \sum_{\beta(w) = b} \beta(M_b) \otimes \tau(M_w)
$$

$$
= \sum_{b = c \setminus s} \left( \sum_{\beta(u) = c} \beta(M_u) \right) \otimes \left( \sum_{\tau(v) = s} \tau(M_t) \right) + \sum_{\beta(w) = b} 1 \otimes \tau(M_w).
$$

By Lemma 3.7 and (1.4), the first sum becomes $\sum_{b = c \setminus s} M_c \otimes M_s$ and the second sum vanishes unless $b = \beta(\text{max}(t))$. This completes the proof. □

### 4. Hopf Variations

#### 4.1. The $\mathcal{V} Sym$–Hopf module $\mathcal{M} Sym_+$. Let $\mathcal{M}_+ := (\mathcal{M}_n)_{n \geq 1}$ be the bi-leveled trees with at least one internal node and define $\mathcal{M} Sym_+$ to be the positively graded part of $\mathcal{M} Sym$, which has bases indexed by $\mathcal{M}_+$. A **restricted splitting** of $b \in \mathcal{M}_+$ is a splitting $b \xrightarrow{\chi_t} (b_0, \ldots, b_m)$ with $b_0 \in \mathcal{M}_+$, i.e., $b_0 \neq 1$. Given $b \xrightarrow{\chi_t} (b_0, \ldots, b_m)$ and $t \in \mathcal{V}_m$, form the bi-leveled tree $(b_0, \ldots, b_m)/t$ by grafting the ordered forest $(b_0, \ldots, b_m)$ onto the leaves of $t$, with order ideal consisting of the nodes of $t$ together with the nodes of the forest coming from the order ideal of $b$, as in (3.3)(ii).

We define an action and coaction of $\mathcal{V} Sym$ on $\mathcal{M} Sym_+$ that are similar to the product and coaction on $\mathcal{M} Sym$. They come from a second collection of polytope
maps $\mathcal{M}_n \to \mathcal{Y}_{n-1}$ arising from viewing the vertices of $\mathcal{M}_n$ as painted trees on $n-1$ nodes (see [5, 8]). For $b \in \mathcal{M}_+$ and $t \in \mathcal{Y}_m$, set

$$F_b \cdot F_t = \sum_{b \mathrel{\overset{\gamma}{\to}} (b_0, \ldots, b_m) \mathrel{\overset{\gamma}{\to}} t} F_{(b_0, \ldots, b_m)/t},$$

(4.1)

$$\rho_+(F_b) = \sum_{b \mathrel{\overset{\gamma}{\to}} (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}.$$

For example, in the fundamental bases of $\mathcal{M} \text{Sym}_+$ and $\mathcal{Y} \text{Sym}$, we have

$$F_{b_\gamma} \cdot F_{r_\gamma} = F_{b_\gamma} \gamma' + F_{r_\gamma} = F_{b_\gamma \gamma'},$$

$$\rho_+(F_{b_\gamma \gamma'}) = F_{b_\gamma \gamma'} \otimes 1 + F_{r_\gamma \gamma'} = F_{b_\gamma \gamma'} \otimes F_{r_\gamma \gamma'} + F_{r_\gamma \gamma'} \otimes F_{b_\gamma \gamma'}.$$

**Theorem 4.1.** The operations in (4.1) define a $\mathcal{Y} \text{Sym}$–Hopf module structure on $\mathcal{M} \text{Sym}_+$.

**Proof.** The unital and counital properties are immediate. We check only that the action is associative, the coaction is coassociative, and the two structures commute with each other.

**Associativity.** Fix $b = (t; T) \in \mathcal{M}_+$, $r \in \mathcal{Y}_m$, and $s \in \mathcal{Y}_n$. A term in the expression $(F_b \cdot F_r) \cdot F_s$ corresponds to a restricted splitting and grafting $b \overset{\gamma}{\to} (b_0, \ldots, b_m) \overset{\gamma}{\to} (b_0, \ldots, b_m)/r = c$, followed by another $c \overset{\gamma}{\to} (c_0, \ldots, c_n) \overset{\gamma}{\to} (c_0, \ldots, c_n)/t$. The order ideal for this term equals $T \cup \{\text{the nodes of } r \text{ and } s\}$. Note that restricted splittings of $c$ are in bijection with pairs of splittings

$$\left( b \overset{\gamma}{\to} (b_0, \ldots, b_{m+n}), r \overset{\gamma}{\to} (r_0, \ldots, r_n) \right).$$

Terms of $F_b \cdot (F_r \cdot F_s)$ also correspond to these pairs of splittings. The order ideal for this term is again $T \cup \{\text{the nodes of } r \text{ and } s\}$. That is, $(F_b \cdot F_r) \cdot F_s$ and $F_b \cdot (F_r \cdot F_s)$ agree term by term.

**Coassociativity.** Fix $b = (t; T) \in \mathcal{M}_+$. Terms $F_c \otimes F_r \otimes F_s$ in $\mathcal{M}_+$ and $(\mathbb{1} \otimes \Delta) \rho_+(F_b)$ both correspond to restricted splittings $b \overset{\gamma}{\to} (c, c_1, c_2)$, where $\phi(c_1) = r$ and $\phi(c_2) = s$. In either case, the order ideal on $c$ is $T|_c$.

**Commuting structures.** Fix $b = (s; S) \in \mathcal{M}_+$ and $t \in \mathcal{Y}_m$. A term $F_{c_0} \otimes F_{\phi(c_1)}$ in $\rho_+(F_b \cdot F_t)$ corresponds to a choice of a restricted splitting and grafting $b \overset{\gamma}{\to} (b_0, \ldots, b_m) \overset{\gamma}{\to} (b_0, \ldots, b_{t})/t = c$, followed by a restricted splitting $c \overset{\gamma}{\to} (c_0, c_1)$. The order ideal on $c_0$ equals the nodes of $c_0$ inherited from $S$, together with the nodes of $c_0$ inherited from $t$. The restricted splittings of $c$ are in bijection with pairs of splittings $\left( b \overset{\gamma}{\to} (b_0, \ldots, b_{m+n}), t \overset{\gamma}{\to} (t_0, t_1) \right)$. If $t_0 \in \mathcal{Y}_n$, then the pair of graftings $c_0 = (b_0, \ldots, b_{m+n})/t_0$ and $c_1 = (b_{n+1}, \ldots, b_m)/t_1$ are precisely the terms appearing in $\rho_+(F_b \cdot \Delta(F_t))$. The similarity of (4.1) to the coaction (3.4) of $\mathcal{Y} \text{Sym}$ on $\mathcal{M} \text{Sym}$ gives the following result, whose proof we leave to the reader.
Corollary 4.2. For $b \in \mathcal{M}_+$, we have
\[ \rho_+(M_b) = \sum_{s \in \mathcal{Y}'} M_{b \backslash s} \otimes M_s. \]

This elucidates the structure of $\mathcal{M}Sym_+$. Let $\mathcal{B} \subset \mathcal{M}_+$ be the indecomposable bi-leveled trees—those with only trivial decompositions, $b = b \backslash \emptyset$. Then $(t; T) \in \mathcal{B}$ if and only if $T$ contains the rightmost node of $t$. Every tree $c$ in $\mathcal{M}_+$ has a unique decomposition $c = b \backslash s$ where $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\ast$. Indeed, pruning $c$ immediately above the rightmost node in its order ideal gives a decomposition $c = b \backslash s$ where $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\ast$. This induces a bijection of graded sets,
\[ \mathcal{M}_+ \leftrightarrow \mathcal{B} \times \mathcal{Y}_\ast. \]
Moreover, if $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\ast$, then Corollary 4.2 and (1.6) together imply that
\[ \rho_+(M_b \backslash s) = \sum_{s \in \mathcal{Y}'} M_{b \backslash s} \otimes M_t. \]

Note that $\mathbb{Q}\mathcal{B} \otimes \mathcal{Y}Sym$ is a graded right $\mathcal{Y}Sym$-comodule with structure map,
\[ b \otimes M_s \mapsto b \otimes (\Delta M_s), \]
for $b \in \mathcal{B}$ and $s \in \mathcal{Y}_\ast$. Comparing this with (4.2), we deduce the following algebraic and combinatorial facts.

Corollary 4.3. The map $\mathbb{Q}\mathcal{B} \otimes \mathcal{Y}Sym \to \mathcal{M}Sym_+$ defined by $b \otimes M_s \mapsto M_b \backslash s$ is an isomorphism of graded right $\mathcal{Y}Sym$ comodules.

The quotient of enumerating series $\mathbf{M}(q)_+ / \mathbf{Y}(q)$ is equal to the enumerating series of the graded set $\mathcal{B}$.

In particular, if $\mathcal{B}_n := \mathcal{B} \cap \mathcal{M}_n$, then $|\mathcal{B}_n| = B_n$ by (2.5).

Remark 4.4. The coinvariants in a right comodule $M$ over a coalgebra $C$ are $M^\text{co} := \{ m \in M \mid \rho(m) = m \otimes 1 \}$. We identify the vector space $\mathbb{Q}\mathcal{B}$ with $\mathcal{M}Sym^\text{co}$ via $b \mapsto M_b$. The isomorphism $\mathbb{Q}\mathcal{B} \otimes \mathcal{Y}Sym 
\to \mathcal{M}Sym_+$ is a special case of the Fundamental Theorem of Hopf Modules [19, Theorem 1.9.4]: If $M$ is a Hopf module over a Hopf algebra $H$, then $M \simeq M^\text{co} \otimes H$ as Hopf modules.

4.2. Hopf module structure on $\mathcal{M}Sym$. We use Theorem 3.6 to identify the $\mathcal{Y}Sym$-coinvariants in $\mathcal{M}Sym$. Let $\mathcal{B}'$ be those indecomposable bi-leveled trees which are not of the form $\beta(\max(t))$, for some $t \in \mathcal{Y}_+$, together with $\{1\}$.

Corollary 4.5. The $\mathcal{Y}Sym$-coinvariants of $\mathcal{M}Sym$ have a basis $\{ M_b \mid b \in \mathcal{B}' \}$.

For $n > 0$, the difference $\mathcal{B}_n \setminus \mathcal{B}'_n$ consists of indecomposable bi-leveled trees with $n$ nodes of the form $\beta(\max(t))$. If $\beta(\max(t)) \in \mathcal{B}_n$, then $t = s \setminus 1$, for some $s \in \mathcal{Y}_{n-1}$, and so $|\mathcal{B}_n| = B_n - C_{n-1}$, which we saw in the proof of Theorem 2.7.

For $t \in \mathcal{Y}_1$, set $1 \setminus t := \beta(\max(t))$, and if $1 \neq b \in \mathcal{B}'$, set $b \setminus 1 := b \setminus t$. Every bi-leveled tree uniquely decomposes as $b \setminus t$ with $b \in \mathcal{B}'$ and $t \in \mathcal{Y}_1$. By Theorem 3.6, $M_b \otimes M_t \mapsto M_{b \setminus t}$ induces an isomorphism of right $\mathcal{Y}Sym$-comodules,

\[ \mathcal{M}Sym^\text{co} \otimes \mathcal{Y}Sym \to \mathcal{M}Sym, \]
where the structure map on $\mathcal{M} \text{Sym}^\infty \otimes \mathcal{Y} \text{Sym}$ is $M_b \otimes M_t \mapsto M_b \otimes \Delta(M_t)$. Treating $\mathcal{M} \text{Sym}^\infty$ as a trivial $\mathcal{Y} \text{Sym}$-module, $M_b \cdot M_t = \varepsilon(M_t)M_b$, $\mathcal{M} \text{Sym}^\infty \otimes \mathcal{Y} \text{Sym}$ becomes a right $\mathcal{Y} \text{Sym}$-module. As explained in [19, Example 1.9.3], this makes $\mathcal{M} \text{Sym}^\infty \otimes \mathcal{Y} \text{Sym}$ into a $\mathcal{Y} \text{Sym}$–Hopf module.

We express this structure on $\mathcal{M} \text{Sym}$. Let $b \smashprod t \in \mathcal{M}$, and $s \in \mathcal{Y}$, then
\begin{equation}
M_b \smashprod t \cdot M_s = \sum_{r \in t \smashprod s} M_{b \smashprod r} \quad \text{and} \quad \rho(M_b \smashprod t) = \sum_{t = r \smashprod s} M_{b \smashprod r} \otimes M_s,
\end{equation}
where $t \smashprod s$ is the set of trees $r$ indexing the product $M_t \cdot M_s$ in $\mathcal{Y} \text{Sym}$. The coaction is as before, but the product is new. It is not positive in the fundamental basis,
\begin{equation*}
F_\mathcal{Y} \cdot F_\mathcal{Y} = F_\mathcal{Y} - F_\mathcal{Y} + F_\mathcal{Y} + 2F_\mathcal{Y}.
\end{equation*}

We complete the proof of Theorem 2.7.

**Corollary 4.6.** The power series $S(q)/M(q)$ has nonnegative coefficients.

**Proof.** Observe that
\begin{equation*}
S(q)/M(q) = \left( S(q)/Y(q) \right) / \left( M(q)/Y(q) \right).
\end{equation*}
Since both $\mathcal{G} \text{Sym}$ and $\mathcal{M} \text{Sym}$ are right $\mathcal{Y} \text{Sym}$–Hopf modules, the two quotients of enumerating series on the right are generating series for their coinvariants, by the Fundamental Theorem of Hopf modules. Thus
\begin{equation*}
S(q)/M(q) = S^\infty(q)/M^\infty(q),
\end{equation*}
where $S^\infty(q)$ and $M^\infty(q)$ are the enumerating series for $\mathcal{G} \text{Sym}^\infty$ and $\mathcal{M} \text{Sym}^\infty$. To show that $S^\infty(q)/M^\infty(q)$ is nonnegative, we index bases for these spaces by graded sets $S$ and $B'$, then establish a bijection $B' \times S' \to S$ for some graded subset $S' \subset S$.

The set $B'$ was identified in Corollary 4.5. The coinvariants $\mathcal{G} \text{Sym}^\infty$ were given in [2, Theorem 7.2] as a left Hopf kernel. The basis was identified as follows. Recall that permutations $u \in \mathcal{G}$, may be written uniquely in terms of indecomposables, $u = u_1 \smashprod \cdots \smashprod u_r$ (taking $r = 0$ for $u = \emptyset$). Let $S \subset \mathcal{G}$, be those permutations $u$ whose rightmost indecomposable component has a 132-pattern, and thus $u \neq \max(t)$ for any $t \in \mathcal{Y}_+$. (Note that $u = \emptyset \in S$.) Then $\{M_u \mid u \in S\}$ is a basis for $\mathcal{G} \text{Sym}^\infty$.

Fix a section $g: \mathcal{M} \to \mathcal{G}$, of the map $\beta: \mathcal{G} \to \mathcal{M}$, and define a subset $S' \subset S$ as follows. Given the decomposition $u = u_1 \smashprod \cdots \smashprod u_r$ in (4.5) with $r \geq 0$, consider the length $\ell \geq 0$ of the maximum initial sequence $u_1 \smashprod \cdots \smashprod u_\ell$ of indecomposables belonging to $g(B')$. Put $u \in S'$ if $\ell$ is even. Define the map of graded sets
\begin{equation*}
\kappa: B' \times S' \longrightarrow S \quad \text{by} \quad (b, v) \longmapsto g(b) \smashprod v.
\end{equation*}

The image of $\kappa$ lies in $S$ as the last component of a nontrivial $g(b) \smashprod v$ is either $g(b)$ or the last component of $v$, neither of which can be $\max(t)$ for $t \in \mathcal{Y}_+$.

We claim that $\kappa$ is injective. If $u \in S'$, then $u = \kappa(1, u)$. If $u \in S \setminus S'$, then $u$ has an odd number of initial components from $g(B')$. Letting its first factor be $g(b)$,
we see that $u = g(b) \backslash u' = \kappa(b, u')$ with $u' \in S'$. This surjective map is injective as the expressions $\kappa(1, u')$ and $\kappa(b, u')$ with $b \in B'_+ \cup B''$ and $u' \in S'$ are unique.

This isomorphism of graded sets identifies the enumerating series of the graded set $S'$ as the quotient $S^{co}(q)/M^{co}(q)$, which completes the proof. □

References


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