Coideals Coming from Coactions
Aaron Lauve, June 01, 2005

Let \( K \) be a field and \((C, \Delta, \epsilon)\) be a coalgebra over \( K \). Let \((M, \rho)\) be a right \( C \)-comodule over \( K \). Let \( TC \) (resp. \( TM \)) be the tensor algebra on \( C \) (resp. \( M \)).

**Proposition 1.** The (unital) free algebras \( TC \) and \( TM \) satisfy:

1. \( TC \) is a bialgebra;
2. \( TM \) is a \( TC \)-comodule algebra.

**Proof.** For \( \Delta_{TC} \) and \( \epsilon_{TC} \), just take \( \Delta \) and \( \epsilon \) (and demand they be algebra maps). Similarly, let \( \rho_{TM} \) be the multiplicatively extended \( \rho \). Denote multiplication in \( TC \) and \( TM \) by juxtaposition. Denote the multiplication in \( TC \otimes TC \) and \( TM \otimes TC \) by “\( \bullet \)”. Using the usual definition of \( \bullet \), e.g. \((c \otimes d) \bullet (c' \otimes d') = cc' \otimes dd' \in TC \otimes TC \), it is easy to check that (i) and (ii) hold.

Now, fix an ideal \( A \in TM \) and build a basis \( \{ x_\alpha | \alpha \in A \} \) for \( A \) (here \( A \) is some index set, not nec. finite, not nec. countable, etc.). Next, extend \( \{ x_\alpha \} \) to a basis \( \{ x_\alpha \} \cup \{ w_\beta \} \) for \( TM \) (the \( \beta \) running over some index set \( B \)). Let us also fix how the elements \( x_\alpha \) and \( w_\beta \) diagonalize:

\[
\rho(x_\alpha) = \sum_{\alpha' \in A} x_{\alpha'} \otimes f_{\alpha'}(\alpha) + \sum_{\beta' \in B} w_{\beta'} \otimes f_{\beta'}(\alpha), \tag{1}
\]

\[
\rho(w_\beta) = \sum_{\alpha' \in A} x_{\alpha'} \otimes g_{\alpha'}(\beta) + \sum_{\beta' \in B} w_{\beta'} \otimes g_{\beta'}(\beta). \tag{2}
\]

Here the \( f \)'s and \( g \)'s are some elements of \( TC \). Let \( B \) be the two-sided ideal in \( TC \) generated by \( \{ f_{\beta}(\alpha) | \alpha \in A, \beta \in B \} \).

**Proposition 2.** \( TM/A \) is a \( TC/B \) comodule algebra, and \( B \) is the unique minimal ideal of \( TC \) with this property.

**Remark.** If we are given generators \( x_\alpha \) for \( A \), then it is sufficient to take as generators for \( B \) those \( f \)'s coming from diagonalizing the \( x_\alpha \)'s.

The key to the proof is understanding how the comodule structure of \( TM \) interacts with the diagonalization of \( B \). For \( x_\alpha \in A \), let us compare \((id \otimes \Delta)\rho(x_\alpha)\) and \((\rho \otimes id)\rho(x_\alpha)\ldots\)
\[(\text{id} \otimes \Delta)\rho(x_\alpha) = \sum_{\alpha' \in A} x_{\alpha'} \otimes \Delta f_{\alpha'}(\alpha) + \sum_{\beta' \in B} w_{\beta'} \otimes \Delta f_{\beta'}(\alpha)\]

(relabeling) \[= \sum_{\alpha''} x_{\alpha''} \otimes \left(\Delta f_{\alpha''}(\alpha)\right) +\]
\[\sum_{\beta''} w_{\beta''} \otimes \left(\Delta f_{\beta''}(\alpha)\right),\] (3)

while
\[(\rho \otimes \text{id})\rho(x_\alpha) = \sum_{\alpha'} \left(\sum_{\alpha''} x_{\alpha'} \otimes f_{\alpha''}(\alpha') + \sum_{\beta''} w_{\beta''} \otimes f_{\beta''}(\alpha')\right) \otimes f_{\alpha'}(\alpha) +\]
\[\sum_{\beta'} \left(\sum_{\alpha''} x_{\alpha''} \otimes g_{\alpha''}(\beta') + \sum_{\beta''} w_{\beta''} \otimes g_{\beta''}(\beta')\right) \otimes f_{\beta'}(\alpha)\]
\[= \sum_{\alpha''} x_{\alpha''} \otimes \left(\sum_{\alpha'} f_{\alpha''}(\alpha') \otimes f_{\alpha'}(\alpha) + \sum_{\beta'} g_{\alpha''}(\beta') \otimes f_{\beta'}(\alpha)\right) +\]
\[\sum_{\beta''} w_{\beta''} \otimes \left(\sum_{\alpha'} f_{\beta''}(\alpha') \otimes f_{\alpha'}(\alpha) + \sum_{\beta'} g_{\beta''}(\beta') \otimes f_{\beta'}(\alpha)\right).\] (4)

Proof of Proposition. Most of the work has been done. We proceed in steps.

(Coideal): The calculation above shows that \(\Delta(B) \subseteq B \otimes TC + TC \otimes B\) (compare expressions (3) and (4) and use the linear independence of \(\{x_\alpha, w_{\beta'}\}\)). Next, again using the linear independence of \(\{x_\alpha, w_{\beta'}\}\), one writes out the identity
\[(\text{id} \otimes \epsilon)\rho(x_\alpha) = x_\alpha\]
in \(M \otimes K\) explicitly (cf. equation (1)) and deduces that \((\forall \alpha, \beta') \Delta(f_{\beta'}(\alpha)) = 0\).

(Comodule algebra): We need \(\rho\) to be well-defined on the quotient \(TM/A\); in particular, \(\rho(x_\alpha)\) must be zero. From equation (1) and the linear independence of \(\{x_\alpha, w_{\beta'}\}\), we see that \(f_{\beta'}(\alpha)\) must be zero for all \(\alpha, \beta'\). No problem, these were our generators for \(B\) and hence they are zero in \(TC/B\).

(Minimality): If \(B' \subsetneq B\) is a smaller ideal, then there exists an \(f_{\beta'}(\alpha)\) which belongs to \(B \setminus B'\) and hence, which is not zero in the quotient \(TC/B'\). Clearly this will fail to make \(\rho\) well-defined.

(Uniqueness): If \(B_1\) and \(B_2\) are two minimal (co)ideals, then it isn’t difficult to show that \(B_1 \cap B_2\) is a coideal and moreover causes \(\rho\) to be well-defined. By the minimality of either of the \(B_i\) this is a contradiction.