QUASIDETERMINANTS AND $q$-COMMUTING MINORS

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Abstract. We present two new proofs of the $q$-commuting property holding among certain pairs of quantum minors of an $n \times n$ $q$-generic matrix. The first uses elementary quasideterminantal arithmetic; the second involves paths in an edge-weighted directed graph. Together, they indicate a means to build the multihomogeneous coordinate rings of flag varieties in other noncommutative settings.

1. Introduction & Main Theorem

This paper arose from an attempt to understand the “quantum shape algebra” of Taft and Towber [Taft and Towber, 1991], which we call the quantum flag algebra $\mathcal{F}_q^\ell(n)$ here. One goal was to find quasideterminantal justifications for the relations chosen for $\mathcal{F}_q^\ell(n)$. A second goal was to find some hidden relations, within $\mathcal{F}_q^\ell(n)$, known to hold in an isomorphic image. To more quickly reach a statement of the theorem, we save further remarks on the goals for later.

Definition 1. Given two subsets $I, J \subseteq [n]$, we say $J$ surrounds $I$, written $J \rightarrow I$, if (i) $|J| \leq |I|$, and (ii) there exist disjoint subsets $\emptyset \subseteq J', J'' \subseteq J$ such that:

a. $J \setminus I = J' \cup J''$,

b. $j' < i$ for all $j' \in J'$ and $i \in I \setminus J$,

c. $i < j''$ for all $i \in I \setminus J$ and $j'' \in J'$,

In this case, we put $\langle \langle J, I \rangle \rangle = |J''| - |J'|$.

Given an $n \times n$ $q$-generic matrix $X$ and a subset $I \subseteq [n]$ with $|I| = d$, we write $[I]$ for the quantum minor built from $X$ by taking row-set $I$ and column-set $[d]$.

Theorem 1 ($q$-Commuting Minors). If the subsets $I, J \subseteq [n]$ satisfy $J \rightarrow I$, the quantum minors $[J]$ and $[I]$ $q$-commute. Specifically,

\begin{equation}
[J][I] = q^{\langle \langle J, I \rangle \rangle}[I][J].
\end{equation}

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1In the literature, sets $J$ and $I$ sharing this relationship are called “weakly separated.” We avoid this terminology because it does not indicate who separates whom.
A proof in the case \( I \cap J = \emptyset \) may be found in [Krob and Leclerc, 1995], while Leclerc and Zelevinsky [Leclerc and Zelevinsky, 1998] show that \([J][I] = q^\alpha [I][J]\) for some \( \alpha \in \mathbb{Z} \) if and only if \( J \sim I \). By now, many more commutation formulas are known for much larger collections of quantum minors, cf. [Fioresi, 1999, Goodearl, 2006]. The impetus for finding such results has been two-fold: (i) from the point of view of representation theory, such questions are intimately tied to the study of the canonical (or crystal) bases of Lustzig and Kashiwara, cf. [Berenstein and Zelevinsky, 1993, Hong and Kang, 2002, Scott, 2005]; (ii) from the point of view of noncommutative algebraic geometry, the study of quantum determinantal ideals provides noncommutative versions of the classical determinantal varieties, cf. [Goodearl and Lenagan, 2000, Kelly et al., 2004]. Our goal is different.

Given a noncommutative algebra \( \mathcal{A} \) with a ‘quantum’ determinant \( D \), can we readily define an \( \mathcal{A} \)-analog of \( \mathcal{F}_{\ell_q}(n) \) by specializing quasideterminantal identities to the pair \((\mathcal{A}, D)\)? Toward this goal, we analyze the gold standard, \( \mathcal{F}_{\ell_q}(n) \), from a quasideterminantal point of view. This idea leads to two new proofs of Theorem 1. The first proof \((Q)\) uses simple arithmetic involving quasideterminants; the second \((G)\) involves counting weighted paths on a directed graph. Taken together, they imply that if \((\mathcal{A}, D)\) satisfies some version of Theorem 5, then quasi-Plücker relations indicate how to define the flag algebra for \( \mathcal{A} \).

1.1. Useful notation. The reader has already encountered our notation \([n]\) for the set \(\{1, 2, \ldots, n\}\); let \(\binom{[n]}{d}\) denote the set of all subsets of \([n]\) of size \(d\). Given a set \(I = \{i_1 < i_2 < \cdots < i_d\} \in \binom{[n]}{d}\) and any \(I' \subseteq I\), we write \(I'\) for the subset built from \(I\) by deleting \(I'\) (i.e. \(I \setminus I'\)) and \(I''\) for the complement (i.e. a fancy way of saying keep \(I'\)).

We would like to extend this delete/keep notation to \([n]^d\), the set of all \(d\)-tuples chosen from \([n]\), but the notations \(I''\) and \(I'\) are not well-defined (the entries of \(I'\) may occur in more than one place within \(I\)). Restricting our attention to \(I \in [n]^d\) (those \(d\)-tuples with distinct entries), we interpret \(I''\) and \(I'\) in the obvious way. If \(I, J\) are two sets or tuples of sizes \(d, e\) respectively, we define \(A|B\) to be the \((d+e)\)-tuple \((i_1, \ldots, i_d,j_1,\ldots,j_e)\). For \(I \in [n]^d\), we define the length of \(I\) to be \(\ell(I) = \#\text{inv}(I) = \#\{ (j, k) : j < k \text{ and } i_j > i_k \}\). Fix \(i \in [n]\) and \(I = i_1, i_2, \ldots, i_d\) (viewed either as a set or a \(d\)-tuple without repetition); if there is a \(1 \leq k \leq d\) with \(i_k = i\), then \(k\) is the position of \(i\) and we write \(\text{pos}_I(i) = k\).

We extend our delete/keep notation to matrices. Let \(A\) be an \(n \times n\) matrix whose rows and columns are indexed by \(R\) and \(C\), respectively. For any \(R' \subseteq R\) and \(C' \subseteq C\), we let \(A_{R',C'}\) denote the submatrix built from \(A\) by deleting row-indices \(R'\) and column-indices \(C'\). Let \(A_{R',C'}\) be the complementary
submatrix. In case \( R' = \{ r \} \) and \( C' = \{ c \} \), we may abuse notation and write, e.g., \( A^{rc} \). We will also need a means to construct matrices from \( A \) whose rows (columns) are repeated or are not in their natural order. If \( I \in R^d \) and \( J \in C^e \), let \( A_{I,J} \) denote the obvious new matrix built from \( A \).

2. Preliminaries for \( Q \)-Proof

2.1. Quasideterminants. The quasideterminant [Gelfand and Retakh, 1991, Gelfand et al., 2005] was introduced by Gelfand and Retakh as a replacement for the determinant over noncommutative rings \( R \). Given an \( n \times n \) matrix \( A = (a_{ij}) \) over \( R \), the quasideterminant \( |A|_{ij} \) (there is one for each position \((i, j)\) in the matrix) is not a polynomial in the entries \( a_{ij} \) but rather a rational expression, as we will soon see. Consequently, quasideterminants are not always defined. Below is a sufficient condition (cf. loc. cit. for more details).

**Definition 2.** Given \( A \) and \( R \) as above, if \( A_{ij} \) is invertible over \( R \), then the \((ij)\)-quasideterminant is defined and given by

\[
|A|_{ij} = a_{ij} - \rho_i \cdot (A^{ij})^{-1} \cdot \chi_j,
\]

where \( \rho_i \) is the \( i \)-th row of \( A \) with column \( j \) deleted and \( \chi_j \) is the \( j \)-th column of \( A \) with row \( i \) deleted.

**Remark 1.** One deduces that \( |A|_{ij}^{-1} = (A^{-1})_{ji} \) when both sides are defined.

Details on this remark and the following three theorems may be found in [Gelfand et al., 2005, Krob and Leclerc, 1995, Lauve, 2006]. Note that the phrase ‘when defined’ is implicit throughout.

**Theorem 2** (Homological Relations). Let \( A \) be a square matrix and let \( i \neq j \) \((k \neq l)\) be two row (column) indices. We have

\[
-|A^{jk}|_{il}^{-1} \cdot |A|_{ik} = |A^{ik}|_{jl}^{-1} \cdot |A|_{jk}.
\]

**Theorem 3** (Muir’s Law of Extensible Minors). Let \( A = A_{R,C} \) be a square matrix with row (column) indices \( R \) \((C)\). Fix \( R_0 \subseteq R \) and \( C_0 \subseteq C \). Say an algebraic, rational expression \( I = I(A, R_0, C_0) \) involving the quasi-minors \( \{ |A_{R',C'}|_{rc} : r \in R' \subseteq R_0, c \in C' \subseteq C_0 \} \) is an identity if the equation \( I = 0 \) is valid. For any \( L \subseteq R \setminus R_0 \) and \( M \subseteq C \setminus C_0 \), the expression \( I' \) built from \( I \) by extending all minors \( |A_{R',C'}|_{rc} \) to \( |A_{L \cup R', M \cup C'}|_{rc} \) is also an identity.

**Definition 3.** Let \( B \) be an \( n \times d \) matrix. For any \( i, j, k \in [n] \) and \( M \subseteq [n] \setminus \{i\} \) \((|M| = d - 1)\), define \( r^M_{ji} = r^M_{ij}(B) := |B_{(j|M), [d]}|_{jk} |B_{(i|M), [d]}|_{ik}^{-1} \). Gelfand and Retakh [Gelfand and Retakh, 1997] show this ratio is independent of \( k \), and call it a right-quasi-Plücker coordinate for \( B \).
Remark 2. In case \( B \) is \( n \times m \) for some \( m > d \), we choose the first \( d \) columns of \( B \) to form the above ratio unless otherwise indicated.

**Theorem 4** (Quasi-Plücker Relations). Fix an \( n \times n \) matrix \( A \), subsets \( M, L \subseteq [n] \) with \( |M| + 1 \leq |L| \), and \( i \in [n] \setminus M \). We have the quasi-Plücker relation \((P_{L,M,i})\)

\[
1 = \sum_{j \in L} r_{ij}^{L \setminus j} r_{ji}^M.
\]

2.2. Quantum determinants. An \( n \times n \) matrix \( X = (x_{ab}) \) is said to be \( q \)-generic if its entries satisfy the relations

\[
\begin{align*}
(\forall i, \forall k < l) x_{il} x_{ik} &= q x_{ik} x_{il} \\
(\forall i < j, \forall k) x_{jk} x_{ik} &= q x_{ik} x_{jk} \\
(\forall i < j, \forall k < l) x_{jk} x_{il} &= x_{il} x_{jk} \\
(\forall i < j, \forall k < l) x_{jl} x_{ik} &= x_{ik} x_{jl} + (q - q^{-1}) x_{il} x_{jk}.
\end{align*}
\]

Notice that every submatrix of a \( q \)-generic matrix is again \( q \)-generic.

Fix a field \( k \) of characteristic 0 and a distinguished invertible element \( q \in k \) not equal to a root of unity. Let \( M_q(n) \) be the \( k \)-algebra with \( n^2 \) generators \( x_{ab} \) subject to the relations making \( X \) a \( q \)-generic matrix. It is known [Kelly et al., 2004] that \( M_q(n) \) is a (left) Ore domain with (left) field of fractions \( D_q(n) \).

**Definition 4.** Given any \( d \times d \) matrix \( A \), define \( \det_q A \) by

\[
\det_q A = \sum_{\sigma \in S_d} (-q)^{-\ell(\sigma)} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(d),d}.
\]

When \( A = X_{R,C} \) is a submatrix of \( X \), we have: (i) this quantity agrees with the analogous quantity modeled after the column-permutation definition of the determinant, (ii) swapping two adjacent rows of \( A \) introduces a \( q^{-1} \), and (iii) allowing any row of \( A \) to appear twice yields zero. Properties (ii) and (iii) allow us to uniquely define the determinant of \( A = X_{I,C} \) for any \( I \in [n]^d \) and \( C \in \binom{[n]}{d} \). In case \( C = \{1, 2, \ldots, d\} \), we introduce the shorthand notation \( \det_q A = [I] \). We will also need the case \( C = s + [d] := \{s + 1, s + 2, \ldots, s + d\} \) for some \( s > 0 \), which we write as \([I; s]\).

Properties (i)–(iii) give us the important

**Theorem 5** (Quantum Determinantal Identities). Let \( A = X_{R,C} \) be a \( d \times d \) submatrix of \( X \). Then for all \( i, j \in R \) and \( k \in C \), we have:

\[
\sum_{c \in C} A_{jc} \cdot \left\{ (-q)^{\text{pos}_j(i) - \text{pos}_c(c)} \det_q A^{ic} \right\} = \delta_{ij} \cdot \det_q A
\]
In particular every submatrix of $X$ is invertible in $D_q(n)$ and (after Remark 1) we are free to use the preceding quasideterminantal formulas on matrices built from $X$. The important formula follows: for all $I \in [n]^d$

\begin{equation}
|X_{I,(s+1,\ldots,s+d)}|_{i,s+d} = (-q)^{d-\text{pos}_I(i)}[I;s] \cdot [I^i;s]^{-1},
\end{equation}

where the factors on the right commute. Theorems 2 and 5 are combined with (2) in [Lauve, 2006] to prove

**Theorem 6.** Given any $i, j \in [n]$, $\{j\} \prec \{i\}$. For any $M \subseteq [n]$, the quantum minors $[j|M]$ and $[i|M]$ $q$-commute according to equation (1).

### 3. $Q$-Proof of Theorem

Our first proof of Theorem 1 proceeds by induction on $|J|$ and rests on two key lemmas.

**Lemma 1.** If $I \subseteq [n]$ and $j \in [n] \setminus I$ satisfy $\{j\} \prec I$. Then $[j][I] = q^{\langle \langle j,I \rangle \rangle} [I][j]$.

**Proof.** From $(\mathcal{P}_{I,\emptyset,j})$ and (2) we have

$$1 = \sum_{i \in I} [j|I \setminus i][i|I \setminus i]^{-1} [i][j]^{-1},$$

or

\begin{equation}
[j] = \sum_{i \in I} [j|I^i][i|I^i]^{-1}[i].
\end{equation}

Theorem 6 tells us that $[j|I^i]$ and $[i|I^i]$ $q$-commute, so we may clear the denominator in (3) on the left and get

\begin{equation}
[I][j] = \sum_{i \in I} (-q)^{\ell(i|I^i)} q^{-\langle \langle j,i \rangle \rangle} [j|I^i][i].
\end{equation}

In the other direction, Theorem 5 tells us that $[i|I^i]$ and $[i]$ commute; clearing (3) on the right yields

\begin{equation}
[j][I] = \sum_{i \in I} (-q)^{\ell(i|I^i)} [j|I^i][i].
\end{equation}

Compare (4) and (5) to conclude that $[j]$ and $[I]$ $q$-commute as desired. $\square$

**Lemma 2.** Fix $J, I \subseteq [n]$ satisfying $J \prec I$. For all $M \subseteq [n] \setminus (I \cup J)$, one has $J \cup M \prec I \cup M$ and $[J \cup M][I \cup M] = q^{\langle \langle J,I \rangle \rangle}[I \cup M][J \cup M]$.

**Proof.** The first statement is clear from the definition of ‘surrounds.’ The second statement is a consequence of Muir’s Law and (2). $\square$

We are now ready for the first advertised proof of Theorem 1.
Proof of Theorem. Given \( J, I \subseteq [n] \) with \( d = |J| \leq |I| = e \), put \( s = |J \cap I| \). After Lemma 2, we may assume \( s = 0 \). We proceed by induction on \( d \), the base case being handled in Lemma 1.

Let \( j \) be the least element of \( J \), i.e. \( \ell(j|J) = 0 \), and consider \((P_{I,J,j,J})\):

\[
1 = \sum_{i \in I} r_{ji} r_{iJ}^{-1} r_{ij} r_{Ji}^{-1}.
\]

In terms of quantum determinants, we have

\[
[j|J] = \sum_{i \in I} [j|I] [i|I]^{-1} [i|J].
\]

By induction, we may clear the denominator to the right and get

\[
(j|J)[I] = q^{[j|J]} \sum_{i \in I} (-q)^{\ell(i|I)} [j|I] [i|J].
\]

On the other hand, we may clear the denominator on the left at the expense of \( q^{-\langle \langle j,i \rangle \rangle} \):

\[
[I][j|J] = q^{-\langle \langle j,i \rangle \rangle} \sum_{i \in I} (-q)^{\ell(i|I)} [j|I] [i|J].
\]

We are nearly done. First observe the following three facts.

\[
q^{[j|J]} = q^{\langle \langle j,i \rangle \rangle} \quad q^{-\langle \langle j,i \rangle \rangle} = q^{\langle \langle j,i \rangle \rangle} \quad q^{\langle \langle j,i \rangle \rangle} = q^{\langle \langle j,i \rangle \rangle} q^{\langle \langle j,i \rangle \rangle}
\]

Using these observations to compare (6) and (7) finishes the proof. \( \square \)

4. Preliminaries for \( \mathcal{G} \)-Proof


Definition 5 (Quantum Flag Algebra). The quantum flag algebra \( \mathcal{F}_{\ell}(n) \) is the \( k \)-algebra generated by symbols \( \{ f_I : I \in [n]^d, 1 \leq d \leq n \} \) subject to the relations indicated below.

- **Alternating relations** \((A_I)\): For any \( I \in [n]^d \) and \( \sigma \in \mathfrak{S}_d \),

\[
f_I = \begin{cases} 
0 & \text{if } I \text{ contains repeated indices} \\
(-q)^{-\ell(\sigma)} f_{\sigma I} & \text{if } \sigma I = (i_1 < i_2 < \cdots < i_d)
\end{cases}
\]

- **Young symmetry relations** \((Y_{l,j}(a))\): Fix \( 1 \leq a \leq d \leq e \leq n - a \). For any \( I \in \binom{[n]}{d+a} \) and \( J \in [n]^{d-a} \),

\[
0 = \sum_{\Lambda \subseteq I, |\Lambda| = a} (-q)^{-\ell(I \setminus \Lambda|\Lambda)} f_{I \setminus \Lambda} f_{\Lambda|J}
\]
Monomial straightening relations ($M_{J,I}$): For any $J, I \subseteq [n]$ with $|J| \leq |I|$, 

$$f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda| = |J|} (-q)^{\ell(\Lambda \setminus \Lambda)} f_{J \setminus \Lambda} f_{\Lambda}$$

Remark 3. Technically, we should have taken $I, J$ to be tuples instead of sets in (9) and (10). Identify, e.g. $I = \{i_1 < i_2 < \cdots < i_d\}$ with $(i_1, i_2, \ldots, i_d)$. This abuse of notation will reoccur without further ado.

In their article, Taft and Towber construct an algebra map $\phi : \mathcal{F}_q(n) \to M_q(n)$ taking $f_I$ to $[I]$ and show that $\phi$ is monic, with image the subalgebra of $M_q(n)$ generated by the quantum minors $\{[I] : I \in \mathcal{P}[n], 1 \leq d \leq n\}$.

We have already seen that the minors $[I]$ often $q$-commute. This relation does not appear above, and so must be a consequence of relations (8)–(10). Abbreviate the right-hand side of (9) by $Y_{I,J; (a)}$. Also, we abbreviate the difference ($\text{lhs} - \text{rhs}$) in (10) by $M_{J,I}$, and the difference ($\text{lhs} - \text{rhs}$) in (1) by $C_{J,I}$ (replacing $\llbracket \cdot \rrbracket$ by $f$). As (1),(9),(10) are all homogeneous, a likely guess is that $C_{J,I}$ is some $k$-linear combination of a certain number of expressions $M_{K,L}$ and $Y_{M,N; (a)}$ (modulo the alternating relations). As illustrated in the example below, this simple guess works.

Example ($\{1\} \prec \{2, 3, 4\}$). We calculate the expressions $C_{1,234}$, $M_{1,234}$, and $Y_{1234; \emptyset (1)}$ and arrange them as rows in Table 1. Viewing the table column by column, deduce $C_{1,234} = M_{1,234} + q^2 Y_{1234; \emptyset (1)}$.

<table>
<thead>
<tr>
<th>$C_{1,234}$</th>
<th>$f_1 f_{234}$</th>
<th>$-q^{-1} f_{234} f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{1,234}$</td>
<td>$f_1 f_{234}$</td>
<td>$-q^2 f_{123} f_4 + q^1 f_{124} f_3 - q^0 f_{134} f_2$</td>
</tr>
<tr>
<td>$Y_{1234; \emptyset (1)}$</td>
<td>$f_{123} f_4 - q^{-1} f_{124} f_3 + q^{-2} f_{134} f_2 - q^{-3} f_{234} f_1$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Finding the relation $f_1 f_{234} - q^{-1} f_{234} f_1 = 0$.

While the proof idea will be simple (“perform Gaussian elimination”), the proof itself is not. We separate out the more interesting steps below.

4.2. POset paths. Given a set $X$, the elements of the power set $\mathcal{P}X$ have a partial ordering: for $A, B \in \mathcal{P}X$, we say $A < B$ if $A \subseteq B$. We are interested in the case $X \subseteq [n]$ and we think of this POset as an edge-weighted, directed graph as follows.

Definition 6. Given $I, J \subseteq [n]$ such that $J \prec I$, the graph $\Gamma(J; I)$ has vertex set $\mathcal{V} = \mathcal{P}J$ and edge set $\mathcal{E} = \{(A, B) : A, B \in \mathcal{V}, A \subseteq B\}$. Each
edge \((A, B)\) of \(\Gamma\) has a weight \(\alpha^B_A\) given by the function \(\alpha : \mathcal{E} \to \mathbb{R}\),

\[
\forall (A, B) \in \mathcal{E} : \quad \alpha^B_A = (-q)^{-\ell(J \setminus B \setminus A) - \ell(B \setminus A | A) + (2|J \setminus B| - |I|)} \min((B \setminus A) \cap J')
\]

for \(J'\) as in Definition 1.

**Example.** If \(|J| = m\), then \(\Gamma(J)\) has \(2^m\) vertices and \(\sum_{k=1}^{m} \binom{m}{k} (2^m - 1)\) edges. In Figure 1, we give an illustration of \(\Gamma(\{1, 5, 6\})\), omitting two edges and many edge weights for legibility.

![Graph](image)

**Figure 1.** The graph \(\Gamma(\{1, 5, 6\})\) (partially rendered).

For the remainder of the subsection, we assume \(J \cap I = \emptyset\). Write \(J = J' \cup J'' = \{j_1 < \ldots < j_r' \} \cup \{j_{r'+1} < \ldots < j_{r''}\}\); also, put \(|J| = r' + r'' = r\), \(|I| = s\), and \(s - r = t\).

In the graph \(\Gamma(J; I)\), we consider paths and path weights defined as follows:

\[
\mathfrak{P}_0 = \left\{(A_1, A_2, \ldots, A_p) \mid A_i \subseteq J \text{ s.t. } \emptyset \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_p \subseteq J\right\}
\]

and \(\mathfrak{P} = \mathfrak{P}_0 \cup \hat{0} \cup \hat{1}\), where \(\hat{0} = (\emptyset)\), and

\[
\hat{1} = \left(\{j_{r'+1}, j_{r'+2}, \ldots, j_{r''}\}, \{j_{r'+1}, j_{r'+2}, \ldots, j_{r''}\}, \ldots, \{j_2, j_3, \ldots, j_r\}, J\right).
\]

The weight \(\alpha(\pi)\) of a path \(\pi = (A_1, \ldots, A_p) \in \mathfrak{P}_0\) is the product of edge weights of the augmented path \((\emptyset, \pi, J)\):

\[
\alpha^{A_1}_{\emptyset} \cdot \alpha^{A_2}_{A_1} \cdots \alpha^{A_p}_{A_{p-1}} \cdot \alpha^J_{A_p}.
\]

We extend the definition of \(\alpha\) to all of \(\mathfrak{P}\) as follows. Notice that if \(B = A\) in (11), we get \(\alpha^A_A = 1\). With this broader definition of the weight function \(\alpha\), we may define \(\alpha(\pi) = \alpha(\emptyset, \pi, J)\) for \(\pi = \hat{0}, \hat{1}\) as well.

**Definition 7.** Given a subset \(K \subseteq J\), define \(mM(K)\) as follows. If \(K \cap J' \neq \emptyset\), put \(mM(K) = \min(K \cap J')\). Otherwise, put \(mM(K) = \max(K \cap J'')\).
For any path \( \pi = (A_1, \ldots, A_p) \), put \( A_0 = \emptyset \) and \( A_{p+1} = J \). Notice that \( \hat{1} \) has the property that \( A_k \setminus A_{k-1} \neq mM(A_{k+1} \setminus A_{k-1}) \) for all \( 1 \leq k < r \), but \( A_r = mM(A_{r+1} \setminus A_{r-1}) \).

**Definition 8.** Fix a length \( 1 \leq p \leq r - 1 \). A path \( (A_1, \ldots, A_p) \in \mathfrak{P}_0 \) shall be called *regular* (or *regular at position* \( i_0 \)), if

(a) \( |A_i| = i (\forall 1 \leq i \leq i_0) \);
(b) \( A_{i_0} \setminus A_{i_0-1} = mM(A_{i_0+1} \setminus A_{i_0-1}) \) (again, taking \( A_0 = \emptyset \) and \( A_{p+1} = J \) if necessary). A sequence is called *irregular* if it is nowhere regular. Extend the notion of regularity to \( \mathfrak{P} \) by calling \( \hat{0} \) irregular and \( \hat{1} \) regular.

**Remark 4.** The set \( \mathfrak{P} \) is the disjoint union of its regular and irregular paths. We point out this tautology only to emphasize its importance in the coming proposition. Write \( \mathfrak{P}' \) for the irregular paths, and \( \mathfrak{P}'' \) for the regular paths.

**Proposition 7.** The subsets \( \mathfrak{P}' \) and \( \mathfrak{P}'' \) of \( \mathfrak{P} \) are equinumerous.

We will build a bijective map \( \varphi \) between the two sets. Given an irregular path \( \pi = (A_1, \ldots, A_p) \in \mathfrak{P}_0 \), we insert a new set \( B \) so that \( \varphi(\pi) \) is regular at \( B \):

1. Find the unique \( i_0 \) satisfying: \((|A_i| = i (\forall 1 \leq i \leq i_0) \land (|A_{i_0+1}| > i_0 + 1))\).
2. Compute \( b = mM(A_{i_0+1} \setminus A_{i_0}) \).
3. Put \( B = A_{i_0} \cup \{b\} \).
4. Define \( \varphi(\pi) := (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A_p) \).

For the remaining irregular path \( \hat{0} \), we put \( \varphi(\hat{0}) = (\{j_1\}) \), which agrees with the general definition of \( \varphi \) if we think of \( \hat{0} \) as the empty path (\( () \) instead of the path consisting of the empty set.

**Example.** Table 2 illustrates the action of \( \varphi \) on \( \mathfrak{P} \) when \( J = \{1, 5, 6\} \).

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>( \hat{0} )</th>
<th>(5)</th>
<th>(6)</th>
<th>(15)</th>
<th>(16)</th>
<th>(56)</th>
<th>(5, 56)</th>
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<tbody>
<tr>
<td>( \varphi(\pi) )</td>
<td>(1)</td>
<td>(5, 15)</td>
<td>(6, 16)</td>
<td>(1, 15)</td>
<td>(1, 16)</td>
<td>(6, 56)</td>
<td>( \hat{1} )</td>
</tr>
</tbody>
</table>

**Table 2.** The pairing of \( \mathfrak{P}' \) and \( \mathfrak{P}'' \) via \( \varphi \).

**Proof of Proposition.** We reach a proof in three steps.

**Claim 1:** \( \varphi(\mathfrak{P}') \subseteq \mathfrak{P}'' \).

Take a path \( \pi \in \mathfrak{P}' \) (i.e. a path with no regular points). The effect of \( \varphi \) is to insert a regular point at position \( i_0 + 1 \) (the spot where \( B \) sits), so the claim is proven if we can show \( \varphi(\pi) \in \mathfrak{P} \).

As \( \varphi(\hat{0}) \) clearly belongs to \( \mathfrak{P} \), we may focus on those \( \pi \in \mathfrak{P}_0 \). Also, it is plain to see that \( \pi^1 \) is irregular, and \( \varphi(\pi^1) = \hat{1} \). If \( \varphi \) is to be a bijection, we are left with the task of showing that \( \varphi(\mathfrak{P}' \cap \mathfrak{P}_0 \setminus \pi^1) \subseteq \mathfrak{P}_0 \).
When \( |A_p| < r - 1 \), any \( B \) that is inserted will result in another path in \( \mathfrak{P}_0 \) (because \( |B| \) must be less than \( r \)). When \( |A_p| = r - 1 \), there is some concern that we will have to insert a \( B \) at the end of the path, resulting in \( J \) being the new terminal vertex—disallowed in \( \mathfrak{P}_0 \). This cannot happen:

Case \( p < r - 1 \): At some point \( 1 \leq i_0 < p \), there is a jump in set-size greater than one when moving from \( A_{i_0} \) to \( A_{i_0+1} \). Hence, the \( B \) to be inserted will not come at the end, but rather immediately after \( A_{i_0} \) to \( A_{i_0+1} \).

Case \( p = r - 1 \): The only path \( (A_1,A_2,\ldots,A_{r-1}) \in \mathfrak{P}_0 \) which is nowhere regular is the path \( \hat{\alpha}1 \).

Claim 2: \( \varphi \) is 1-1.

Suppose \( \varphi(A_1,\ldots,A_p) = \varphi(A'_1,\ldots,A'_{p'}) \), and suppose we insert \( B \) and \( B' \) respectively. By the nature of \( \varphi \), we have \( p = p' \) and \( i_0 \neq i'_0 \). Take \( i_0 < i'_0 \).

Also notice that \( (A'_1,\ldots,A'_{p'}) = (A_1,\ldots,A_{i_0},B,A_{i_0+1},\ldots,A'_{p'},\ldots,A'_{p'}) \) In particular, \( B \) is a regular point of \( (A'_1,\ldots,A'_{p'}) \), and consequently, \( (A'_1,\ldots,A'_{p'}) \notin \mathfrak{P}' \).

Claim 3: \( \varphi \) is onto.

Consider a path \( \pi = (A_1,\ldots,A_p) \in \mathfrak{P}' \). If \( p = 1 \), then it is plain to see that the only irregular path is \( \pi = (\{j_1\}) \), which is the image of (\( \emptyset \)) under \( \varphi \). So we consider \( \pi \in \mathfrak{P}' \) with \( p > 1 \). Note that \( |A_1| = 1 \), for otherwise \( \pi \) cannot have any regular points. Now, locate the first \( 1 \leq i_0 \leq p \) with (a) \( |A_{i_0}| = i_0 \); and (b) \( A_{i_0} \setminus A_{i_0-1} = \mathfrak{m}(A_{i_0+1} \setminus A_{i_0-1}) \). The path \( \pi' = (A_1,\ldots,A_{i_0-1},A_{i_0+1},\ldots,A_k) \) is in \( \mathfrak{P}' \) and moreover, \( \varphi(\pi') = \pi \). \( \square \)

Certainly one could cook up other bijections between the regular and irregular paths in \( \mathfrak{P} \). The map we have used has an additional nice property.

**Proposition 8.** **The bijection \( \varphi \) from the proof of Proposition 7 is path-weight preserving.**

The result rests on

**Lemma 3.** Let \( \emptyset \subseteq A \subseteq B \subseteq C \subseteq J \). Writing \( \hat{B} = B \setminus A \) and \( \hat{C} = C \setminus B \), we have

\[
(12) \quad \alpha_B^A \alpha_C^B = \left[ (-q)^{2\ell(\hat{B} \cap J' \hat{C}) - 2\ell(\hat{C} \cap J')} \right] \alpha_A^C.
\]

**Proof.** From the definition of \( \alpha^*_A \), we have

\[
\alpha_B^A = (-q)^{-\ell(J \setminus B \setminus B) - \ell(B \setminus A) + \left\{ 2|J \setminus A| - 2|B| - |I| \right\} \ell(\hat{B} \cap J')}
\]

\[
\alpha_C^B = (-q)^{-\ell(J \setminus C \hat{B}) - \ell(C \setminus B \cup A) + \left\{ 2|J \setminus A| - 2|B \cup C| - |I| \right\} \ell(\hat{C} \cap J')}
\]

\[
\alpha_A^C = (-q)^{-\ell(J \setminus C \hat{B} \cup C) - \ell(B \cup C \setminus A) + \left\{ 2|J \setminus C| - 2|B \cup C| - |I| \right\} \ell(\hat{B} \cup C \cap J')}
\]

Now compare exponents on each side of (12), using facts such as \( |\hat{C}| \ell(\hat{B} \cap J') = \ell(\hat{C} \cap \hat{B} \cap J') + \ell(\hat{B} \cap J' \hat{C}) \) and \( \ell(\hat{C} \cap \hat{B}) = \ell(\hat{C} \cap B \cap J') \ell(\hat{C} \cap \hat{B} \cap J''). \( \square \)
Now the proposition follows by comparing $\alpha(A_{i_0}, A_{i_0+1})$ and $\alpha(A_{i_0}, B, A_{i_0+1})$.

**Proof of Proposition.** Suppose that $\pi = (\ldots, A, C, \ldots)$, and that $\varphi(\pi)$ inserts $B$ immediately after $A$. Then $B = A \cup mM(C \setminus A)$. Writing $b = mM(C \setminus A)$, (12) implies

$$\alpha(\varphi(\pi)) = \left[ (-q)^{2\ell(b \cap J' \cap \hat{C}) - 2\ell(\hat{C} | b \cap J'')} \right] \cdot \alpha(\pi).$$

Now, if $b \cap J' \neq \emptyset$, then $b$ is the smallest element in $C \setminus A$, and in particular, $\ell(b \cap \hat{C}) = 0$. In this same case, $b \cap J'' = \emptyset$, so $\ell(\hat{C} | b \cap J'') = 0$ too. An analogous argument works for the case $b \cap J' = \emptyset$. □

One more interesting fact about $\Gamma(J; I)$ and $\mathfrak{P}$ is worth mentioning. When calculating $\alpha(\pi^\dagger)$ using (12), the twos introduced in the exponents there all disappear.

**Proposition 9.** Given, $J, J', J''$, and $\pi^\dagger$ as above, we have

$$\alpha(\pi^\dagger) = (-q)^{|J'|(|J'|-1) - |J''||(|J''|-1)} \times \alpha_0^{J'}. \tag{13}$$
Proof. Applying (12) repeatedly to the expression $\alpha(\pi^1)$ we see that

$$\begin{align*}
\alpha(\pi^1) &= \left[ (-q)^{2\ell(j_{r,t+1} \cap J'_{j_{r,t+2}})} - 2(\ell(j_{r,t+2} \cap J'_{j_{r,t+3}}) - 2(\ell(j_{r,t+3} \cap J'_{j_{r,t+4}})) \times \\
&\quad \alpha_j^{J_{j_{r,t+1}}, J'_{j_{r,t+2}}} \alpha_j^{J_{j_{r,t+2}}, J'_{j_{r,t+3}}} \cdots \alpha_j^{J_{j_{r,t+d}}, J'_{j_{r,t+d}}}
\right] \times \\
&= (-q)^{-2(1)} \left[ (-q)^{2\ell(j_{r,t+1} \cap J'_{j_{r,t+2}})} - 2(\ell(j_{r,t+2} \cap J'_{j_{r,t+3}}) - 2(\ell(j_{r,t+3} \cap J'_{j_{r,t+4}})) \times \\
&\quad \alpha_j^{J_{j_{r,t+1}}, J'_{j_{r,t+2}}} \alpha_j^{J_{j_{r,t+2}}, J'_{j_{r,t+3}}} \cdots \alpha_j^{J_{j_{r,t+d}}, J'_{j_{r,t+d}}}
\right] \times \\
&= (-q)^{-2(1)} \cdots (-q)^{-2(|J''| - 1)} \left[ (-q)^{2\ell(j_{r,t+1} \cap J'_{j_{r,t+2}})} - 2(\ell(j_{r,t+2} \cap J'_{j_{r,t+3}}) - 2(\ell(j_{r,t+3} \cap J'_{j_{r,t+4}})) \times \\
&\quad \alpha_j^{J_{j_{r,t+1}}, J'_{j_{r,t+2}}} \alpha_j^{J_{j_{r,t+2}}, J'_{j_{r,t+3}}} \cdots \alpha_j^{J_{j_{r,t+d}}, J'_{j_{r,t+d}}}
\right] \times \\
&= \cdots
\end{align*}$$

5. $G$-Proof of Theorem

We keep the notations $J', J'', r', r'', r, s, t$ from Section 4.2, and as we did there, we only consider the case $J \cap I = \emptyset.$\textsuperscript{2} Before we begin, we define a new quantity $CM_{J,I}(\theta)$.

$$C_{J,I} = M_{J,I} = -q^{|J||J'|} f_J f_J + \left( \sum_{\Lambda \subseteq I, |\Lambda| = r} (-q)^{\ell(\Lambda | I \setminus \Lambda)} f_J f_{I \setminus \Lambda} f_{\Lambda} \right)$$

$$= \sum_{\Lambda \subseteq I} (-q)^{|J'| |J''| |J|} (-q)^{-\ell((J \cup J') \Lambda | \Lambda)} f_{(J \cup J') \setminus \Lambda} f_{\Lambda} - q^{|J||J'|} f_J f_J .$$

\textsuperscript{2}Only minor changes to this proof are needed to prove the theorem in the general setting (e.g. replacing every instance of $J$ below with $J_0 := J \setminus I$). In the interest of avoiding even more notation, we leave this work to the reader.
Here, we have replaced $\ell(\Lambda|I^\Lambda)$ with $|I^\Lambda| - \ell(I^\Lambda|\Lambda)$ and $\ell(J|\Lambda^J)$ with $|J^\Lambda| - \ell(I^\Lambda|J)$. Now factor out a power of $-q$ and forget what coefficient appears in front of $f_{I,J}$:

$$CM_{I,J}(\theta) := (-q)^{|J||t| + |J''||J|} \left( \sum_{\Lambda \subseteq I} (-q)^{-\ell((J\cup J')^\Lambda|\Lambda)} f_{(J\cup J')\setminus\Lambda} - \theta f_{I,J} \right).$$

We prove the theorem in steps:

**Proposition 10.** Suppose $I, J \subseteq [n]$ are such that $J \cap I$. With $CM_{I,J}(\theta)$ and $Y_{L,K;\langle\cdot\rangle}$ as defined above,

$$\sum_{\emptyset \subseteq K \subseteq J} \eta_K \cdot Y_{(I\cup J)\setminus K,K;\langle r-|K|\rangle} = CM_{I,J}(\theta)$$

for some constants $\{\eta_K \in \mathbb{Z}[q,q^{-1}] : \emptyset \subseteq K \subseteq J\}$ and $\theta \in \mathbb{Z}[q,q^{-1}]$.

**Proposition 11.** In the notation above, $\theta = (-q)^{|J||t| + |J''||J|} \cdot v^\theta$.

The alternating property of the symbols $f_K$ and the product in $\mathcal{F}I_q(n)$ play no role in our proof, so we begin by eliminating these distractions. Let $V$ be the vector space over $\mathbb{F}$ with basis $\{e_{(A,B)} : A \cup B = I \cup J, A \cap B = \emptyset, \text{and } |B| = r\}$. There is a $\mathbb{F}$-linear map $\mu : V \rightarrow \mathcal{F}I_q(n)$, sending $e_{A,B}$ to $f_A f_B$. The vectors

$$v^\theta := \sum_{\Lambda \subseteq I} (-q)^{-\ell((I\cup J)|\Lambda)} e_{(I\cup J)|\Lambda,\Lambda} - \theta e_{I,J}$$

and (for each $\emptyset \subseteq K \subseteq J$)

$$v^K := \sum_{\Lambda \subseteq (I\cup J), |\Lambda| = r - |K|} (-q)^{-\ell((I\cup J)^K|\Lambda)} (-q)^{-\ell(\Lambda|K)} e_{(I\cup J)|\Lambda,\Lambda}$$

have familiar images. Check that $\mu((-q)^{|J||t| + |J''||J|} \cdot v^\theta) = CM_{I,J}(\theta)$ and

$$\mu(v^K) = Y_{(I\cup J)\setminus K,K;\langle r-|K|\rangle}.$$ 

Proposition 10 will be proven if we can show that $v^\theta$ is a linear combination of $v^K$ for some $\theta$. This is not immediate as the span of the vectors $v^K$ has dimension (at most) $2^r - 1$, while $V$ is $\binom{r+s}{r}$ dimensional.

**Definition 9.** For each $K \in \mathcal{P}J$, let $V_{(K)} = \text{span}_{\mathbb{F}} \{e_{A,B} : B \cap J = K\}$. Clearly, $V$ is graded by the POset $\mathcal{P}J$, i.e., $V = \bigoplus_{K \in \mathcal{P}J} V_{(K)}$. For each $K \in \mathcal{P}J$, define the distinguished element $e^K$ by

$$e^K = \sum_{\Lambda \subseteq I, |\Lambda| = r - |K|} (-q)^{-\ell((I\cup J)^K|\Lambda)} (-q)^{-\ell(\Lambda|K)} e_{(I\cup J)|\Lambda,\Lambda}.$$ 

For any $v \in V$, write $(v)_{(K)}$ for the component of $v$ in $V_{(K)}$, that is, $v = \sum_K (v)_{(K)}$. 


Notice that $e^J = e_{I,J}$, and that

$$e^\emptyset = \sum_{\Lambda \subseteq I, |\Lambda| = r} (-q)^{-\ell((I \cup J)^\Lambda|\Lambda)} e_{(I \cup J) \setminus \Lambda, \Lambda}$$

In other words, $v^\emptyset = e^\emptyset - \theta e^J$. Good fortune provides that the $v^{K'}$ may also be expressed in terms of the $e^K$.

**Lemma 4.** For each $K' \in \mathcal{P}J \setminus J$, there are constants $\alpha^K_{K'} \in \mathbb{K}$ satisfying

$$v^{K'} = \sum_{K \in \mathcal{P}J} \alpha^K_{K'} e^K.$$

**Remark 5.** As the proof will show, these $\alpha^K_{K'}$ are precisely the edge-weights of $\Gamma(J; I)$ from Section 4.2, in particular $\alpha^K_{K'} = 1$. It will also show that $\alpha^K_{K'} = 0$ if $K' \not\leq K$ in the POset $\mathcal{P}J$, a critical ingredient in the approaching Gaussian elimination argument.

**Proof of Lemma.** Fixing a subset $K'$, if $K \supseteq K'$, we write $\hat{K} = K \setminus K'$. Similarly, let $\hat{\Lambda} = \Lambda \setminus J$. Studying $v^{K'}$, we see that

$$v^{K'} = \sum_{\Lambda \subseteq (I \cup J) \setminus K'} (-q)^{-\ell((I \cup J)^\Lambda|\Lambda)} (-q)^{-\ell(\Lambda|K')} e_{(I \cup J) \setminus (\Lambda \cup K'), \Lambda \cup K'}$$

$$= \sum_{K \in \mathcal{P}J} (v^{K'})_{(K)}$$

$$= \sum_{K \in \mathcal{P}J} \sum_{\Lambda \subseteq (I \cup J) \setminus K'} (-q)^{-\ell((I \cup J)^\Lambda \cup K|\Lambda \cup K)} \times$$

$$(-q)^{-\ell(\hat{\Lambda} \cup \hat{K}|K')} e_{(I \cup J) \setminus (\hat{\Lambda} \cup \hat{K}), \hat{\Lambda} \cup K}$$

$$= \sum_{K \in \mathcal{P}J} (-q)^{-\ell((I \hat{\Lambda}) \cup (J K))}(\hat{K}) (-q)^{-\ell(\hat{K}|K')} \times$$

$$\left( \sum_{\Lambda \subseteq I, |\Lambda| = r - |K|} (-q)^{-\ell((I \cup J)^{\hat{\Lambda} \cup K}|\Lambda)} (-q)^{-\ell(\hat{\Lambda}|K')} e_{(I \cup J) \setminus (\hat{\Lambda} \cup K), \hat{\Lambda} \cup K} \right).$$

Why can we perform this last step? Since $J \preceq I$, the expression $\ell(I \hat{\Lambda}|\hat{K})$ does not actually depend on $\hat{\Lambda}$, only on $|\hat{\Lambda}|$. Indeed, it equals $|I \setminus \hat{\Lambda}| \cdot |\hat{K} \cap J'|$. 
Multiplying and dividing by \((-q)^{-\ell(\hat{\Lambda}|K)}\), we rewrite this last expression as

\[
v^K' = \sum_K (-q)^{-\ell((I \cup J) \cup K)|K}} \left( \sum_{\hat{\Lambda} \subseteq I, |\hat{\Lambda}| = r - |K|} (-q)^{-\ell((I \cup J) \cup \hat{\Lambda}|\hat{\Lambda})} e_{(I \cup J) \setminus (\hat{\Lambda} \cup K), \hat{\Lambda} \cup K)} \right)
\]

\[
= \sum_{K' \leq K} (-q)^{2|J \setminus K| - |I| - \ell(J \setminus K)} e_{J \setminus K} \alpha_{K'} \cdot e^K.
\]

\[\square\]

**Corollary 12.** For any \(v^{K'}, v^K\) with \(K' < K\) in the POset \(\mathcal{P}J\), and for the same constants \(\alpha_{K'}^{K}\) as defined above, we have

\[\langle v^{K'} - \alpha_{K'}^{K} v^K \rangle_K = 0.\]

**Proof of Proposition 10.** We use the corollary to perform a certain Gaussian elimination on the “matrix” of the vectors \(v^K\). Table 3 displays this matrix for the POset \(\mathcal{P} \{1, 5, 6\}\) and should make our intentions clear.

<table>
<thead>
<tr>
<th>(v)</th>
<th>(e^0)</th>
<th>(e^1)</th>
<th>(e^5)</th>
<th>(e^6)</th>
<th>(e^{15})</th>
<th>(e^{16})</th>
<th>(e^{56})</th>
<th>(e^{156})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v^{15})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\alpha_{15}^{156})</td>
</tr>
<tr>
<td>(v^{16})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\alpha_{16}^{156})</td>
</tr>
<tr>
<td>(v^{56})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\alpha_{56}^{156})</td>
</tr>
<tr>
<td>(v^1)</td>
<td>1</td>
<td>(\alpha_1^{15})</td>
<td>(\alpha_1^{16})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v^5)</td>
<td>1</td>
<td>(\alpha_5^{15})</td>
<td>(\alpha_5^{56})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v^6)</td>
<td>1</td>
<td>(\alpha_6^{15})</td>
<td>(\alpha_6^{56})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v^0)</td>
<td>1</td>
<td>(\alpha_0^{15})</td>
<td>(\alpha_0^{16})</td>
<td>(\alpha_0^{56})</td>
<td>(\alpha_0^{156})</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 3.** Writing the vectors \(v^{K'}\) in terms of the \(e^K\).

Performing Gaussian elimination between the rows in the first two layers of the matrix, we see that the new rows in the second layer—who began their life with \(|J| + 1\) nonzero entries—now have exactly two nonzero entries.

\[
\left\langle v^{J \setminus \{k,l\}} \right\rangle' = v^{J \setminus \{k,l\}} - \alpha_{J \setminus \{k,l\}}^{J} v^{J \setminus k} - \alpha_{J \setminus \{k,l\}}^{J} v^{J \setminus l} + e^{J \setminus \{k,l\}} + \left( \alpha_{J \setminus \{k,l\}}^{J} - \alpha_{J \setminus \{k,l\}}^{J} \alpha_{J \setminus k} - \alpha_{J \setminus \{k,l\}}^{J} \alpha_{J \setminus l} \right) e^{J},
\]

\[\square\]
Proof of Proposition 11. Careful bookkeeping shows that

\[ \theta = \alpha^J_0 - \left( \sum_{\emptyset \subseteq K \subseteq J} \alpha^K_0 \alpha^K_J \right) + \left( \sum_{\emptyset \subseteq K_1 \subseteq K_2 \subseteq J} \alpha^K_1 \alpha^K_2 \alpha^K_J \right) - \cdots \]

\[ \cdots + (-1)^{|J|-1} \left( \sum_{\emptyset \subseteq K_1 \subseteq \cdots \subseteq K_{|J|-1} \subseteq J} \alpha^K_1 \alpha^K_2 \cdots \alpha^K_{|J|-1} \right) . \]

In other words, \( \theta \) is a signed sum of path weights \( \alpha(\pi) \), \( \pi \) running over all paths in \( \mathcal{P} \) save for \( \hat{1} \). As the sign attached to \( \pi \) is the same as the length of \( \pi \), and as the bijection \( \varphi \) from Section 4.2 increases length by one but preserves path weight, we immediately conclude

\[ \theta = (-1)^{|J|-1} \alpha(\pi^1) \]

\[ = (-1)^{|J|-1} (-q)^{|J'|||J'|+|J''||J'|} \times \alpha^J_0 \]

\[ = (-1)^{|J|-1} (-q)^{|J'|||J'|} \times \alpha^J_0 \]

\[ = q^{|J''|-|J'|} (-q)^{|J'|||J'|} \times \alpha^J_0 \]

With Proposition 11 proven, Theorem 1 is finally demonstrated (modulo the Taft-Towber isomorphism \( \phi \)). Moreover, we achieve the second goal stated in the introduction. A brief discussion of the first goal follows.

6. On Quantum- and Quasi- Flag Varieties

The algebra \( \mathcal{F}_{\ell_q}(n) \) is a quantum deformation of the classic multihomogeneous coordinate ring of the full flag variety over \( \text{GL}_n \). In [Taft and Towber, 1991], it is admitted that finding the proper form of the relations was somewhat difficult. In [Fiorese, 1999] we see a completely different (equivalent) set of relations. One hopes to proceed in a less ad-hoc manner. Perhaps a theory of noncommutative flag varieties using quasi-Plücker coordinates could help explain the choices for the relations in \( \mathcal{F}_{\ell_q}(n) \). In [Lauve, 2006], it is shown that any relation \( (\mathcal{Y}_{i,j})(a) \) has a quasi-Plücker coordinate origin. Section 3 shows that \( (1) \) does too. The second proof of Theorem 1 shows that a great many instances of \( (\mathcal{M}_{i,j}) \) do as well; to see this, note that the roles of \( M_{i,j} \) and \( C_{i,j} \) were interchangeable there. The question of whether and to what extent the gap (the case \( J \not\preceq I \)) may be filled by finding new quasi-Plücker
coordinate identities is an interesting one. Toward a partial answer, we leave the reader to verify that

$$(P_{I,J}^j) \Rightarrow (M_{I,J})$$

whenever $I, J \subseteq [n]$ are such that $|J| \leq |I|$ and $J \setminus j \subseteq I$.

References


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