Lagrange's Theorem for Hopf Monoids in Species

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SIAM Conference on Discrete Mathematics, Austin, TX

Pre-History: Finite Groups

Theorem (Lagrange, 1770)

If K, H are finite groups with $K \subseteq H$, then $|K|$ divides $|H|$.

Even better: Fix a field k. In language of algebras and modules, we have: $\exists C \subseteq H$ (*coset representatives*) satisfying

$$kH \simeq kK \otimes kC$$

as kK modules, i.e., kH is a free left kK-module.

(kK action on $kK \otimes kC$ given by $a * (b \otimes c) := (ab) \otimes c$.)

History: Hopf Algebras

Conjecture (Kaplansky, 1973)

If K, H are Hopf algebras with $K \subseteq H$, then H is a free left K-module.

Special case: group algebras $H = kG$, which are Hopf algebras under the coproduct $\Delta(g) = g \otimes g$ for all $g \in G$.

Worth noting: if H is finite dimensional, then $\dim K$ divides $\dim H$.

History: Hopf Algebra Results

Theorem (Oberst–Schneider, 1974)

There are infinite dimensional counter-examples $K \subseteq H$ to Kaplansky's conjecture.

(even when H is *relatively nice:* commutative, cocommutative and cosemisimple)

Theorem (Nichols–Zoeller, 1989)

If $K \subseteq H$ are finite dimensional Hopf algebras then H is a free left K-module, i.e.,

$$H \simeq K \otimes C$$

for some coset representatives C.

History: Hopf Algebra in Combinatorics

finite dimensional . . . $H = \bigoplus_{n \geq 0} H_n$ . . . *infinite dimensional*

✓ ? ✗

- H_n is finite dimensional; basis indexed by combinatorial structures on $[n]$
- *product* and *coproduct* describe ways to *"join"* and *"break"* these structures

Theorem (Radford, 1977)

If $K \subseteq H$ are graded Hopf algebras with $H_0 \subseteq K$, then H is a free left K-module.

New Result: Hopf Monoids in Species

Theorem (Aguiar–L, 2010)

If $\mathbf{K}, \mathbf{H}$ *are Hopf monoids in the category* $(\mathrm{Sp}, \cdot)$ *of linear species, with* $\mathbf{K} \subseteq \mathbf{H}$, *then* $\mathbf{H}$ *is a free left* $\mathbf{K}$*-module.*

Proof: Follow Radford.

Key ingredients:

- a fundamental theorem for Hopf modules in $(\mathrm{Sp}, \cdot)$
- properties of the wedge $\mathbf{P} \wedge \mathbf{Q}$ of two comonoids in $(\mathrm{Sp}, \cdot)$

Species: Definition (Joyal)

A species of combinatorial structures is a functor

$$\mathbf{P}\colon \big(\text{finite sets and bijections}\big) \longrightarrow \big(\text{vector spaces}\big).$$

A species $\mathbf{P}$ consists of:

- a vector space $\mathbf{P}[I]$ for each finite set I;
- a linear isomorphism
$$\mathbf{P}[\sigma]\colon \mathbf{P}[I] \to \mathbf{P}[J]$$
for each bijection $\sigma\colon I \to J$.

Worth noting: Let $\mathbf{n} = \{1, 2, \ldots, n\}$. For each $n \geq 1$, $\mathfrak{S}_n$ acts on $\mathbf{P}[\mathbf{n}]$. Write $\mathbf{P}[\mathbf{n}]_{\mathfrak{S}_n}$ for the quotient by this action.

Species: Examples

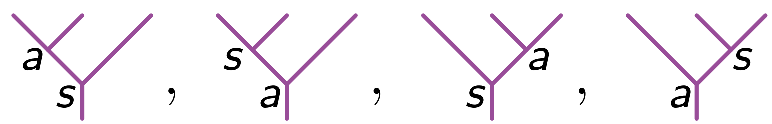
1. The species **L** of linear orders, e.g.,

$$\mathbf{L}[\{a, s, x\}] = \mathrm{span}_k\Big\{[a, s, x],\ [s, a, x],\ [s, a, x],\ [s, x, a],\ [x, a, s],\ [x, s, a]\Big\}$$

2. The species $\mathbf{\Pi}$ of set partitions., e.g.,

$$\mathbf{\Pi}[\{a, s, x\}] = \mathrm{span}_k\Big\{\{\{a, s, x\}\},\ \{\{a\}, \{s, x\}\},\ \ldots,\ \{\{a\}, \{s\}, \{x\}\}\Big\}$$

3. The species **Y** of (labeled, rooted) planar binary trees, e.g.,

$\mathbf{Y}[\{a, s\}] = \mathrm{span}_k\Big\{$ 

$\Big\}$

Species: Monoidal Structure

The category Sp of linear species is a monoidal category under the Cauchy Product, $(\mathbf{P}, \mathbf{Q}) \rightsquigarrow \mathbf{P} \cdot \mathbf{Q}$, defined by

$$\big(\mathbf{P} \cdot \mathbf{Q}\big)[I] := \bigoplus_{S \sqcup T = I} \mathbf{P}[S] \otimes \mathbf{Q}[T] \,.$$

Compare: in the category gVec of graded vector spaces, "$\cdot$" is the tensor product:

$$\big(V \cdot W\big)_i := \bigoplus_{s+t=i} V_s \otimes W_t \,.$$

We can speak of *monoids, comonoids, bimonoids,* and *Hopf monoids* in each category.

Species: Hopf Monoids

A Hopf monoid $(\mathbf{P}, \mu, \Delta)$ in $(\mathrm{Sp}, \cdot)$ consists of:

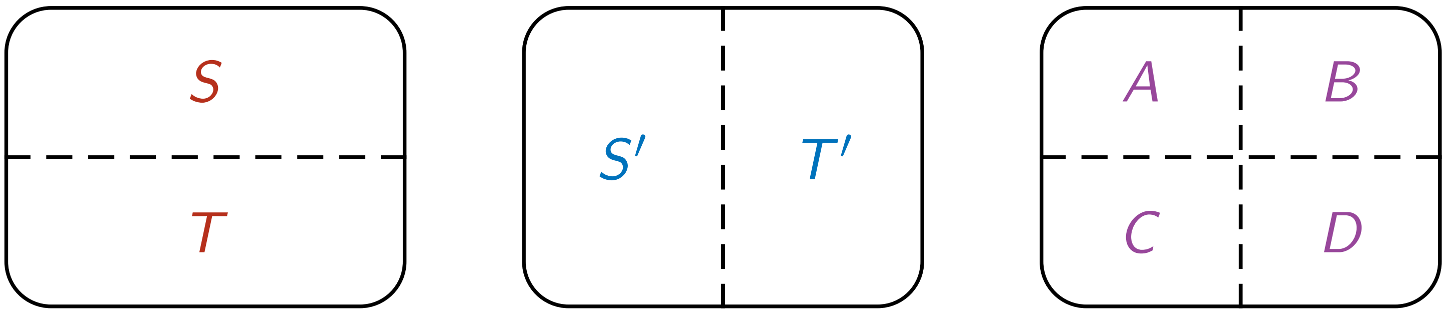
- a species $\mathbf{P}$;
- for each *ordered decomposition* $S \sqcup T = I$, maps

$$\mu_{S,T} \colon \mathbf{P}[S] \otimes \mathbf{P}[T] \to \mathbf{P}[I] \qquad \text{and} \qquad \Delta_{S,T} \colon \mathbf{P}[I] \to \mathbf{P}[S] \otimes \mathbf{P}[T]\,;$$

- additional maps (unit, counit, antipode);
- various compatibility axioms, including the following . . .

Species: Compatibility of μ and Δ

Fix decompositions $S \sqcup T = I = S' \sqcup T'$. Let A, B, C, D be as below.



Then the following diagram commutes.

$$\begin{array}{ccccc}
\mathbf{P}[A] \otimes \mathbf{P}[B] \otimes \mathbf{P}[C] \otimes \mathbf{P}[D] & \xrightarrow{\mathbb{1} \otimes \mathrm{tw} \otimes \mathbb{1}} & & & \mathbf{P}[A] \otimes \mathbf{P}[C] \otimes \mathbf{P}[B] \otimes \mathbf{P}[D] \\
\uparrow {\scriptstyle \Delta_{A,B} \otimes \Delta_{C,D}} & & & & \downarrow {\scriptstyle \mu_{A,C} \otimes \mu_{B,D}} \\
\mathbf{P}[S] \otimes \mathbf{P}[T] & \xrightarrow{\mu_{S,T}} & \mathbf{P}[I] & \xrightarrow{\Delta_{S',T'}} & \mathbf{P}[S'] \otimes \mathbf{P}[T']
\end{array}$$

(expresses compatibility between “join” and “break” for the combinatorial structures)

Species: Hopf Monoids

Examples:

1. The species $\mathbf{L}$ of linear orders

$$\mathbf{L}[S] \otimes \mathbf{L}[T] \xrightarrow{\mu} \mathbf{L}[I] \qquad \mathbf{L}[I] \xrightarrow{\Delta} \mathbf{L}[S] \otimes \mathbf{L}[T]$$

$$\ell_1 \otimes \ell_2 \longmapsto \ell_1 | \ell_2 \qquad \ell \longmapsto \ell|_S \otimes \ell|_T$$

(extend by letting S precede T) (restrict the ordering of I)

2. The species $\mathbf{\Pi}$ of set partitions

$$\mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T] \xrightarrow{\mu} \mathbf{\Pi}[I] \qquad \mathbf{\Pi}[I] \xrightarrow{\Delta} \mathbf{\Pi}[S] \otimes \mathbf{\Pi}[T]$$

$$\pi_1 \otimes \pi_2 \longmapsto \pi_1 \cup \pi_2 \qquad \pi \longmapsto \pi|_S \otimes \pi|_T$$

(the union partitions I) (intersect blocks of π with S and T)

Back to the New Result: Enumerative Corollary

- Fix a species $\mathbf{P}$.
- Define the exponential and ordinary generating series by

$$\mathcal{E}_{\mathbf{P}}(x) = \sum_{n\geq 0} (\dim \mathbf{P}[\mathbf{n}])\, \frac{x^n}{n!} \qquad \text{and} \qquad \mathcal{P}_{\mathbf{P}}(x) = \sum_{n\geq 0} \left(\dim \mathbf{P}[\mathbf{n}]_{\mathfrak{S}_n}\right) x^n \,.$$

Corollary (via Joyal)

If $\mathbf{K} \subseteq \mathbf{H}$ are Hopf monoids in $(\mathrm{Sp}, \cdot)$ then there is a subspecies $\mathbf{C} \subseteq \mathbf{H}$ and a left $\mathbf{K}$-module isomorphism $\mathbf{H} \simeq \mathbf{K} \cdot \mathbf{C}$. In particular,

$$\mathcal{E}_{\mathbf{H}}(x) = \mathcal{E}_{\mathbf{K}}(x)\, \mathcal{E}_{\mathbf{C}}(x) \qquad \textit{and} \qquad \mathcal{P}_{\mathbf{H}}(x) = \mathcal{P}_{\mathbf{K}}(x)\, \mathcal{P}_{\mathbf{C}}(x)\,.$$

(A similar result holds in $(\mathrm{gVec}, \cdot)$ by Radford's theorem.)

Back to the New Result: Application

Test for Hopf submonoids:

1. Define $\mathbf{\Pi}^{\mathrm{e}}$ by $\mathbf{\Pi}^{\mathrm{e}}[I] := \begin{cases} \mathbf{\Pi}[I], & \text{if } |I| \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$

   $$\mathcal{E}_{\mathbf{\Pi}^{\mathrm{e}}}(x) \nmid \mathcal{E}_{\mathbf{\Pi}}(x)$$ (so no Hopf submonoid can exist)

2. Define $\mathbf{\Pi}^{\mathrm{ob}}$ by $\mathbf{\Pi}^{\mathrm{ob}}[I] := \mathrm{span}_k\{\textit{partitions of } I \textit{ into odd \# of blocks}\}$.

   $$\mathcal{E}_{\mathbf{\Pi}^{\mathrm{ob}}}(x) \nmid \mathcal{E}_{\mathbf{\Pi}}(x)$$ (so no Hopf submonoid can exist)

Back to the New Result: Applications

Test for Hopf submonoids & subalgebras:

3. Note that $\mathcal{E}_{\mathbf{\Pi}}(x)$ divides $\mathcal{E}_{\mathbf{L}}(x)$. *Perhaps there is a Hopf map* $\mathbf{\Pi} \hookrightarrow \mathbf{L}$ *between species?* (see [Aguiar–Mahajan], Thm. 12.57)

4. Put $\Lambda := \{\textit{integer partitions}\}$ and $\Lambda^{\mathrm{e}} := \{\textit{integer partitions of even \#s}\}$. Recall that $k\Lambda$ possesses a Hopf structure (the Hopf algebra of symmetric functions).

$$\mathcal{P}_{k\Lambda^{\mathrm{e}}}(x) \nmid \mathcal{P}_{k\Lambda}(x) \qquad \text{(so no Hopf subalgebra can exist)}$$

Back to the New Result: Caveat Emptor

- There are <u>many</u> factorization results in the Hopf literature. (The fundamental theorem of Hopf modules is an example.)
- Divisibility of generating series does not guarantee Hopf subalgebras.
- I'll leave you with a speculative one just for fun...

Let kD denote the span of Dyck paths (certain monomials of even length) inside $H = k\langle x, y\rangle$. There is a natural Hopf algebra structure on H. Moreover, $\mathcal{E}_{kD}(x)$ divides $\mathcal{E}_H(x)$.

Might there be a Hopf subalgebra of Dyck paths?

Thank You

(and happy hunting!)

References

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