Yangian Flags via Quasideterminants

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Key Idea	Proceed with Caution	Watch Out!

Yangian flags via quasideterminants

Abstract

The Yangian Y_n, by construction, is a deformation of the universal enveloping algebra of the general linear Lie algebra. Analogous to its classical counterpart, Y_n has a nice (i.e., Ore) localization. As there is a (nonzero)(Yangian) determinant in Y_n, one may also view Y_n as a deformation of the (coordinate ring of the) general linear group.

From this point of view, existence of the localization allows us to use the quasideterminant to construct the coordinate ring $\mathcal{F}\ell(Y_n)$ for a flag variety for Y_n. Analogous to its classical counterpart, $\mathcal{F}\ell(Y_n)$ is simultaneously a comodule algebra for Y_n (viewed it as a general linear group) and a model for irreducible representations for Y_n (viewed as a general linear algebra).

Overview

I. **Two actions of $\mathrm{GL}_n(\mathbb{C})$ on polynomials $\mathbb{C}[X_{n\times n}]$**

A. one, a model for the action of GL_n on flags

B. the other, a model for irreducible representations of $\mathfrak{gl}_n$

II. **Flag varieties for $\mathrm{GL}_n(D)$ over division rings D**

A. noncommutative flags

B. coordinatization via the quasideterminant

C. recaptures the classical story upon "specialization"

III. **Yangians Y_n for $\mathfrak{gl}_n$**

A. defined as a deformation of $U(\mathfrak{gl}_n)$

B. view Y_n as a deformation of $\mathbb{C}[\mathrm{GL}_n]$ and use (*II.*) to mimic (*I.A.*)

C. treating Y_n as usual, . . . (*I.B.*) works too!

Notation

- SETS & SEQUENCES

  $[n] = \{1, 2, \ldots, n\}$

  $\binom{[n]}{d}$ = sets of size d

  $[n]^d$ = sequences of size d

- SET/SEQUENCE OPERATIONS

  $I|J = (i_1, \ldots, i_r, j_1, \ldots, j_s)$

- MATRIX OPERATIONS

  A^{ij} *delete* row i and column j

  $A_{I,J}$ *keep* rows I and columns J

  $A_J = A_{[d],J}$ for $J \in [n]^d$

- SEQUENCES & PERMUTATIONS

  $\ell(\sigma)$ *length* of permutation σ

  $\ell(I|J)$ *length* of permutation $I|J$

I.A. **Flag Varieties & Coordinate Rings**

- Let $X = (x_{ij})_{1\leq i,j\leq n}$ be a matrix of indeterminants. Write $\mathbb{C}[X]$ for the polynomials in the x_{ij}.
- Define an action $\mathrm{GL}_n \times \mathbb{C}[X] \to \mathbb{C}[X]$ by $g \cdot F(X) = F(g^{-1} \cdot X)$.

Example. Fix $n = 3$, and consider $g^{-1} = \begin{bmatrix} 2 & & \\ 7 & 4 & \\ & & 3 \end{bmatrix}$. Then,

$$F(X) = x_{23} \quad \mapsto \quad [g \cdot F](X) = 7x_{13} + 4x_{23}.$$

Moreover,

$$\det X_{\{1,3\}} \overset{g}{\mapsto} (2x_{11})(7x_{13} + 4x_{23}) - (7x_{11} + 4x_{21})(2x_{13}) = 8 \cdot \det X_{\{1,3\}}$$

$$\det X_{\{1,2,3\}} \overset{g}{\mapsto} 24 \cdot \det X_{\{1,2,3\}}.$$

I.A. Flag Varieties & Coordinate Rings

Observation. The minors $\left\{\Delta(I) := \det X_{[d],I} \mid I \in \binom{[3]}{d};\ d = 2, 3\right\}$ in $\mathbb{C}[X]$ are *projective invariants* for the subgroup $\begin{bmatrix} \blacksquare & 0 \\ * & \blacksquare \end{bmatrix} \subseteq \mathrm{GL}_3$.

- This is not surprising. . .

I.A. Flag Varieties & Coordinate Rings

Definition. *A **flag** Φ of shape γ is an increasing chain of subspaces of V:*

$$\Phi \,:\, 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r = V \qquad (\dim W_i/W_{i-1} = \gamma_i).$$

Definition. ***Flags*** *are right cosets:* $Fl(\gamma) = \mathrm{P}_\gamma \backslash \mathrm{GL}_n(\mathbb{C})$, *where*

$$\mathrm{P}_\gamma = \begin{bmatrix} \blacksquare & & & 0 \\ & \blacksquare & & \\ & & \ddots & \\ * & & & \blacksquare \end{bmatrix}$$

- Focus on $\gamma = (1, 1, \ldots, 1)$ for clarity. Write $Fl(n)$ in this case.
- $Fl(n)$ is made into a variety by the **Plücker embedding:**

$$\eta : A \mapsto \left\{ \det A_I \,:\, I \in [n]^d,\, 1 \leq d < n \right\},$$

a map into $\mathbb{P}_\gamma := \mathbb{PC}^n \times \mathbb{PC}^{n^2} \times \cdots \times \mathbb{PC}^{n^{n-1}}$.

I.A. Flag Varieties & Coordinate Rings

Theorem (Schur '01; Hodge '43)**.** *Let* $\mathbb{C}[\mathbb{P}_\gamma] = \mathbb{C}[f_I : I \in \binom{[n]}{d};\ 1 \leq d < n]$ *be the homogeneous coordinate ring for* $\mathbb{P}_\gamma$*. The image of* $\eta\big(Fl(n)\big)$ *in* $\mathbb{P}_\gamma$ *is cut out by the homogeneous relations below:*

($\mathcal{A}_I$) ***Alternating*****:** *For all* $I \in [n]^d$*, all permutations* $\sigma \in \mathfrak{S}_d$*:*

$$f_I = (-1)^{\ell(\sigma)} f_{\sigma I} \quad (= 0 \text{ when } I \text{ has repetitions})$$

($\mathcal{Y}_{I,J}$) ***Young Symmetry*****:** *For all* $1 \leq r \leq d \leq e$*,* $I \in \binom{[n]}{d-r}$*, and* $J \in \binom{[n]}{e+r}$

$$\sum_{\Lambda \subset J,\, |\Lambda| = r} (-1)^{\ell(\Lambda|J\setminus\Lambda)} f_{I|\Lambda} f_{J\setminus\Lambda} = 0$$

e.g. $(\mathcal{Y}_{\emptyset,\{124\}}) : \quad 0 = f_1 f_{24} - f_2 f_{14} + f_4 f_{12}\,.$

- Call the quotient (of $\mathbb{C}[\mathbb{P}_\gamma]$ modulo these relations) the **flag algebra** $\mathcal{F}\ell(n)$.

Theorem. $\mathcal{F}\ell(n)$ *is isomorphic to the subalgebra of* $\mathbb{C}[X]$ *generated by the minors* $\{\Delta(I) : I \in \binom{[n]}{d};\ 1 \leq d < n\}$ *of* $X = (x_{ij})$*.*

Taft and Towber's Work (Briefly)

"It would be very important to define flag spaces for quantum groups." [Manin '88]

Their Goal. Build a **quantum flag algebra** $\mathcal{F}\ell_q(n)$ with the following properties:

1. $\mathcal{F}\ell_q(n)$ reduces to $\mathcal{F}\ell(n)$ when $q \to 1$.

1′. e.g., $\mathcal{F}\ell_q(n)$ and $\mathcal{F}\ell(n)$ share the same basis (*semi-standard Young tableaux*).

2. $\mathcal{F}\ell_q(n)$ is a comodule algebra for the quantum group $\mathbb{C}[\mathrm{GL}_q(n)]$.

3. $\mathcal{F}\ell_q(n)$ is isomorphic to the subalgebra of $\mathbb{C}[\mathrm{GL}_q(n)]$ generated by the minors $\{\det_q X_I \ : \ I \in \binom{[n]}{d},\ 1 \le d < n\}$ of its matrix of generators $X = (x_{ij})$.

Theorem (T-T '91). Success! Put $\mathcal{F}\ell_q(n) := \mathbb{C}\langle f_I \ : \ I \in [n]^d,\ 1 \le d < n\rangle$ modulo

$(\mathcal{A}_I)^q$ **:** $f_I = (-q)^{\ell(\sigma)} f_{\sigma I}$

$(\mathcal{Y}_{I,J})^q$ **:** $\sum_\Lambda (-q)^{-\ell(\Lambda|J\setminus\Lambda)} f_{I|\Lambda} f_{J\setminus\Lambda} = 0$

$(\mathcal{C}_{I,J})^q$ **:** For all $1 \le d < e$, $I \in \binom{[n]}{d}$, and $J \in \binom{[n]}{e}$:

$$f_J f_I = \sum_{\substack{\Lambda \subseteq J \\ |J\setminus\Lambda| = d}} (-q)^{\ell(\Lambda|J\setminus\Lambda)} f_{J\setminus\Lambda} f_{\Lambda|I}$$

I.B. **Polynomial Representations of $\mathfrak{gl}_n$**

- Keep X and $\mathbb{C}[X]$ as above.
- Define an action $\mathrm{GL}_n \times \mathbb{C}[X] \to \mathbb{C}[X]$ by $g \cdot F(X) = F(X \cdot g)$.
- This action also looks nice on the minors $\Delta(I)$.

Example. Fix $n = 3$, and consider $g = \begin{bmatrix} 2 & & \\ 7 & 4 & \\ & & 3 \end{bmatrix}$.

Then $X = (\mathbf{a} \mid \mathbf{b} \mid \mathbf{c})$ maps to $X \cdot g = (2\mathbf{a} + 7\mathbf{b} \mid 4\mathbf{b} \mid 3\mathbf{c})$.

So g acts on minors as follows:

$$\Delta(1,3) \stackrel{g}{\mapsto} \Delta(2\mathbf{a} + 7\mathbf{b}\,,\, 3\mathbf{c}) = 6\Delta(1,3) + 21\Delta(2,3)$$

$$\Delta(1,2,3) \stackrel{g}{\mapsto} 24\Delta(1,2,3) + 84\Delta(2,2,3) = 24\Delta(1,2,3)$$

I.B. **Polynomial Representations of $\mathfrak{gl}_n$**

- Take the derivative of this action, and get a familiar module for $\mathfrak{gl}_n$
- Map semi-standard Young tableaux T onto polynomials $\Delta_T \in \mathbb{C}[X]$ as follows:

$$\begin{array}{|c|c} \hline 6 & \\ \hline 4 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \mapsto \qquad \Delta(2,4,6)\Delta(4)$$

$$\begin{array}{|c|c|c|c|} \hline 3 & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} 2 & 4 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 4 & 4 \\ \hline \end{array} \qquad \mapsto \qquad \Delta(1,2,3)\Delta(1,4,6)\Delta(4,6)\Delta(6)$$

- **Fact.** For λ a partition with at most n parts, the irreducible $\mathfrak{gl}_n$ module V_λ is exactly the $\mathbb{C}$-span of the Δ_T, the set running over semi-standard tableaux T of shape λ with fillings from $[n]$.
- That is, $\mathcal{F}\ell(n) \simeq \bigoplus_\lambda V_\lambda$ as $\mathfrak{gl}_n$-modules.

II.A. Noncommutative Flags

"Follow" the work of Taft and Towber

- Start from noncommutative flags, not from the algebra of functions $\mathcal{F}\ell(\gamma)$.
- Hopefully arrive at correct generalizations of $\mathcal{F}\ell(\gamma)$.

Preliminary steps are identical to $\mathrm{GL}_n(\mathbb{C})$ case

- Fix a skew-field D and a free (left) D-module $V = D^n$
- The two competing definitions for flags are again equivalent.

Definition. *The* ***noncommutative flags*** $Fl(\gamma)$ *of shape* γ *over* D *are the cosets* $\mathrm{P}_\gamma \backslash \mathrm{GL}_n(D)$.

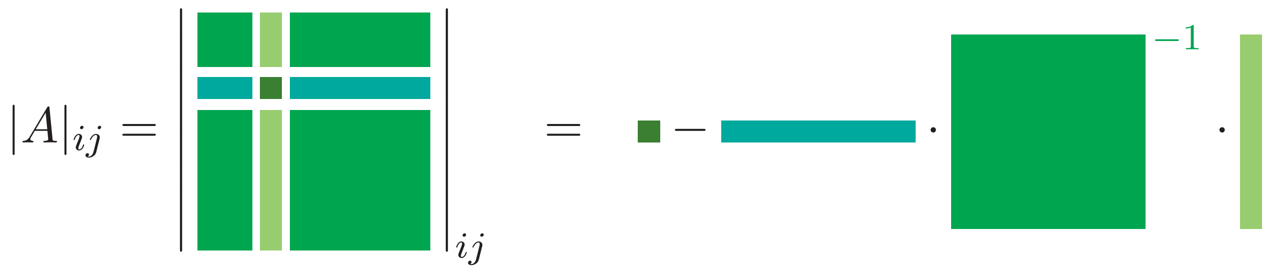
Questions

1. Can we find good coordinates for $Fl(\gamma)$?
2. Can we characterize $Fl(\gamma)$ via relations on these coordinates?
3. Do specializations yield appropriate versions of $(\mathcal{A}_I)$, $(\mathcal{Y}_{I,J})$, and $(\mathcal{C}_{I,J})$?

II.B. **Quasideterminantal Coordinatization**

"A main organizing tool in noncommutative algebra." [G-G-R-W '05]

Definition (Gelfand-Retakh '91)**.** *Given an* $n \times n$ *matrix* A *over some ring* R*, the* (ij)*-quasideterminant* $|A|_{ij}$ *is defined whenever* A^{ij} *is invertible, and in that case,*



Example ($n = 2$**).**

$$|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21} \qquad |A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22}$$

Properties.

- In the commutative case, it looks like $\pm\det A / \det A^{ij}$.
- Has a Cramer's rule.
- Is zero (or undefined) when A is not full rank.
- $\cdots$

II.B. Quasideterminantal Coordinatization

Definition. *The (left)* ***quasi-Plücker coordinate*** *of* A *associated to* (i,j,K)*, for* $i,j \in [n]$ *and* $K \subseteq [n] \setminus \{i,j\}$*, is the ratio*

$$p_{ij}^{K}(A) := |A_{i\cup K}|_{si}^{-1} |A_{j\cup K}|_{sj} \quad (s \in [n]).$$

Properties (G-R '97; L. '04). The quasi-Plücker coordinates $p_{ij}^{K}(A)$ satisfy

- $p_{ij}^{K}(A)$ is independent of s appearing in definition.
- $p_{ij}^{K}(g \cdot A) = p_{ij}^{K}(A)$ for all $g \in \mathrm{GL}_n(D)$.
- **(Question 1: Yes)** If $F(A)$ is some rational function in the a_{ij} which is $\mathrm{P}_\gamma(D)$-invariant, then F is a rational function in the $p_{ij}^{K}(A)$.
- **(Question 2: Maybe)** For $|I| \leq |J| - 1$, pairs of coordinates satisfy

  ($\mathcal{A}_I^{i,j,k}$) ***Alternating*****:** $\quad p_{ij}^{k\cup I} \cdot p_{jk}^{i\cup I} = -p_{ik}^{j\cup I}$.

  ($\mathcal{Y}_{I,J}^{\,i}$) ***Young Symmetry*****:** $\quad \sum_{\ell\in J} p_{i\ell}^{I}(A) \cdot p_{\ell i}^{J\setminus\ell}(A) = 1$.

  ($\mathcal{C}_{??}^{??}$) ***Any More?***

II.C. Specialization to T-generic Flags

- Fix a $\mathbb{K}$-algebra $\mathcal{A}(n)$ on n^2 generators $T = (t_{ij})$.

Definition. *A ring map $\mathcal{A}(n) \hookrightarrow D$ into a division ring D is called T-**inverting** if T and its submatrices are invertible over D. In this case, we say that D has T-**generic flags**, and that $\mathcal{A}(n)$ is the **ring of coordinate functions** for the T-generic matrices.*

Definition. *Let Det be a map from square submatrices of T to $\mathcal{A}(n)$. Call Det an **amenable determinant** if: (i) $\mathrm{Det}\, T_{i,j} = t_{ij}$; (ii) $(\forall I, J)$ some version of row- or column-expansion of $\mathrm{Det}\, T_{I,J}$ holds; (iii) $(\forall I' \subseteq I,\ J' \subseteq J)$ $\mathrm{Det}\, T_{I',J'}$ and $\mathrm{Det}\, T_{I,J}$ almost commute.*

(Question 3: To some extent)

Theorem (L. '05). *If $\mathcal{A}(n)$, D, and Det are as above, then the quasi-Plücker relations give $(\mathcal{A}_I)$, and $(\mathcal{Y}_{I,J})$, and (some) $(\mathcal{C}_{I,J})$ relations in $\mathcal{A}(n)$ for minors $\{\mathrm{Det}\, T_I\}$.*

II.C. Specialization to T-generic Flags

Example. Let T be a matrix of commuting indeterminants. Then

- the quasi-Plücker coordinates become ratios of determinants.
- the quasi-Plücker relations reduce to those which cut out $Fl(n)$ in $\mathbb{P}_\gamma$.

Example. Let T be the **q-generic** matrix from quantum group theory. Then $\mathcal{A}(n) = \mathbb{C}_q[\mathrm{GL}_n]$ has an (Ore) field of fractions, so quasi-Plücker minors make sense when specializing a matrix of formal noncommuting variables to T. Moreover

- the quasi-Plücker coordinates become ratios of quantum determinants.
- the quasi-Plücker relations reduce to those used to define $\mathcal{F}\ell_q(\gamma)$ in case $\gamma = (d, n-d)$.
- for arbitrary γ, some of the commuting relations $(\mathcal{C}_{I,J})^q$ are missing, but . . .

III.A. The Yangian $\boldsymbol{Y_n}$

- Let $t_{ij}^{(r)}$ be a collection of noncommuting variables $(1 \le i, j \le n;\ r = 1, 2, \ldots)$.
- For convenience, write $t_{ij}^{(0)} = \delta_{ij}$.

Definition. *The **Yangian** Y_n for $\mathfrak{gl}_n$ is the algebra $\mathbb{C}\big\langle t_{ij}^{(r)} \mid i, j \in [n];\ r \in \mathbb{N}\big\rangle$ modulo the relations*

$$(\forall\, r, s \ge 0) \quad \big[t_{ij}^{(r+1)}\,,\, t_{kl}^{(s)}\big] - \big[t_{ij}^{(r)}\,,\, t_{kl}^{(s+1)}\big] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}.$$

In particular,

$$\big[t_{ij}^{(1)}\,,\, t_{kl}^{(1)}\big] - \big[\delta_{ij}\,,\, t_{kl}^{(2)}\big] = \delta_{kj} t_{il}^{(1)} - \delta_{il} t_{kj}^{(1)}.$$

Theorem. *The mapping $E_{ij} \mapsto t_{ij}^{(1)}$ is an embedding $U(\mathfrak{gl}_n) \hookrightarrow Y_n$.*

III.B. **The Yangian as deformation of** $\mathbb{C}[\mathrm{GL}_n]$

- Collect the $t_{ij}^{(r)}$ as power series: $t_{ij}(u) := \delta_{ij} + \sum_{r\geq 1} t_{ij}^{(r)} u^{-r}$.
- Collect as a matrix of generators: $T(u) := (t_{ij}(u))$.

Definition. *The* ***Yangian*** Y_n *is the* $\mathbb{C}$*-algebra generated by* $T(u)$ *modulo the relations*

$$\left[t_{ij}(u)\,,\,t_{kl}(v)\right] = \frac{1}{u-v}\left(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)\right).$$

(i.e., equate the coefficients of all $u^{-r}v^{-s}$ *occuring on each side, letting* u, v *commute).*

- a determinant function $\mathbf{t}\square(u)$ is defined for $T(u)$ which is invertible in $Y_n[[u^{-1}]]$.
- Y_n—more precisely $Y_n[[u^{-1}]]$—starts to look like our $\mathcal{A}(n)$.
- Even better, $\mathbf{t}\square(u)$ has the properties required of our $\mathrm{Det}\ldots$

III.B. **The Yangian as deformation of** $\mathbb{C}[\mathrm{GL}_n]$

- For any $a \in \mathbb{C}$, the ***shift*** $t_{ij}(u+a)$ of $t_{ij}(u)$ is defined by $\sum_r t_{ij}^{(r)}(u+a)^{-r}$, i.e., the power series got by expanding $(u+a)^{-r} = \sum_{p \geq 0} \binom{-r}{p} a^p u^{-p-r}$ for each r.

Definition. *For any submatrix $T_{I,J}(u)$ of $T(u)$ of order d, the* ***quantum determinant*** $\mathbf{t}_J^I(u)$ *is given by*

$$\mathbf{t}_J^I(u) = \sum_{\sigma \in \mathfrak{S}_d} (-1)^{\ell(\sigma)} t_{i_{\sigma 1} j_1}(u) t_{i_{\sigma 2} j_2}(u-1) \cdots t_{i_{\sigma d} j_d}(u-d+1).$$

Define $\mathbf{t}_J^I(u+a)$ analogous to $t_{ij}(u+a)$.

Properties. Det is **amenable:**

- $\mathbf{t}_J^I(u)$ is alternating in I, J;
- $\mathbf{t}_J^I(u) = \sum_{i \in I} (-1)^{i-1} t_{ij}(u) \mathbf{t}_{J \setminus j}^{I \setminus i}(u-1)$;
- $\left[\mathbf{t}_{J'}^{I'}(u)\, ,\, \mathbf{t}_J^I(v)\right] = 0.$

III.B. Toward $\mathcal{F}\ell(Y_n)$ for the Yangians

- Simplify notation: If $|I| = d$, write $\mathbf{t}_I(u)$ for $t_I^{[d]}(u)$.

Corollary. *Fix $I \in \binom{[n]}{d}$ and $J \in \binom{[n]}{e}$ for any $1 \leq d \leq e$. The quantum column minors of $T(u)$ satisfy the following relations in $Y_n[[u^{-1}]]$:*

$(\mathcal{A}_I)^u$ **:**

$$\mathbf{t}_I(u) = (-1)^{\ell(\sigma)}\mathbf{t}_{\sigma I}(u)$$

$(\mathcal{Y}_{I,J})^u$ **:**

$$\sum_{\Lambda \subseteq J,\, |\Lambda| = r} (-q)^{-\ell(\Lambda | J \setminus \Lambda)} \mathbf{t}_{I|\Lambda}(u{+}d)\mathbf{t}_{J \setminus \Lambda}(u{+}e{+}r) = 0$$

$(\mathcal{C}_{I,J})^u$ **:**

$$\mathbf{t}_J(u{+}e)\mathbf{t}_I(u{+}d) = \sum_{\Lambda \subseteq J,\, |J \setminus \Lambda| = d} (-1)^{\ell(\Lambda | J \setminus \Lambda)} \mathbf{t}_{J \setminus \Lambda}(u{+}e)\mathbf{t}_{\Lambda | I}(u{+}d)$$

- Can we reproduce Taft and Towber's goals?

Taft and Towber's Work (Briefly)

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Their Goal. Build a **quantum flag algebra** $\mathcal{F}\ell_q(n)$ with the following properties:

1. $\mathcal{F}\ell_q(n)$ reduces to $\mathcal{F}\ell(n)$ when $q \to 1$.

1′. e.g., $\mathcal{F}\ell_q(n)$ and $\mathcal{F}\ell(n)$ share the same basis (*semi-standard Young tableaux*).

2. $\mathcal{F}\ell_q(n)$ is a comodule algebra for the quantum group $\mathbb{C}[\mathrm{GL}_q(n)]$.

3. $\mathcal{F}\ell_q(n)$ is isomorphic to the subalgebra of $\mathbb{C}[\mathrm{GL}_q(n)]$ generated by the minors $\left\{\det_q X_I \,:\, I \in \binom{[n]}{d},\, 1 \leq d < n\right\}$ of its matrix of generators $X = (x_{ij})$.

Theorem (Taft-Towber '91). Success! Put $\mathcal{F}\ell_q(n) := \ldots$

III.B. Toward $\mathcal{F}\ell(Y_n)$ for the Yangians

Partial results

- Collect the formal noncommuting variables $f_I^{(r)}$ (coordinate functions for the $\mathbf{t}_I^{(r)}$) as a power series $f_I(u) := \sum_r f_I^{(r)} u^{-r}$.

Theorem. *Put* $\mathcal{F}(Y_n) := \mathbb{C}\left\langle f_I^{(r)} \right\rangle$ *modulo* $(\mathcal{A}_I)^u$, $(\mathcal{Y}_{I,J})^u$, *and* $(\mathcal{C}_{I,J})^u$. *We have:*

2. $\mathcal{F}(Y_n)$ *is a right comodule algebra for* Y_n.

3′. *There is a ring map* $\mathcal{F}(Y_n) \longrightarrow Y_n$ *onto the subalgebra of* Y_n *generated by the (coefficients of powers of* u^{-1} *appearing in the power series) quantum minors* $\mathbf{t}_I(u)$.

Limits to the analogy (missing relations)

3. $\mathcal{F}(Y_n) \longrightarrow Y_n$ is not injective. For example

$$\left[(\mathbf{t}_I)^{(r)}\,,\, (\mathbf{t}_I)^{(s)} \right] = 0 \qquad (\forall r, s).$$

1. $\mathcal{F}(Y_n)$ does not reduce to $\mathcal{F}\ell(n)$ in any sense. One at least needs relations $(\mathcal{Y}_{I,J})^{v,w}$ and $(\mathcal{C}_{I,J})^{v,w}$ for any v, w, not just $v = u + a, w = u + b$.

III.C. Highest Weight Representations of Y_n

Highest weight modules/vectors

Definition. *A module $M^{\lambda(u)}$ for Y_n is called a highest weight module, with weight(s) $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$, if:*

- *there is a vector $\xi \in M^{\lambda(u)}$ such that $Y_n \cdot \xi = M^{\lambda(u)}$;*
- $t_{aa} \cdot \xi = \lambda_a(u) \cdot \xi$ *for all* $1 \leq a \leq n$ *and moreover,* $\lambda_a(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$*;*
- $t_{ab} \cdot \xi = 0$ *for all* $1 \leq a < b \leq n$*.*

- Criteria for irreducibility and finite-dimensionality of the $M^{\lambda(u)}$ are known (Drinfeld '88, Billig-Futorny-Molev '05).
- There are highest weight vectors all over the place inside $\mathcal{F}(Y_n)\ldots$

III.C. **Highest Weight Representations of** Y_n

- For $f_J^{(r)} \in \mathcal{F}(Y_n)$, define a $T(u)$-action by:

$$t_{ab}(u) \cdot f_J^{(r)} = \delta_{ab} f_J^{(r)} + \delta_{b \in J}\, u^{-1} f^{(r)}_{j_1 \cdots a \cdots j_d}\,,$$

  where $j_1 \cdots a \cdots j_d$ indicates replacement of b by a in J.

- Extend this to an action on monomials $f_{J_1}^{(r_1)} \cdots f_{J_k}^{(r_k)}$ by letting $t_{ab}(u)$ act by derivations.

Theorem. *This action respects the multiplicative relations in* Y_n *and* $\mathcal{F}(Y_n)$*, making* $\mathcal{F}(Y_n)$ *a module algebra, decomposing as highest-weight modules.*

Compare with Parabolic Subalgebras, I

- Fix a composition $\gamma = (\gamma_1, \ldots, \gamma_r)$ of n. Denote its partial sums by

$$\|\gamma\| = \{\gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \cdots + \gamma_{r-1}\}.$$

- Factor $T(u)$ over D as $\mathbb{L} \cdot \mathbb{D} \cdot \mathbb{U}$ with $\mathbb{D}$ block-diagonal of shape γ. Then, $\mathbb{L} = (E_{ij}(u))$ and $\mathbb{U} = (F_{ij}(u))$ are matrices with matrix entries. (e.g., $F_{ij}(u)$ has shape $\gamma_i \times \gamma_j$)

Theorem (Brundan-Kleshchev '05). *Let Y_γ^- (resp. Y_γ^+) denote the* ***parabolic subalgebra*** *of D generated by the (coefficients of the powers of u^{-1} occurring in the power-series in the) entries of the E's (resp. F's).*

1. *Y_γ^- and Y_γ^+ are actually subalgebras of Y_n.*
2. *They generate Y_n, together with the coefficients of $\mathbf{t}_{[i]}(u)$ $(i \in \|\gamma\|)$.*

Compare with Parabolic Subalgebras, II

- Fix γ, $\|\gamma\|$, and Y_γ^+ as above.
- Instead of studying $\mathcal{F}_q(Y_n)$, one can study the ***"pre" flag algebra:***

  The subalgebra of D generated by the quasi-Plücker coordinates $p_{ij}^K(T(u))$, for all choices (i, j, K) with $|K| + 1 \in \|\gamma\|$.
- (Or rather, generated by the coefficients of the powers of u^{-1} appearing therein.)

Corollary. *The algebra Y_γ^+ is the subalgebra of D generated by (the coefficients of ...) the left quasi-Plücker coordinates p_{ab}^K for $K = [k] \setminus a$, $a \leq k < b$, and $k \in \|\gamma\|$.*

- Conclude: the relations $\sum_{\ell \in [j]} p_{i\ell}^{[i-1]} \cdot p_{\ell k}^{[j] \setminus \ell} = 1$ give some new relations among the generators of Y_γ^+.

Questions

- Are there any remaining quasi-Plücker identites to discover?
- Find enough relations to make $\mathcal{F}_q(Y_n) \to Y_n$ injective (and a comodule map).
- Can modifications to the definition of $t_{ab}(u) \cdot f_J^{(r)}$ be made to recover all irreducible (highest weight) Y_n-modules inside $\mathcal{F}(Y_n)$?

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