Hopf structures on binary trees (*variations on a theme*)

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The (known) cast of characters

Polytopes and their Hopf Algebras

Permutahedra:
vertices are *permutations*.
Malvenuto-Reutenauer, ‘95

$$\mathcal{S}Sym$$

Associahedra:
vertices are *planar binary trees*.
Stasheff, ‘63
Loday-Ronco, ‘98
Hivert-Novelli-Thibon, ‘02
Reading, ‘05
Aguiar-Sottile, ‘06

Hypercubes:
vertices are *compositions*.
Gessel, ‘84
Another Family of Polytopes

**Multiplihedra:**
a family nestled between permutahedra and associahedra, vertices are *bi-leveled trees.*

- Stasheff, ‘70
- Iwase-Mimura, ‘89
- Forcey, ‘07
Another Hopf Algebra?

Multiplihedra:

Main Results

- a graded algebra
- an $\mathcal{G}Sym$-module
- a $\mathcal{V}Sym$–Hopf module

and, in an unrelated way, a Hopf algebra!
The Multiplihedra

Fix a map $f$ between tensor categories (not necessarily associative):

$$(C,\cdot) \xrightarrow{f} (D,\cdot)$$

How many ways to **multiply** and **map** three objects $a, b, c$?

$$f(a \cdot (b \cdot c)) \quad f((a \cdot b) \cdot c) \quad f(a) \cdot (f(b) \cdot f(c)) \quad (f(a) \cdot f(b)) \cdot f(c)$$

$$f(a) \cdot f(b \cdot c) \quad f(a \cdot b) \cdot f(c)$$

Stasheff’s Multiplihedra
The Multiplihedra (Poset) $\mathcal{M}_3 \ldots$

$f(a\cdot(b\cdot c))$

\[ f((a\cdot b)\cdot c) \quad f(a)\cdot f(b\cdot c) \]

\[ f(a\cdot b)\cdot f(c) \quad f(a)\cdot(f(b)\cdot f(c)) \]

\[ (f(a)\cdot f(b))\cdot f(c) \]

\[ \ldots \text{of bi-leveled trees} \]
Permutations as Ordered Trees

Permutations:

1 3 2 2 4 1 3 3 1 4 5 2

Read between the leaves.

Ordered Trees (total orders):

Planar Binary Trees (partial orders):

The map $\mathfrak{S}_n \to \mathcal{M}_n \to \mathcal{Y}_n$
Understanding the map $\mathfrak{S}_n \to \mathcal{M}_n \to \mathcal{Y}_n$

Here, $\beta$ forgets most of the total ordering (remembering only relation to the leftmost node); the map $\phi$ forgets even that information.

Write $\tau = \phi \circ \beta$. 
Hopf Structures on Binary Trees

The Hopf algebra $\mathcal{Y}Sym = \bigoplus_{n \geq 0} \text{span}\{F_t \mid t \in \mathcal{Y}_n\}$ is defined via "splitting" and "grafting" operations on trees.

Definition (Splitting)

A $p$-splitting of a tree $t$ is a forest $(t_0, \ldots, t_p)$, gotten by choosing $p$ leaves of $t$ and splitting their branchings down to the root.

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Hopf Structures on Binary Trees

The Hopf algebra $\mathcal{V} \text{Sym} = \bigoplus_{n \geq 0} \text{span}\{F_t \mid t \in \mathcal{V}_n\}$ is defined via “splitting” and “grafting” operations on trees.

Definition (Grafting)

A grafting of a forest $(t_0, \ldots, t_p)$ onto a tree $s$ with $p$ nodes is the tree gotten by grafting the root of $t_i$ to the $i^{th}$ leaf of $s$.
Hopf Structures on Binary Trees

The Hopf algebra $\mathcal{Y} \text{Sym} = \bigoplus_{n \geq 0} \text{span}\{F_t \mid t \in \mathcal{Y}_n\}$ is defined via splitting $(\cdot \overset{\gamma}{\rightarrow} \cdot)$ and grafting $(\cdot / \cdot)$ operations on trees.

Definition (Multiplication and Comultiplication)

Given two binary trees $s$ and $t$ ($s$ having $p$ nodes), put

$$F_t \cdot F_s = \sum_{t \overset{\gamma}{\rightarrow} (t_0, t_1, \ldots, t_p)} F_{(t_0, t_1, \ldots, t_p) / s} \quad \text{and} \quad \Delta(F_t) = \sum_{t \overset{\gamma}{\rightarrow} (t_0, t_1)} F_{t_0} \otimes F_{t_1}.$$ 

(Loday-Ronco, '98, '02) Analogous operations on ordered trees give the Hopf algebra $\mathcal{G} \text{Sym}$ of permutations.

(Loday-Ronco, '98, '02) The map $\tau : \mathcal{G} \text{Sym} \rightarrow \mathcal{Y} \text{Sym} \quad (F_t \mapsto F_{\tau(t)})$ is a Hopf algebra map.
Let $\mathcal{M} \text{Sym} = \bigoplus_{n \geq 0} \text{span}\{F_t \mid t \in \mathcal{M}_n\}$ be the space of bi-leveled trees.

**Proposition (F-L-S)**

*Splitting & grafting may be extended to bi-leveled trees, making $\mathcal{M} \text{Sym}$ into an algebra . . . but not a coalgebra.*
Let $\mathcal{M} \text{Sym} = \bigoplus_{n \geq 0} \text{span}\{F_t \mid t \in \mathcal{M}_n\}$ be the algebra of bi-leveled trees.

**Theorem (F-L-S)**

- $\mathcal{M} \text{Sym}$ is a right (and left) module over $\mathcal{V} \text{Sym}$ and $\mathcal{G} \text{Sym}$.
- The maps $\mathcal{G} \rightarrow \mathcal{M} \rightarrow \mathcal{V}$ extend to algebra maps respecting the structures above.
- $\mathcal{M} \text{Sym}$ and $\mathcal{M} \text{Sym}_+$ are right $\mathcal{V} \text{Sym}$–Hopf modules . . .
  
  . . . in several ways.
- Neither is a $\mathcal{G} \text{Sym}$–Hopf module.
**MSym in the Monomial Basis**

- Define new basis \( \{ M_t \mid t \in M. \} \) via *Möbius Inversion* in the posets \( M. \).
- (Aguiar-Sottile, ’05, ’06) hidden structure was revealed in analogous bases for \( \mathcal{G}Sym \) and \( \mathcal{V}Sym \).

**Main Theorem (F-L-S)**

\( (\rho : MSym_+ \rightarrow MSym_+ \otimes VSym) : \rho(M_s) \) is the sum of ways to “prune” along right branch without clipping between circled nodes.

\[
\rho(\quad) = \quad \otimes | + \quad \otimes \quad + \quad \otimes \quad
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\[
\rho(\quad) = \quad \otimes |
\]
\( M \text{Sym} \text{ in the Monomial Basis} \)

- The **Fundamental Theorem of Hopf Modules** states that
  \[ M \simeq M^\text{co} \otimes H, \]
  where \( M^\text{co} := \{ m : \rho(m) = m \otimes 1 \} \) are the coinvariants.

**Corollary (F-L-S)**

*In the Hopf module structure on \( M \text{Sym}_+ \), a basis of coinvariants is given by \( M_s \), with \( s \) ranging over bi-leveled trees with no un-circled nodes on the rightmost branch.*

- Enumeration of coinvariants [OEIS, A127632]
  \[
  \frac{\text{Hilb}(M \text{Sym}_+)}{\text{Hilb}(Y \text{Sym})} = t + t^2 + 3t^3 + 11t^4 + 44t^5 + 185t^6 + \cdots
  \]
Future Questions

- Coinvariants for the other $\mathcal{Y}Sym$–Hopf module structure?
  (Much harder; seems to involve a restricted Möbius function.)

- What happened to $QSym$?
  (We expect to find many new Hopf modules for $\mathcal{Y}Sym$ and $QSym$.)

- Connections between the algebra and the geometry?
  (See also graph multiplihedra in arXiv:0807.4159.)

References

- Our data:
  http://www.math.tamu.edu/~sottile/pages/HopfAlgebras/MSym/index.html

- An extended abstract:
  http://www.math.tamu.edu/~lauve/papers/MSym_fpsac.pdf
Towards a Proof

- It would be nice to embed $\mathcal{M}_n$ in $\mathfrak{S}_n$.
- A similar idea was used in (Aguiar-Sottile, ‘06).
- They found a section $\iota: \mathcal{Y}_n \rightarrow \mathfrak{S}_n$ of the natural map $\tau: \mathfrak{S}_n \rightarrow \mathcal{Y}_n$.
- They showed that the pair gives a Galois connection between the posets, i.e., a pair of order-preserving maps $(\tau, \iota)$ satisfying
  \[
  \tau(w) \leq t \iff w \leq \iota(t) \quad \text{for all } t \in \mathcal{Y}_n, w \in \mathfrak{S}_n.
  \]
- The rest was “easy.”
Towards an embedding $\mathcal{M}_n \hookrightarrow \mathcal{S}_n$

An interval in $\mathcal{M}_7$ ... and its preimage in $\mathcal{S}_7$.

This choice doesn't give an embedding.

This choice doesn't either.

This choice works.
Towards an embedding $\mathcal{M}_n \hookrightarrow \mathcal{S}_n$

\[\begin{pmatrix}\left(\min \max\right) \end{pmatrix} = \begin{pmatrix}6 & 7 & 1 & 2 & 5 & 4 & 3 & 8 & 9\end{pmatrix} = 674598231.\]

**Proposition (F-L-S)**

The section $\iota : \mathcal{M} \to \mathcal{S}$, given by $\iota(t) = \left(\frac{\max}{\min}\right)(t)$ is an embedding of posets.
Curious aspects of the proof

$(\beta, \iota)$ does not form a Galois connection

$$\beta(3671425) \leq t \ldots \text{but } 3671425 \not\leq \iota(t).$$
... but \((\beta, \iota)\) does form an interval retract

Two posets \(P\) and \(Q\) are related by an **interval retract** if there are order-preserving maps \(\beta : P \to Q\) and \(\iota : Q \to P\) satisfying

\[
\beta(\iota(t)) = t \quad \text{and} \quad \beta^{-1}(t) \text{ is an interval} \quad \forall t.
\]

**Theorem (F-L-S)**

If \(P\) and \(Q\) are two posets related by an interval retract \((\beta, \iota)\), then the Möbius functions \(\mu_P\) and \(\mu_Q\) are related by

\[
\forall s < t \in Q \quad \sum_{v \in \beta^{-1}(s), \ w \in \beta^{-1}(t)} \mu_P(v, w) = \mu_Q(s, t).
\]
Curious aspects of the proof

...but $(\beta, \iota)$ does form an interval retract

- Two posets $P$ and $Q$ are related by an interval retract if there are order-preserving maps $\beta : P \to Q$ and $\iota : Q \to P$ satisfying

$$\beta(\iota(t)) = t \quad \text{and} \quad \beta^{-1}(t) \text{ is an interval} \quad \forall t.$$ 

**Proposition (F-L-S)**

The maps $\beta : \mathcal{S}_n \to \mathcal{M}_n$ and $\iota : \mathcal{M}_n \to \mathcal{S}_n$ form an interval retract between $\mathcal{S}_n$ and $\mathcal{M}_n$. In particular,

$$\beta\left(\sum_{\sigma \in \beta^{-1}(t)} M_\sigma\right) = M_t.$$ 

- This is good enough to prove our main theorem.
The future

A host of new polytopes waiting to be explored.

The polytopes $\mathcal{IG}_d$ and $\mathcal{IG}_r$ also appear in Devadoss-Forcey, ‘08.

(See that paper for notations.)

We focus on $\mathcal{I}(4)$ here.
The future

The vertices of each polytope are indexed by certain types of planar binary trees.

We focus on bi-leveled trees here.
The weak order on $\mathfrak{S}_4$, highlighting the fibers of the maps $\beta$ and $\tau$, respectively.
Appendix

Coinvariant Enumeration

for the two separate \( \mathcal{Y} \text{Sym} \)-Hopf module structure on \( \mathcal{M} \text{Sym} \).

\[
\frac{\text{Hilb}(\mathcal{M}_+)}{\text{Hilb} (\mathcal{Y}_+)} = 0 + t + t^2 + 3t^3 + 11t^4 + 44t^5 + 185t^6 + \cdots
\]

\[
\frac{\text{Hilb}(\mathcal{M}_\ast)}{\text{Hilb} (\mathcal{Y}_\ast)} = 1 + 0t + 0t^2 + t^3 + 6t^4 + 30t^5 + 143t^6 + \cdots
\]
From Painted Trees to Bi-leveled Trees

An $f$-mapping

$$f(a) \cdot (f(b \cdot c) \cdot f(d))$$

A painted tree

1. Add a branch on the left, marking first application of $f$.  
2. Circle new node, and all nodes corresponding to multiplication in $\mathcal{D}$. 
3. Forget the branch paintings.

A bi-leveled tree
The Multiplihedra (Posets) $M_3$ & $M_4$