The Markoff Condition and Central Words

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Abstract. During his study of quadratic forms $AX^2+2BXY+CY^2$, Markoff [3] introduced a condition that a bi-infinite word $\boldsymbol{w}$ must satisfy in order to represent the descriminant $\sqrt{B^2-AC}$. The condition involves certain finite factors of $\boldsymbol{w}$. We prove that these factors are *palindromic* and comprise the *central words* in the theory of Sturmian words.

Markoff Condition

To a bi-infinite word $\boldsymbol{w} = \cdots w_{-2}w_{-1}w_0w_1w_2\cdots$ with $w_i \in \mathbb{Z}_{>0}$, associate the numbers $\lambda_i(\boldsymbol{w}) = [w_i, w_{i+1}, w_{i+2}, \ldots] + [0, w_{i-1}, w_{i-2}, \ldots]$, where

$$[a, b, c, \ldots] := a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{\ddots}}} .$$

Lemma (Markoff [2])**.** *1) If $\sup_i \lambda_i(\boldsymbol{w}) \leq 3$, then $\boldsymbol{w} \in \{1,2\}^{\mathbb{Z}}$ and the blocks of 1s & 2s have even length.*

2) If $\sup_i \lambda_i(\boldsymbol{w}) < 3$, then the word $\boldsymbol{w}' := \cdots w_{-4}w_{-2}w_0w_2w_4\cdots$ satisfies: every factor

$$\boldsymbol{w}' = \cdots xy \cdots ,$$

with $\{x, y\} = \{1, 2\}$, extends to a factor

$$\boldsymbol{w}' = \cdots y \overleftarrow{m}\, xy\, m\, x \cdots$$

for some word m (here, e.g., $\overleftarrow{aab} := baa$).

Example. The following (periodic & aperiodic) words satisfy the Markoff condition at each xy.

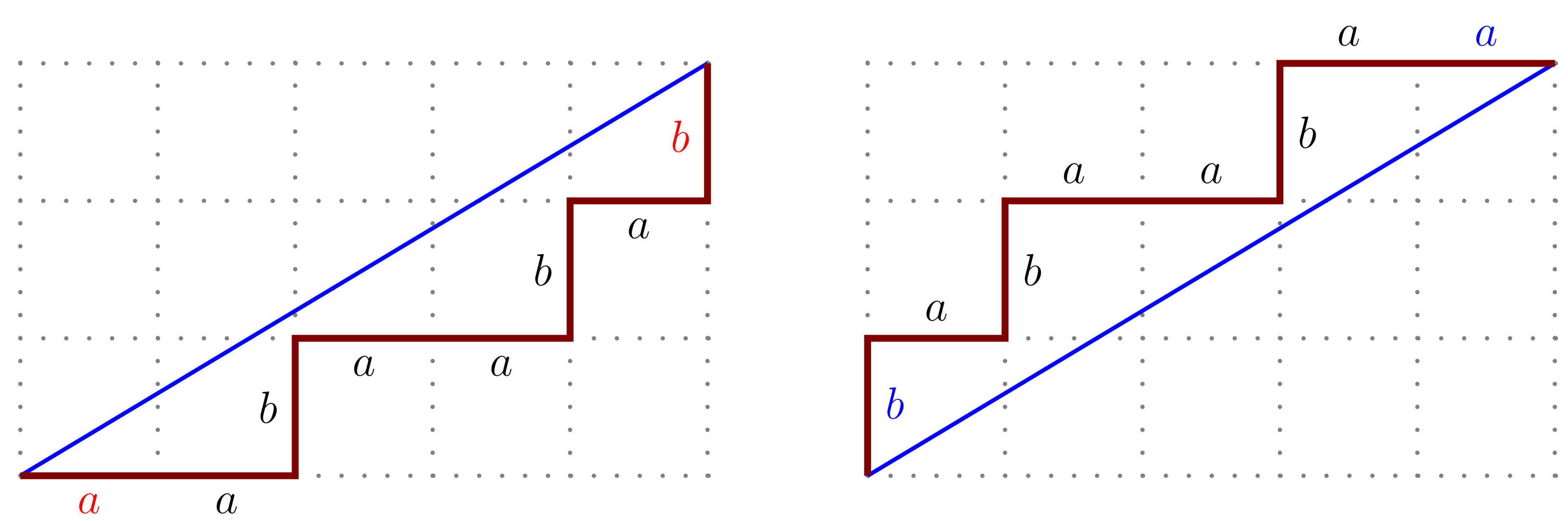
$$\cdots aabaabab\ aabaaba\underline{b\ a}abaabab\ aab \cdots$$

$$\cdots baababababa\underline{ab}ababaababa\underline{ba}baa \cdots$$

Question (Reutenauer)**.** Notice that all of the *Markoff words* m above are palindromes, i.e., $m = \overleftarrow{m}$. Is this always the case?

Christoffel Words

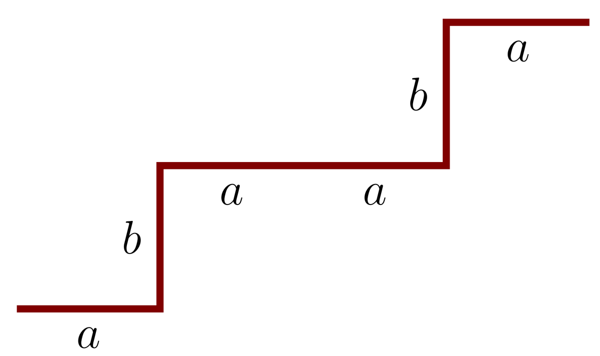
Given relative prime $p, q \in \mathbb{Z}_{>0}$, draw the line segment from $(0,0)$ to (p,q) and the two lattice paths that follow it as closely as possible.



$L(5,3) = aabaabab$ $\quad$ $U(5,3) = babaabaa$

The words $L(p,q)$ and $U(p,q)$ formed by encoding x-steps as a and y-steps as b are called (*lower* and *upper*) *Christoffel words*.

Lemma (Geometric [1])**.** *If $L(p,q) = amb$ then $U(p,q) = bma$ and m is a palindrome.*



$= abaaba = \overleftarrow{abaaba}$

Connections & Results

The "central words" m in the lemma turn out to be *the* central words from the theory of Sturmian words:

A word w is *balanced* if $\big||u|_a - |v|_a\big| \leq 1$ for all factors u and v of w with $|u| = |v|$. A word m is *central* iff amb and bma are balanced.

Theorem (Reutenauer [4])**.** *A bi-infinite word satisfies the Markoff condition iff it is balanced.*

Theorem (G-L-S)**.** *Central words are palindromes. A word m is central if and only if m is a Markoff word.*

Theorem (G-L-S)**.** *A word m is a central word if and only if amb or bma is a Christoffel word.*

References

[1] J. Berstel, A. Lauve, C. Reutenauer, F. V. Saliola, *Combinatorics on words*, CRM Proceedings & Lecture Notes, to appear.

[2] T. W. Cusick, M. E. Flahive, *The Markoff and Lagrange spectra*, v. 30 of Mathematical Surveys and Monographs, AMS, 1989.

[3] A. Markoff, *Sur les formes quadratiques binaires indéfinies*, Math. Ann. 15 (3) (1879) 381–406.

[4] C. Reutenauer, *On Markoff's property and Sturmian words*, Math. Ann. 336 (1) (2006) 1–12.