Invariant and Coinvariant Spaces for Noncommutative Symmetric Polynomials

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(classical)

The Algebra $S^{\mathfrak{S}}$ of (Commutative) Symmetric Polynomials

Setup: Fix a field $\mathbb{K}$ containing $\mathbb{Q}$ and consider the polynomial ring $S = \mathbb{K}[\mathbf{x}]$ over the set of ordered variables $\mathbf{x} = \{x_1 < x_2 < \cdots\}$.

Give an **$\mathfrak{S}_{|\mathbf{x}|}$-module structure** to S by extending $\boldsymbol{\sigma \cdot x_i := x_{\sigma(i)}}$ linearly and multiplicatively.

Question: (An Invariant Theory classic)

What does the space $S^{\mathfrak{S}}$ of $\mathfrak{S}_{|\mathbf{x}|}$ invariants look like?

Answer: (You all know it. . .)

The Algebra $S^{\mathfrak{S}}$ of (Commutative) Symmetric Polynomials

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Question: (An Invariant Theory classic)

What does the space $S^{\mathfrak{S}}$ of $\mathfrak{S}_{|\mathbf{x}|}$ invariants look like?

Answer: (You all know it, if not you should)

- $S^{\mathfrak{S}}$ is a polynomial ring isomorphic to S;
- A basis for $S^{\mathfrak{S}}$ in S is given by the **monomial symmetric functions**

$$\Big\{ m_\lambda \,:\, \lambda \text{ a partition with at most } |\mathbf{x}| \text{ parts} \Big\}.$$

The Coinvariant Space $S_{\mathfrak{S}}$

Let $S_+^{\mathfrak{S}}$ denote the space of **symmetric polynomials without constant term** and let $\langle S_+^{\mathfrak{S}} \rangle$ denote the corresponding ideal inside S.

Motivation: If V is an $\mathfrak{S}_{|\mathbf{x}|}$-submodule of S, then $f(\mathbf{x})V \simeq V$ for each $f(\mathbf{x}) \in S^{\mathfrak{S}}$. Let's remove the redundancies:

The interesting part of S is the structure of the **coinvariant space**

$$S_{\mathfrak{S}} := S/\langle S_+^{\mathfrak{S}} \rangle.$$

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The interesting part of S is the structure of the **coinvariant space**

$$S_{\mathfrak{S}} := S / \langle S_+^{\mathfrak{S}} \rangle.$$

Theorem (Chevalley): *There is an $\mathfrak{S}_{|\mathbf{x}|}$-module isomorphism*

$$S \simeq S_{\mathfrak{S}} \otimes S^{\mathfrak{S}}.$$

Moreover, $S_{\mathfrak{S}}$ is isomorphic to the regular representation $\mathbb{K}\mathfrak{S}_{|\mathbf{x}|}$. Moreover, there is an explicit realization of $S_{\mathfrak{S}}$ inside S as $(S_+^{\mathfrak{S}})^{\perp}$ for a certain inner product (the **"Harmonic Polynomials"**).

Dimension Enumeration (1/2)

Definition: Given a graded vector space $V = \bigoplus_{d \geq 0} V_d$, let $\mathrm{Hilb}_t(V)$ denote the formal series

$$\mathrm{Hilb}_t(V) = \sum_{d \geq 0} v_d t^t \qquad (v_d = \dim V_d)$$

that enumerates the dimensions of the graded pieces of V.

Corollary: *(Immediate from Chevalley). The formal series* $\mathrm{Hilb}_t(S)/\mathrm{Hilb}_t(S^{\mathfrak{S}})$ *has positive integer coefficients.*

Dimension Enumeration (2/2)

No Need for Chevalley: In fact, the quotient is **obviously positive:**

It's easy to see that

$$\mathrm{Hilb}_t(S) = \prod_{x_i \in \mathbf{x}} \frac{1}{1-t} \quad \text{and} \quad \mathrm{Hilb}_t(S^{\mathfrak{S}}) = \prod_{x_i \in \mathbf{x}} \frac{1}{1-t^i},$$

so, the quotient is

$$\mathrm{Hilb}_t(S_{\mathfrak{S}}) = \prod_{x_i \in \mathbf{x}} \frac{1-t^i}{1-t} = \prod_{x_i \in \mathbf{x}} \left(1 + t^1 + \cdots t^{i-1}\right).$$

This is the familiar $[n]_t!$ sum.

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(also nice for other finite refl. groups)

This is the familiar $[n]_t!$ sum.

(Not surprising since $\dim \mathbb{K}\mathfrak{S}_n = n!$.)

(Things won't be so easy in what follows.)

(also classical)

The Algebra $T^{\mathfrak{S}}$ of (NonCommutative) Symmetric Polynomials

Setup: Now consider the noncommutative polynomial ring $T = \mathbb{K}\langle \mathbf{x} \rangle$ over the set of ordered variables $\mathbf{x} = \{x_1 < x_2 < \cdots\}$.

Give an $\mathfrak{S}_{|\mathbf{x}|}$-module structure to T by extending $\sigma \cdot x_i := x_{\sigma(i)}$ linearly and multiplicatively.

Question: (M. C. Wolf (1936) asks)

What does the space $T^{\mathfrak{S}}$ of $\mathfrak{S}_{|\mathbf{x}|}$ invariants look like?

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Give an $\mathfrak{S}_{|\mathbf{x}|}$-module structure to T by extending $\sigma \cdot x_i := x_{\sigma(i)}$ linearly and multiplicatively.

Question: (M. C. Wolf (1936) asks)

What does the space $T^{\mathfrak{S}}$ of $\mathfrak{S}_{|\mathbf{x}|}$ invariants look like?

Answer: (We might have expected it)

- $T^{\mathfrak{S}}$ is a polynomial ring (though not isomorphic to T);
- A basis for $T^{\mathfrak{S}}$ is given by the **monomial symmetric functions**

$$\left\{ M_{\mathbf{A}} \;:\; \mathbf{A} \text{ a set partition with at most } |\mathbf{x}| \text{ parts} \right\}.$$

The Basis of $T^{\mathfrak{S}}$ (Set Partitions)

Notation: Given a set partition, e.g., of [5], drop the braces and order the blocks by minimum elements to simplify/standardize notation:

$$\{\{2,3\},\{1,4\},\{5\}\} \rightsquigarrow \mathbf{14.23.5}.$$

The Basis: Set partitions define monomial symmetric functions by recording which positions in a monomial carry the same variable:

$$M_{14.23.5} = x_1 x_2 x_2 x_1 x_3 + x_7 x_4 x_4 x_7 x_3 + \cdots$$
$$\quad\uparrow\uparrow$$

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$$\uparrow\uparrow \qquad \uparrow \quad \uparrow \qquad\qquad \uparrow \quad \uparrow$$

A $\mathfrak{S}_{|\mathbf{x}|}$-Module Decomposition of T

Let $T_+^{\mathfrak{S}}$ denote the symmetric polynomials without constant term, and let $\langle T_+^{\mathfrak{S}} \rangle$ denote the corresponding **left ideal** inside T.

As in the commutative case, define the **coinvariant space** by

$$T_{\mathfrak{S}} := T / \langle T_+^{\mathfrak{S}} \rangle.$$

Theorem (N. Bergeron-Reutenauer-Rosas-Zabrocki): *There is an $\mathfrak{S}_{|\mathbf{x}|}$-module isomorphism*

$$T \simeq T_{\mathfrak{S}} \otimes T^{\mathfrak{S}}.$$

Moreover, $T_{\mathfrak{S}}$ is isomorphic to the $\mathfrak{S}_{|\mathbf{x}|}$-submodule of T given by the **"Noncommutative Harmonic Polynomials"** inside T (i.e., $(T_+^{\mathfrak{S}})^{\perp}$ for a certain bilinear form on T).

How Common is This Decomposition?

Setup: Return to S for a moment. Fix a finite group G and a multiplicative G-action on S, i.e.,

$$\sigma(ab) = \sigma(a)\sigma(b) \qquad (\forall\, a, b \in S)\,(\forall\, \sigma \in G).$$

Then:

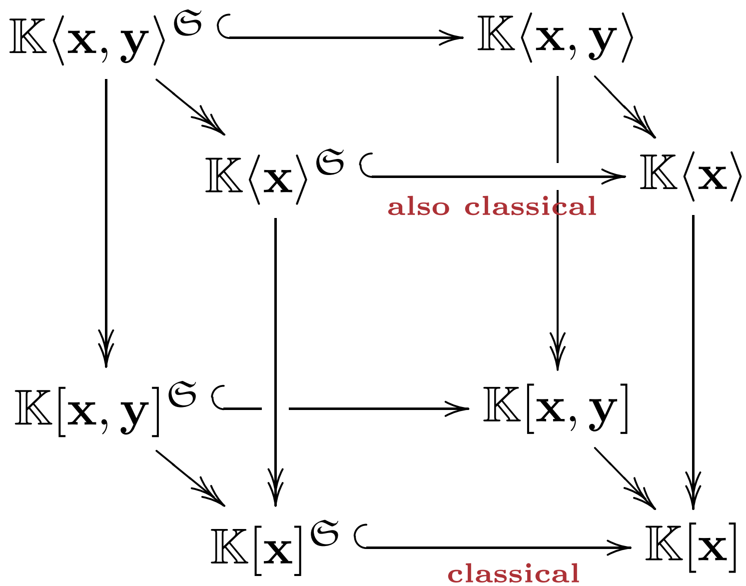
Theorem (Shephard-Todd): *The ring S decomposes as*

$$S \simeq S_G \otimes S^G$$

if and only if *G is a (pseudo-) reflection group and the action on S_1 is the action of G on its defining vector space.*

A Motivating Picture

A Cube of Hopf Algebras: François once showed me this:



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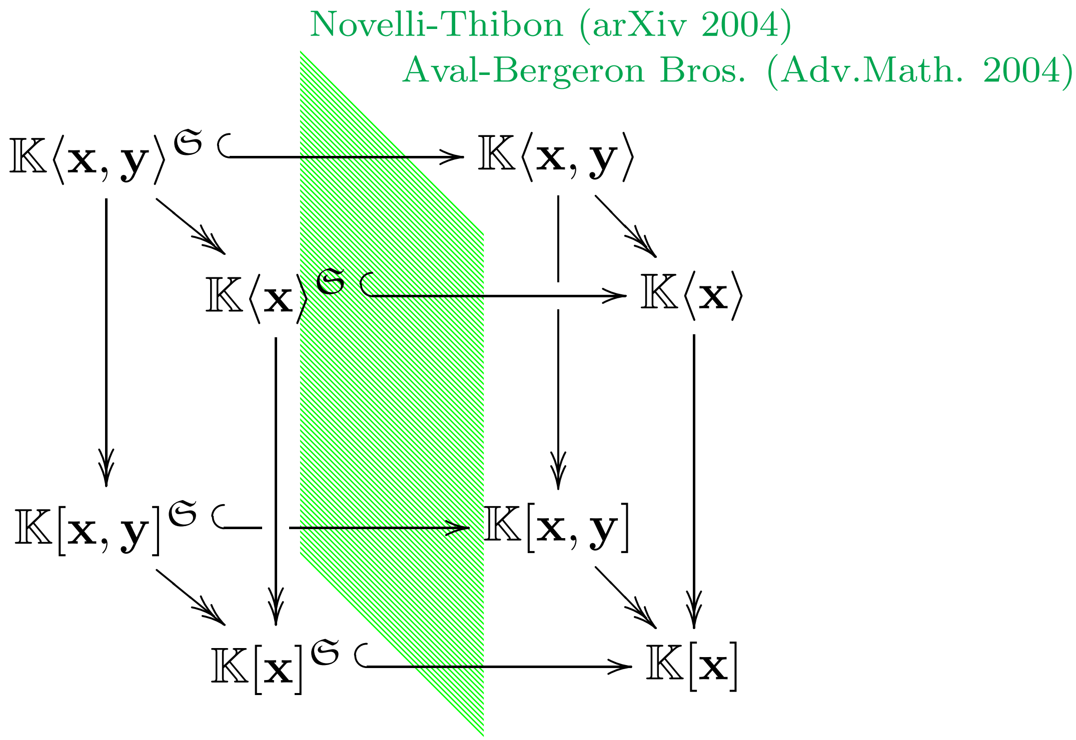

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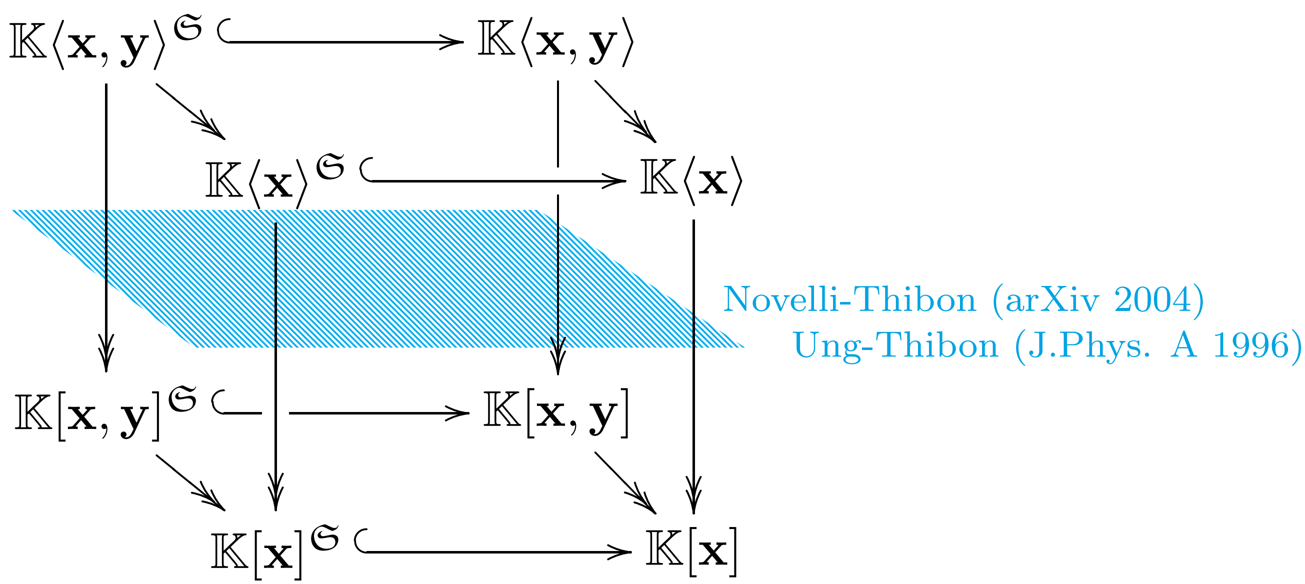

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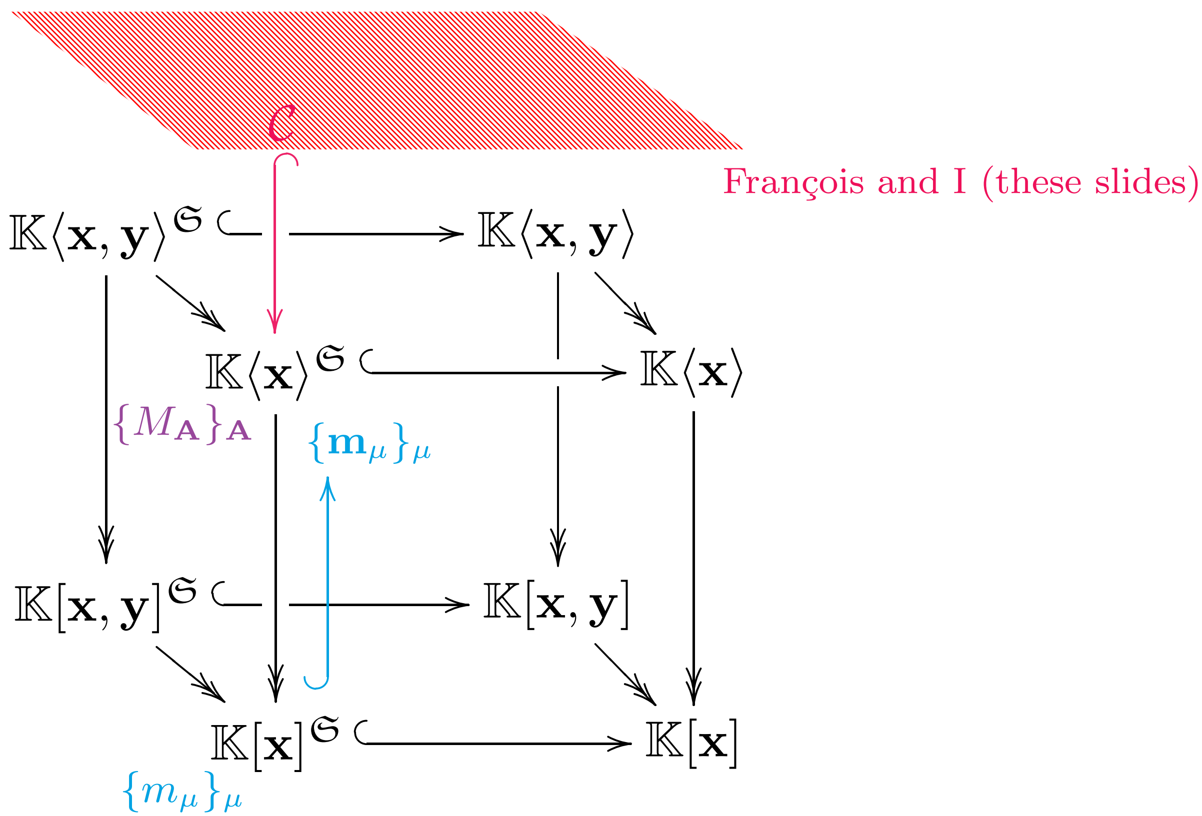

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Main Question

Is there a subspace $\mathcal{C}$ within the invariants $T^{\mathfrak{S}}$ so that

$$\boldsymbol{T}^{\mathfrak{S}} \simeq \boldsymbol{\mathcal{C}} \otimes \boldsymbol{S}^{\mathfrak{S}}?$$

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Is there a subspace $\mathcal{C}$ *within the invariants* $T^{\mathfrak{S}}$ *so that*

$$\boldsymbol{T}^{\mathfrak{S}} \simeq \boldsymbol{\mathcal{C}} \otimes \boldsymbol{S}^{\mathfrak{S}} ?$$

↑

and if so, what interpretation should be given to this symbol?

The "Place Action" on T

Definition: Let T_d denote the homogeneous degree d polynomials in T. Define an **$\mathfrak{S}_d$ action on T_d** by permuting "places."

Examples: (in cycle notation)

$$(23) * x_1x_4x_7x_2 = x_1x_7x_4x_2$$

$$(1324) * x_1x_4x_7x_2 = x_2x_7x_1x_4.$$

Fact (Rosas-Sagan): Under the place action, the $M_{\mathbf{A}}$'s satisfy

$$\rho * M_{\mathbf{A}} = M_{\rho \cdot \mathbf{A}},$$

with $\rho \cdot \mathbf{A}$ the usual permutation action on set partitions.

(still classical?)

The "Place Action" on $\mathcal{N}$

Notation: Each set partition has an underlying **shape:** an integer partition gotten by recording the set sizes ($14.23.5 \mapsto \lambda(14.23.5) = \mathbf{221}$).

An Invariant Function: Let $\mathcal{N} = T^{\mathfrak{S}}$ and $\mathcal{N}_\mu$ be the subspace of monomial functions of shape μ. The sum

$$\mathbf{m}_\mu := \sum_{\lambda(\mathbf{A})=\mu} M_{\mathbf{A}}$$

is a symmetric function (under the place action) inside $\mathcal{N}_\mu$.

Fact: These are the only invariants, i.e., the place invariants $\mathcal{N}^{\mathfrak{S}}$ are in 1-1 correspondence with $S^{\mathfrak{S}}$.

(probably classical)

The "Place Action" on $\mathcal{N}_\mu$

Definition: The **Frobenius characteristic** of an $\mathfrak{S}_d$-module $\mathcal{V}$ is the symmetric function

$$\mathrm{Frob}(\mathcal{V}) = \sum_{\mu \vdash d} v_\mu s_\mu,$$

where s_μ is the usual Schur function and v_μ is the multiplicity of the irreducible $\mathfrak{S}_d$ module $\mathcal{V}(\mu)$ inside $\mathcal{V}$.

Proposition: *In light of the action $\rho * M_{\mathbf{A}} = M_{\rho \cdot \mathbf{A}}$, we deduce that*

$$\mathrm{Frob}(\mathcal{N}_\mu) = h_{d_1}[h_1]\, h_{d_2}[h_2] \cdots h_{d_k}[h_k],$$

$\mu = 1^{d_1} 2^{d_2} \cdots k^{d_k}$, *where $f[g]$ denotes the* ***plethysm*** *of two symmetric functions and h_j is the j-th complete symmetric function.*

Existence of $\mathcal{C}$, Case $|\mathbf{x}| = \infty$

Facts (R-S, B-R-R-Z): The map $\boldsymbol{\iota} : S^{\mathfrak{S}} \to \mathcal{N}$ $(m_\mu \mapsto \mathbf{m}_\mu)$ satisfies:

$$\mathbf{ab} \circ \boldsymbol{\iota} = \mathrm{id} \quad \text{and} \quad \Delta \circ \boldsymbol{\iota} = (\boldsymbol{\iota} \otimes \boldsymbol{\iota}) \circ \Delta.$$

That is, $\boldsymbol{\iota}$ is a **"coalgebra splitting"** of the exact sequence of Hopf algebras $\mathcal{N} \to S^{\mathfrak{S}} \to 0$.

Theorem (Blattner-Cohen-Montgomery): *In such a case, the subspace*

$$\mathcal{C} = \{h \,:\, (\mathrm{id} \otimes \mathbf{ab}) \circ \Delta(h) = h \otimes 1\}$$

yields an isomorphism of vector spaces

$$\mathcal{N} \simeq \mathcal{C} \otimes S^{\mathfrak{S}}.$$

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$$\mathcal{C} = \{h \,:\, (\mathrm{id} \otimes \mathbf{ab}) \circ \Delta(h) = h \otimes 1\}$$

yields a ***"shape graded"*** *isomorphism of vector spaces*

$$\mathcal{N} \simeq \mathcal{C} \otimes S^{\mathfrak{S}}.$$

The (Shape Graded) Isomorphism (1/2)

Decomposing $\mathcal{N}$:

$\mathrm{Hilb}_t(\mathcal{N}) = 1 + (1)t + (2)t^2 + (5)t^3 + (15)t^4 + (52)t^5 + \cdots$

The (Shape Graded) Isomorphism (1/2)

Decomposing $\mathcal{N}$:

$$\mathrm{Hilb}_t(\mathcal{N}) = 1 + (z_1)t + (z_2 + z_1^2)t^2 + (z_3 + 3z_1z_2 + z_1^3)t^3$$
$$+ (z_4 + 4z_1z_3 + 3z_2^2 + 6z_2z_1^2 + z_1^4)t^4$$
$$+ (z_5 + 5z_1z_4 + 10z_3z_2 + 10z_3z_1^2 + 15z_1z_2^2 + 10z_2z_1^3 + z_1^5)t^5 + \cdots$$

The (Shape Graded) Isomorphism (1/2)

Decomposing $\mathcal{N}$:

$$\begin{aligned}\mathrm{Hilb}_t(\mathcal{N}) = 1 &+ (z_1)t + (z_2 + z_1^2)t^2 + (z_3 + 3z_1z_2 + z_1^3)t^3 \\ &+ (z_4 + 4z_1z_3 + 3z_2^2 + 6z_2z_1^2 + z_1^4)t^4 \\ &+ (z_5 + 5z_1z_4 + 10z_3z_2 + 10z_3z_1^2 + 15z_1z_2^2 + 10z_2z_1^3 + z_1^5)t^5 + \cdots\end{aligned}$$

$$= \left[1 + (2z_1z_2)t^3 + (2z_2^2 + 3z_1z_3 + 3z_2z_1^2)t^4 + \cdots\right]$$

$$\times \left[1 + (z_1)t + (z_2 + z_1^2)t^2 + (z_{1^3} + z_1z_2 + z_3)t^3 \cdots\right]$$

The (Shape Graded) Isomorphism (2/2)

Decomposing $\mathcal{N}$:

1	$\leftrightarrow$	1	12.34	$\leftrightarrow$	12.34	12345	$\leftrightarrow$	12345
12	$\leftrightarrow$	12		$\cdots$		1234.5	$\leftrightarrow$	1234.5
1.2	$\leftrightarrow$	1.2	134.2	$\leftrightarrow$	134.2	123.45	$\leftrightarrow$	12.34.5
			1.234	$\leftrightarrow$	1.234		$\cdots$	
123	$\leftrightarrow$	123	14.23	$\leftrightarrow$	14.23	1235.4	$\leftrightarrow$	15.23.4
12.3	$\leftrightarrow$	12.3	14.2.3	$\leftrightarrow$	14.2.3	1245.3	$\leftrightarrow$	1245.3
13.2	$\leftrightarrow$	13.2	13.24	$\leftrightarrow$	13.24	1.2345	$\leftrightarrow$	1.24.35
1.23	$\leftrightarrow$	1.23	124.3	$\leftrightarrow$	124.3		$\cdots$	
1.2.3	$\leftrightarrow$	1.2.3	1.2.34	$\leftrightarrow$	1.2.34	13.2.45	$\leftrightarrow$	13.2.45
			1.24.3	$\leftrightarrow$	1.24.3	1.23.45	$\leftrightarrow$	1.23.45
1234	$\leftrightarrow$	1234	13.2.4	$\leftrightarrow$	13.2.4	134.2.5	$\leftrightarrow$	134.2.5
123.4	$\leftrightarrow$	123.4	1.23.4	$\leftrightarrow$	1.23.4	1.234.5	$\leftrightarrow$	1.234.5
							$\cdots$	

Algebraic Description of $\mathcal{C}$

Definition: Denote by $\mathbb{A}$ the **atomic set partitions:** those partitions $\mathbf{A} \vDash [d]$ that cannot be split into two partitions, $\mathbf{A}' \vDash \{1, 2, \ldots, k\}$ and $\mathbf{A}'' \vDash \{k+1, \ldots, d\}$.

Examples: (of atomic set partitions)

14.25.3 (YES) 13.24.7.5.6 (YES) 13.245 | 58.7 (NO)

Theorem (Hivert-Novelli-Thibon): *$\mathcal{N}$ is isomorphic to the universal enveloping algebra of the free Lie algebra on $\mathbb{A}$, $\mathcal{N} \simeq \mathcal{U}(\mathrm{Lie}(\mathbb{A}))$.*

Theorem: *There is a map from $\mathrm{Lie}(\mathbb{A})$ so that $\mathcal{C}$ is isomorphic to the universal enveloping algebra of its kernel K, $\mathcal{C} \simeq \mathcal{U}(K)$.*

Case $|\mathbf{x}| < \infty$

- no Hopf structure to guide us,
- no shape graded isomorphism, but
- our calculations still suggested a vector space isomorphism . . .

Combinatorial Description of $\mathcal{C}$ (1/5)

Definition: A **rhyme-scheme word** is a word on $\mathbb{N} \setminus \{0\}$ which sees no "2" before the first "1", sees no "3" before the first "2", and so on.

Examples: (of rhyme-scheme words)

1221 (YES) 12123 (YES) 12413 (NO)

Fact: The set partitions are rhyme-scheme words (and vice versa):

$$\mathbf{14.23.5} \leftrightarrow \_\_\_\_\_$$

Combinatorial Description of $\mathcal{C}$ (1/5)

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Examples: (of rhyme-scheme words)

1221 (YES)   12123 (YES)   12413 (NO)

Fact: The set partitions are rhyme-scheme words (and vice versa):

$$\mathbf{1}\underset{1}{}\mathbf{4.23.5} \leftrightarrow \mathbf{1}$$

Combinatorial Description of $\mathcal{C}$ (1/5)

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Examples: (of rhyme-scheme words)

1221 (YES) 12123 (YES) 12413 (NO)

Fact: The set partitions are rhyme-scheme words (and vice versa):

$$\mathbf{14.\underset{2}{2}3.5 \leftrightarrow 12}$$

Combinatorial Description of $\mathcal{C}$ (1/5)

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Examples: (of rhyme-scheme words)

1221 (YES) 12123 (YES) 12413 (NO)

Fact: The set partitions are rhyme-scheme words (and vice versa):

$$\mathbf{14.2\underset{2}{3}.5 \leftrightarrow 122}$$

Combinatorial Description of $\mathcal{C}$ (1/5)

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Examples: (of rhyme-scheme words)

1221 (YES) 12123 (YES) 12413 (NO)

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$$\mathbf{1\underset{1}{4}.23.5 \leftrightarrow 1221}$$

Combinatorial Description of $\mathcal{C}$ (1/5)

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Examples: (of rhyme-scheme words)

1221 (YES) 12123 (YES) 12413 (NO)

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$$\mathbf{14.23.\underset{3}{5} \leftrightarrow 12213}$$

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Examples: (of rhyme-scheme words)

1221 (YES) 12123 (YES) 12413 (NO)

Fact: The set partitions are rhyme-scheme words (and vice versa):

$$\mathbf{14.23.5} \leftrightarrow \mathbf{12213}$$

Theorem (Wolf, 1936): *$\mathcal{N}$ is freely generated by "non-splittable" rhyme-scheme words (**verses**).*

Examples: (of verses)

123214 (YES) 122|1 (NO) 12|123 (NO)

Combinatorial Description of $\mathcal{C}$ (2/5)

Definition: Given a polynomial $p = \sum_w \alpha_w M_w \in \mathcal{N}$, define the **leading term** of p to be the unique M_{w_0} with w_0 lexicographically least among all rhyme-scheme words w satisfying $\alpha_w \neq 0$.

Fact: Given two rhyme scheme words u, v, the leading term of the product $M_u \cdot M_v$ is $M_{u\,v}$ (where $u\,v$ is the concatenation of the two words).

Example: (in case $|\mathbf{x}| = 3$).

$$M_{121} \cdot M_{\mathbf{12}} = \underline{M_{12112}} + M_{12113} + M_{12121} + M_{12123} + M_{12131} + M_{12132}\,. \quad (1)$$

Combinatorial Description of $\mathcal{C}$ (3/5)

Descending Rhymes: (Identifying $S^{\mathfrak{S}}$). Given a partition $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$, define the **descending** rhyme-scheme word $\mathsf{w}(\mu)$ by

$$\mathsf{w}(\mu) = 1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}.$$

(Conveniently, the leading term of $\mathbf{m}_\mu$ is $M_{\mathsf{w}(\mu)}$).

Vexillary Rhymes: (Identifying $\mathcal{C}$). A **vexillary rhyme** is a maximal (possibly empty) descending rhyme, followed by one extra verse.

Examples: (of vexillary rhymes)

$\overbrace{11223\,1223}$ (YES) $\overbrace{111\,1232}$ (YES) $\overbrace{1232}$ (YES)

1122 112 (NO) (since 112 is not a single verse)

Combinatorial Description of $\mathcal{C}$ (4/5)

Vexillary Decomposition: The **vexillary decomposition** of a rhyme-scheme word: first write w as a product of verses; then combine, left to right, into vexillary rhymes; you may end with a descending tail.

Examples:

$$112212 \mapsto 1|122|12 \mapsto 1\;122\;12$$

$$1231231411122311 \mapsto 123|12314|1|1|1223|1|1 \mapsto 123\;12314\;1\;1\;1223\;1\;1.$$

$$1231231411123311 \mapsto 123|12314|1|1|1233|1|1 \mapsto 123\;12314\;1\;1\;1223\;1\;1.$$

Combinatorial Description of $\mathcal{C}$ (5/5)

Let $\mathcal{C}$ be the subalgebra generated by $\{M_w : w \text{ is a vexillary rhyme}\}$.

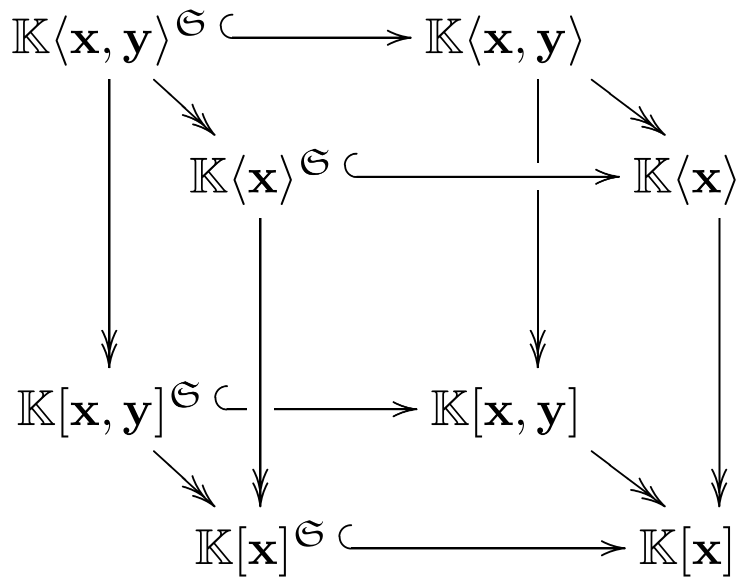
Theorem: *There is an isomorphism of vector spaces*

$$\mathcal{N} \simeq \mathcal{C} \otimes S^{\mathfrak{S}}$$

given by multiplying $M_{v_1} M_{v_2} \cdots M_{v_r}$ *and* $\mathbf{m}_{W(\mu)}$ *in* $\mathcal{N}$.

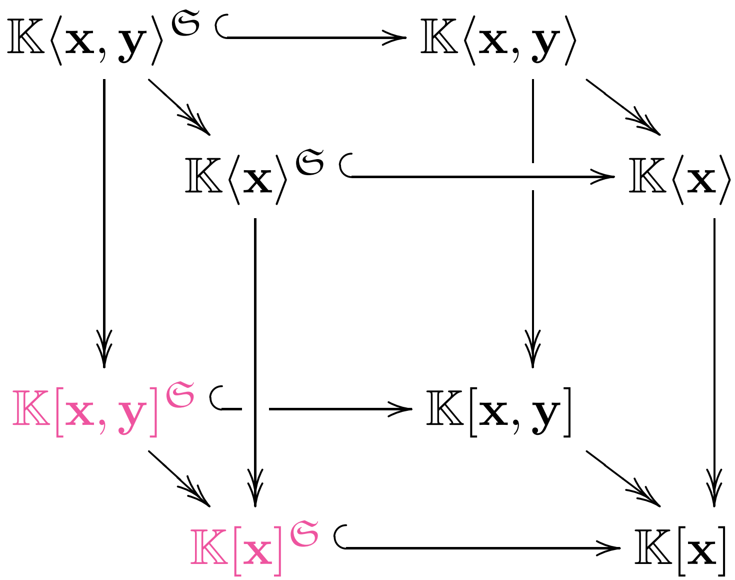
Further Along Our Point of View

This technique is applicable elsewhere in the Hopf cube.



Further Along Our Point of View

This technique is applicable elsewhere in the Hopf cube.



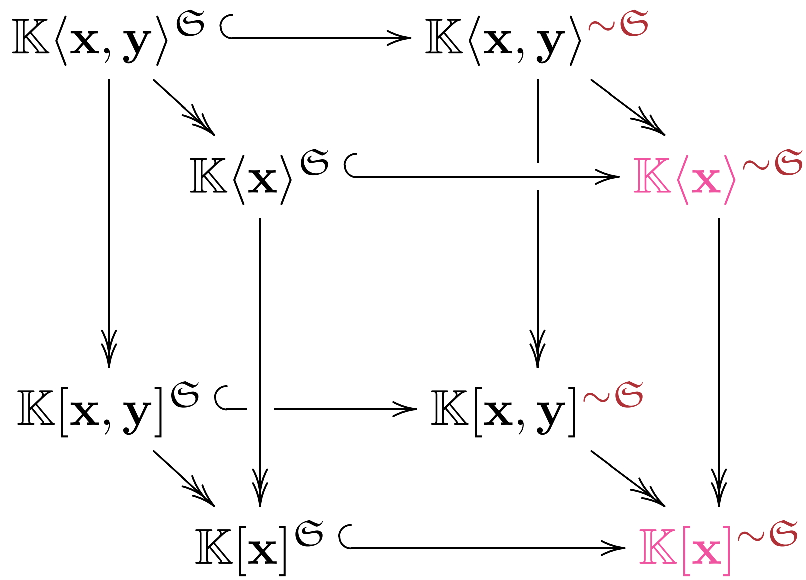
Sometimes, you find obviously positive quotients, like

$$\frac{\mathrm{Hilb}_t(\mathbb{K}[\mathbf{x}, \mathbf{y}]^{\mathfrak{S}})}{\mathrm{Hilb}_t(\mathbb{K}[\mathbf{x}]^{\mathfrak{S}})} = \frac{\prod_{x_i \in \mathbf{x}} {}^{1}\!\big/(1-t^i)^{i+1}}{\prod_{x_i \in \mathbf{x}} {}^{1}\!\big/(1-t^i)}$$

where the result $\prod_{x_i \in \mathbf{x}} (1-t^i)^{-i}$ enumerates **planar partitions.**

Further Along Our Point of View

This technique is applicable elsewhere in the Hopf cube.



Sometimes, you find not-so-obviously positive quotients, like

$$\frac{\mathrm{Hilb}_t(\text{set compositions})}{\mathrm{Hilb}_t(\text{compositions})} = \sum_d (\ell_d)\, t^d,$$

where ℓ_d is the number of **“L-convex polyominoes”** with d cells.

FIN

Muchas Gracias

FIN

Muchas Gracias

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