Rational and Irrational Series over the Free Group

Aaron Lauve

Texas A&M University

joint work with:

Christophe Reutenauer

Université du Québec à Montréal

The Word Problem

(with apologies to group theory)

Problem

Given an algebraic structure $\mathcal{A}$ (group, ring, field, etc.) and expressions $W, W' \in \mathcal{A}$, determine if $W = W'$.

- Fix noncommuting indeterminants x, y and z. Put

$$W = \big(x - z^{-1}\big)\big(1 - yx\big)^{-1}\big(z - y\big) + \big(1 - z^{-1}y\big)\big(y - x^{-1}\big)^{-1}\big(z - x^{-1}\big).$$

- Is $W = 0$?

The Free Skew Field

- Let $k\langle X\rangle$ denote the noncommutative polynomials in X.

- *(informally)* the **free skew field** $k\langle\!\langle X\rangle\!\rangle$ is the set of all well-defined rational expressions in $k\langle X\rangle$.

$$x - x^{-1} \qquad (x-x)^{-1} \qquad \Big(\big((x-x^{-1})^{-1}+y\big)^{-1}-x\Big)^{-1}+y$$

✓ ✗ ✓

- We need a good model of $k\langle\!\langle X\rangle\!\rangle$ for doing computations.

A Model for the Free Skew Field

- Γ – the free group generated by X.
- k^Γ – the formal series over Γ: $\{a = \sum_{\omega\in\Gamma} a_\omega\, \omega\}$.
- $\mathrm{supp}(a)$ – the elements $\omega \in \Gamma$ with nonzero coefficients a_ω.

Definition

Fix a total order $(<)$ of Γ, compatible with multiplication.

The **Malcev–Neumann series** $k(\!(\Gamma)\!)$ w.r.t. $(<)$ are the series $a \in k^\Gamma$ with $\mathrm{supp}(a)$ well ordered.

- *Think:* Laurent series.

Theorem (Lewin '74)

The free skew field is the closure of $k\langle X\rangle$ in $k(\!(\Gamma)\!)$ under $+, \times, (\text{-})^{-1}$.

Euler's Identity

Brion

spot the mistake(?)

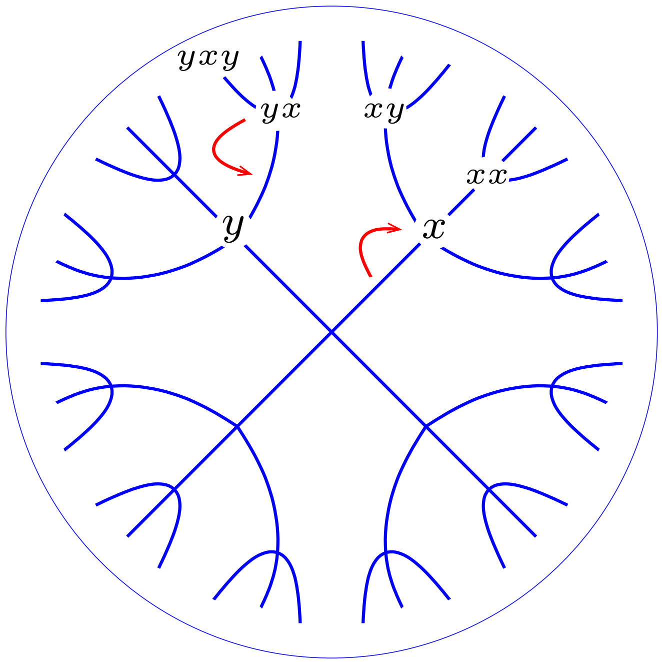
$$
\begin{aligned}
0 &= \sum_{n \in \mathbb{Z}} x^n \\
&= x^{-1} \sum_{n \leq 0} x^n + \sum_{n \geq 0} x^n \quad \longleftarrow \quad x^* \\
&= \frac{x^{-1}}{1 - x^{-1}} + \frac{1}{1 - x} \\
0 &= \frac{1}{x - 1} + \frac{1}{1 - x}
\end{aligned}
$$

- The formal series model for $k\langle\langle X \rangle\rangle$ could similarly be more flexible.
- This is part of the motivation for our work.

Connes' Operator

Pimsner–Voiculescu

- Define an operator $\mathfrak{F}$ on the Cayley Graph $(\Gamma, \mathcal{E})$ of the free group, swapping edges and vertices, as follows.



$$\mathfrak{F}(\nu x) = \{\nu, \nu x\}$$

$$\mathfrak{F}(\{\nu, \nu x\}) = \nu x$$

- k^Γ acts on the (linear span of) vertices and edges:

$$a \cdot \nu = \sum_{\omega \in \Gamma} a_\omega \, \omega\nu \quad \text{and} \quad a \cdot \{\nu, \nu x\} = \sum_{\omega \in \Gamma} a_\omega \, \{\omega\nu, \omega\nu x\}.$$

Connes' Operator

- The **Connes operator** is the commutator $[\mathfrak{F}, a] : k\mathcal{E} \longrightarrow k^{\Gamma}$.

Conjecture (≈ Connes '94)

An element $a \in k\langle\!\langle \Gamma \rangle\!\rangle$ belongs to the free skew field if and only if the operator $[\mathfrak{F}, a]$ has finite rank.

- Essentially proven in Duchamp–Reutenauer '97, save for a small gap.
- *Note:* "$[\mathfrak{F}, W]$ has finite rank" $\neq$ "$W = 0$," but it's a good start.
- This is the rest of the motivation for our work.

A New Characterization of $k\langle\!\langle X \rangle\!\rangle$

Theorem (L–R)

An element $a \in k^{\Gamma}$ belongs to the free skew field if and only if the operator $[\mathfrak{F}, a]$ has finite rank.

- *Content:* like Euler, you can be careless about how you represent an element $a \in k\langle\!\langle X \rangle\!\rangle$ as a series.

A New Characterization of $k\langle\!\langle X \rangle\!\rangle$

Proof.

Induction on $(+, \times, *)$-complexity.

Describe

$$[\mathfrak{F}, a+b], \ \ [\mathfrak{F}, ab] \ \text{ and } \ [\mathfrak{F}, a^*]$$

in terms of

$$[\mathfrak{F}, a] \ \text{ and } \ [\mathfrak{F}, b].$$

□

Expressions without Simplifications

Theorem (L–R)

Every rational expression, taken in k^Γ or $k(\!(\Gamma)\!)$, of $W \in k\langle\!\langle X \rangle\!\rangle$ has an equivalent expression without simplifications.

- A rational expression for $W \in k\langle\!\langle X \rangle\!\rangle$ is an **expression without simplifications** if no subexpression takes the form $\omega \cdot \omega^{-1}$.

Example

The expressions

$$x^{-1}x^* \qquad \text{and} \qquad (y^{-1}xy)^*$$

simplify to

$$x^{-1} + x^* \qquad \text{and} \qquad y^{-1}x^*y.$$

Future Questions

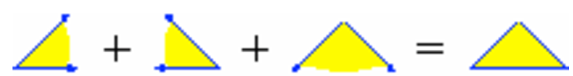
- *The Word Problem:* move from "$[\mathfrak{F}, W]$ has finite rank" to "$W = 0$."
  (We have bounds on generators for IMG $[\mathfrak{F}, W]$*.)*

- *New Problem:* algorithm to find minimal element of $\operatorname{supp}(W + W')$.
  (Deceptively tricky!)

References

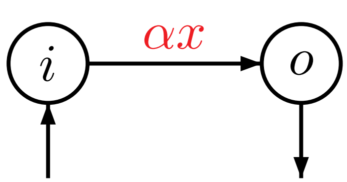
- A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

- G. Duchamp & C. Reutenauer, Un critère de rationalité provenant de la géométrie non commutative, *Invent. Math.*, 128(3):613–622, 1997.

- M. Beck, C. Haase, and F. Sottile, , *Mathematical Intelligencer*, to appear (arXiv:math/0506466v4).

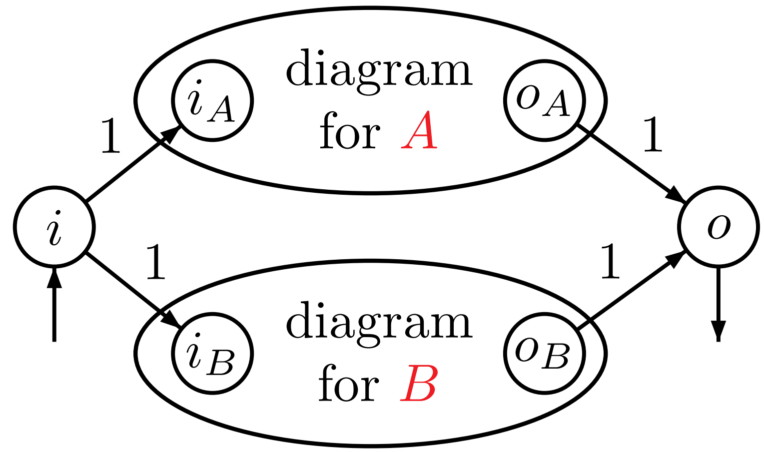
McNaughton–Yamada Algorithm

(for an automaton recognizing a rational expression R)



$R = \alpha x$

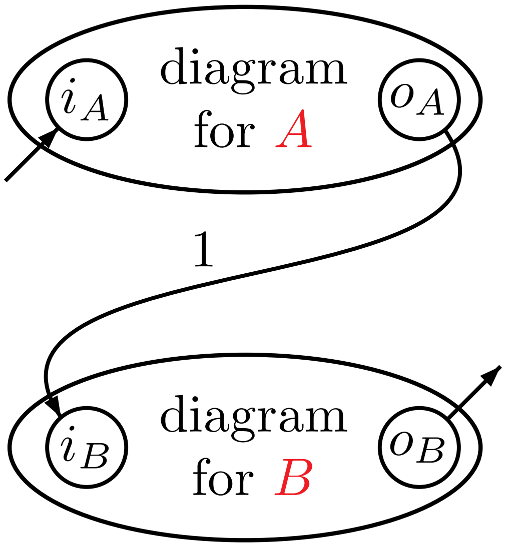
$(x \in X \cup X^{-1}, \alpha \in k)$



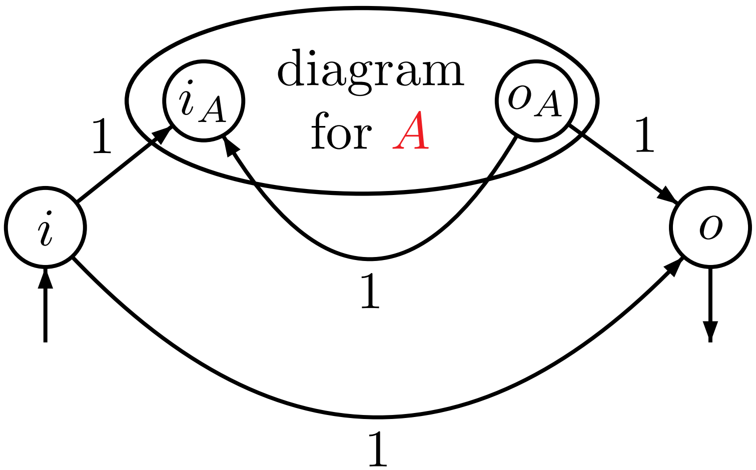
$R = A + B$

McNaughton–Yamada Algorithm

(for an automaton recognizing a rational expression R)



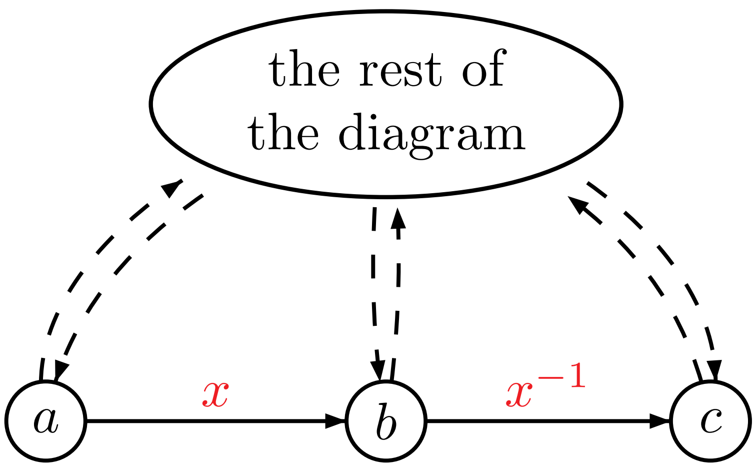
$R = A \cdot B$

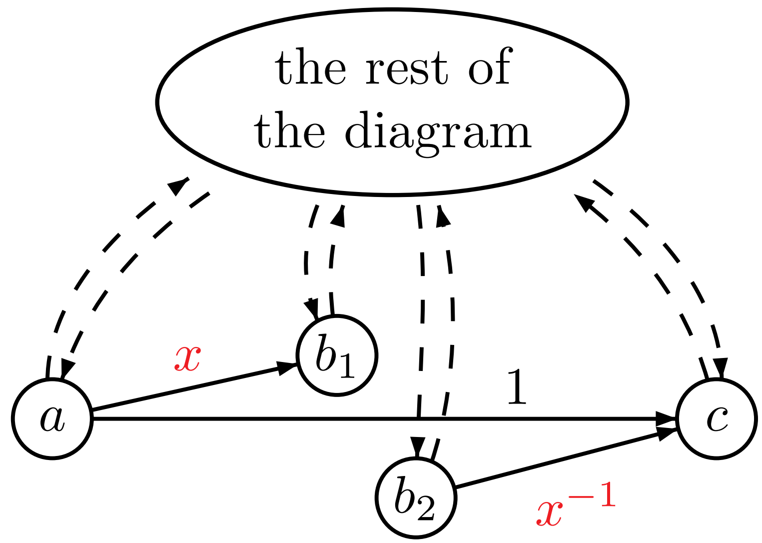


$R = A^*$

Fliess' Idea

(for removing simplifications)





R contains xx^{-1} at b

R does not contain xx^{-1} at b