A QUASIDETERMINANTAL APPROACH TO QUANTIZED FLAG VARIETIES

BY AARON LAUVE

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Vladimir Retakh & Robert L. Wilson

We provide an efficient, uniform means to attach flag varieties, and coordinate rings of flag varieties, to numerous noncommutative settings. Our approach is to use the quasideterminant to define a generic noncommutative flag, then specialize this flag to any specific noncommutative setting wherein an amenable determinant exists.

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Dedication

to Maria

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Chapter 1

Introduction

1.1 Emergence of noncommutative structures

To varying degrees, physicists have been aware of the need for an honest study of noncommutative structures for nearly a century. Heisenberg stumbled upon this need when trying to make sense of the results of light-emission spectra experiments for hydrogen atoms [9]. Faddeev and his Leningrad school found the need in developing an inverse scattering method for quantum field theory [7]. More recently, conformal field theorists are finding a need for noncommutative structures (cf. the instantons of Nekrasov-Schwarz and others [13]).

In these cases and many more, the noncommutative structures are rightly viewed as geometric objects. Using the familiar pairing

$$\{\text{topological spaces } X\} \leftrightarrow \{\text{rings } R \text{ of functions on } X\}\,, \tag{1.1}$$

mathematicians and physicists are developing the subject from the algebraic side. At least three distinct programs for the study of noncommutative (NC) geometry, (which I will call *NC differential geometry*, *NC algebraic geometry*, and *NC Lie theory*) are thriving today.

In NCDG, Alain Connes and others approach noncommutative geometry from the perspective of real analysis and C^*-algebras—the motivation being the passage from classical to quantum mechanics in physics [9]. The geometric data is real: you act on a space of objects X with your operators R, perhaps relaxing the notion of space, and understand that the operators need not commute.

In NCAG, whose pioneers include Michael Artin and Bill Schelter, one studies general noncommutative algebras in a manner suggestive of classic algebraic geometry. The geometric data is transfered via the pairing (1.1), i.e.

$$(\mathrm{Spec}\, R, R\text{-Mod}) \to (\mathrm{X}, \text{line bundles on } X)\,.$$

The current state of the art is most readily applied to "small" algebras, in a sense outlined in the AMS bulletin article [44].

In NCLT, one attempts to deform the Lie group/algebra pairing

$$(G, \mathfrak{g}) \leftrightarrow (\mathcal{O}_G, U(\mathfrak{g}))$$

and get *quantum groups.*

Remark. This term is a misnomer as by "quantum group" we really mean "quantized ring of functions on a Lie group or Lie algebra." First, the abuse of terminology is symptomatic of the notion of geometry here; as in NCAG, it is transfered via the pairing (1.1). Second, the author is inclined to reserve this term only for deformations of $K[G]$ and the like, and indeed we will speak no more of the $U(\mathfrak{g})$ story.

The deformation is carried out with respect to some guiding principle; this may come in the form of R-matrices à la the Faddeev, Reshetikhin, and Takhtajan construction (cf. Chapter 5) or via the closely related construction of Manin (cf. [35]), which attaches a quantum group to any pair A_i of quadratic algebras of the form $K < x_1, \ldots, x_n > /\mathcal{I}_i$, thought of as the polynomials functions on "quantum affine n-space."

1.2 Contributions of this thesis

The prevailing paradigm is that noncommutative mathematics is harder than commutative mathematics. We trust the viewpoint of I. Gelfand and V. Retakh, who admit that noncommutative mathematics is *different*, but argue that it is at least as simple as commutative mathematics. This dissertation lends some weight to this argument in the following sense: after some straightforward, albeit lengthy, calculations in Chapter 4, we are able to easily derive a wealth of results in Chapters 5–8.

Specifically, this thesis provides an efficient, uniform means to attach (coordinate rings of) flag varieties to numerous noncommutative settings. Our approach is to use the quasideterminant to define a generic noncommutative flag, then specialize this flag to any specific noncommutative setting wherein an amenable determinant exists.

Example. The celebrated identity for minors of a 4×2 matrix A

$$p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0,$$

and its quantum counterpart

$$p_{12}p_{34} - q^{-1}p_{13}p_{24} + q^{-2}p_{23}p_{14} = 0,$$

are both consequences of the quasi-Plücker relation

$$r^3_{12}r^4_{21} + r^2_{13}r^4_{31} = 1\,.$$

When compared to the programs outlined in the preceding subsection, this thesis fits most readily within the third program. It distinguishes itself from the present literature in that each flag algebra constructed here may be more concretely attached to a geometric object. A different approach with this same goal, also making use of the quasideterminant, appears in a recent article of Škoda [43].

1.3 Future directions

In Chapter 6, it becomes evident that the quasideterminantal calculus cannot currently capture the entire quantum flag—missing a large portion of the *q-straightening relations*. It may be the case that new quasideterminant identities are waiting to be discovered, identities that will bridge the gap between the pre–flag algebra we introduce below and the quantum flag algebra. On the other hand, it is entirely possible that no quasideterminantal explanation for this current gap exists. Indeed, some preliminary computer calculations the author has performed seem to suggest that there are no *genuinely new* quasi-Plücker identities to discover. Proving this would be a difficult and important result. If accomplished, one would be left with explaining why the quantum setting is special, and under what conditions imposed on a noncommutative setting similar straightening relations may be expected to appear.

In another direction, one need only recall the classic fact that every irreducible representation of GL_n appears within the coordinate algebra for $F\ell((1^n), n)$ to see the importance of studying noncommutative versions of $F\ell$(-,-). Toward this goal, a careful study of the ring of quasi-Plücker coordinates is merited. A modest beginning to this program appears as Section 7.4.

Chapter 2

Preliminaries

2.1 Notation

We fix some notation for the remainder of the thesis.

Sets, Subsets, & Sequences

By $[n]$ we mean the set $\{1, 2, \dots n\}$. If $d \in [n]$, then by $\binom{[n]}{d}$ we mean the set of all subsets of $[n]$ of cardinality d. More generally, if $S \subseteq [n]$, then $\binom{[n]}{S}$ denotes $\bigcup_{s \in S} \binom{[n]}{s}$. For example, in this notation the power set $\mathcal{P}[n]$ of $[n]$ is given by $\emptyset \cup \binom{[n]}{[n]}$. By $[n]^d$ we mean the set of all d*-tuples* $(a_1, a_2, \dots, a_d)$ chosen from $[n]$; and more generally $[n]^S$ denotes $\bigcup_{s \in S} [n]^s$.

We write $\gamma \models n$ when γ is a *composition* of n (any sequence of positive integers summing to n). In the literature, compositions γ are sometimes allowed to have parts $\gamma_i = 0$ for some i. We will not need these "weak-compositions" here. If $\gamma = (\gamma_1, \dots, \gamma_r)$ is a composition of n, we denote the number of parts r by $p(\gamma)$ and we write $|\gamma| = n$. A special subset of $[n-1]$ associated to γ will be important:

$$\|\gamma\| := \left\{ \gamma_1, \gamma_1 + \gamma_2, \dots, \sum_{j \le i} \gamma_j, \dots, \sum_{j \le r-1} \gamma_j \right\}.$$

Notice that the cardinality of the $\|\gamma\|$ is $r - 1$.

Operations on Matrices

Let $A = (a_{ij})$ be an $n \times m$ matrix, and fix subsets $I \subseteq [n]$ and $J \subseteq [m]$. We let $A^{I,J}$ denote the matrix obtained by deleting rows I and columns J from A; by $A_{I,J}$ we mean the matrix obtained by keeping only rows I and columns J of A. With slight abuse of the just-defined notation, A^{ij} will represent the $(n-1)\times(m-1)$ sub-matrix of A obtained by deleting row i and column j. Analogously, we may occasionally write A_{ij} instead of a_{ij}.

When it is clear from the context that I represents row-indices (J represents column-indices), we let A_I (A_J) denote the square matrix obtained from A by taking row-set I (column-set J) and column-set the first $|I|$ columns of A (row-set the first $|J|$ rows of A). Finally, if $I \in [n]^d$, then by A_I we mean the rows (columns) of A indexed by I, put down in the order in which they appear in the sequence I.

Operations on Sets & Tuples

Fix $I \in \binom{[n]}{d}$ and $J \in [n]^d$. *Sets,* e.g. $I = \{i_1, \ldots, i_d\}$, will always be assumed to have the property $i_1 < i_2 < \ldots < i_d$. *Tuples* (or sequences), e.g. $J = (j_1, \ldots, j_d)$, naturally, cannot always be assumed to have this property. For sets or tuples K of length d, we use $|K|$ to denote d.

Remark. In fact, tuples are compositions, but $|\text{-}|$ has different meanings for tuples and compositions. Unless expressly told to view a sequence $A = (a_1, \ldots, a_r)$ as a composition, understand A as a "tuple", with $|A| = r$, not $\sum_i a_i$.

By $\mathsf{tup}(I)$ we mean the sequence $(i_1, \ldots, i_d)$ associated to I; similarly, we let $\mathsf{set}(J)$ denote the set built from J by deleting repetitions and putting the remaining elements in ascending order. Let $\mathsf{rect}(J)$ denote the new tuple built from J by putting its elements in weakly increasing order

Example. $\mathsf{set}(\mathsf{tup}(I)) = I$; if J has no repeated elements, then $\mathsf{rect}(J) = \mathsf{tup}(\mathsf{set}(J))$.

For $k \notin I$, we often write $I \cup k$ for the set $I \cup \{k\}$. To avoid an abundance of $\cup$'s we occasionally write kI or Ik for this same set. For $k \in I$, we may write $I \setminus k$, or I^k for $I \setminus \{k\}$.

Write $k \in J$, if $k \in \mathsf{set}(J)$ and $k \notin J$ otherwise. Analogous to sets, if $k = j_r$ for exactly one $1 \le r \le d$, we write $J \setminus k$ or J^k for the $(d-1)$-tuple built from J by deleting j_r. Conversely, for K in $[n]^d$ or $\binom{[n]}{d}$, we write $K|J$ for the tuple $(k_1, \ldots, k_r, j_1, \ldots, j_d)$. $J|K$ is similarly defined. If $k \in [n]$ we abuse notation and write, e.g., $k|J$ for $(k)|J$.

If A, B are two sets or two tuples ($|A|$ not necessarily equal to $|B|$), we say A *precedes* B (written $A \prec B$), if $\forall a \in A$, $\forall b \in B$, $a < b$. In the case A is a set with subsets $A', A'' \subseteq A$ satisfying $A = A' \cup A''$, we write $A = A' \dot{\cup} A''$ in the case $A' \cap A'' = \emptyset$.

Permutations

Denote the group of permutations of $[n]$ by $\mathfrak{S}_n$. If X is another set with n elements, we will sometimes use $\mathfrak{S}_X$ and sometimes use $\mathfrak{S}_n$ for the permutations of this set. For $\sigma \in \mathfrak{S}_d$, let $\ell(\sigma)$ denote the *length* of σ (the minimum number of adjacent swaps necessary to put the d-tuple $(\sigma 1, \sigma 2, \ldots, \sigma d)$ in increasing order). If a tuple $J = (j_1, \ldots, j_d)$ has no repeated indices, we may view J as a permutation in one-line notation and understand $\ell(J)$ as the minimal number of adjacent swaps necessary to put J in increasing order.

Example. If $I = \{1, 3, 5, 6, 8, 9\}$ and $i = 8$, then $\ell(i|I^i) = \ell((8, 1, 3, 5, 6, 9)) = 4$, while $\ell(iI^i) = \ell(I) = 0$.

Ring Constructions

All rings considered in this note are associative and unital. Throughout this thesis, fields F and skew fields D will always contain $\mathbb{Q}$. For a fixed n, most of our calculations may be carried out fields with characteristic not dividing $n!$, but we make no effort to do so here.

Given a ring A, we denote the ring of $n \times n$ matrices over A by $M_n(A)$ and the set of $n \times m$ matrices over A by $M_{n\times m}(A)$. We denote the identity matrix here by $\mathbb{I}$ or $\mathbb{I}_n$ and the matrix units by $\mathbb{E}_{ij}$: $(\mathbb{E}_{ij})_{ab} = \delta_{ai}\delta_{bj}$. When we are simultaneously working with a ring A—with unit 1—and endomorphisms over A, we also write $\mathbb{I}$ for the identity morphism (e.g. $\mathbb{I} \in \operatorname{End} A^n \otimes A^n = \mathbb{I}_n \otimes \mathbb{I}_n$, $\mathbb{I}_n \in M_n(A)$).

Finally, we often conflate expressions E in a ring R with their associated identities in $R/(E)$. For example, we give the expression

$$\sum_{j\in L} (-1)^{\ell(L\setminus\Lambda|\Lambda)} f_{L\setminus\Lambda} f_{\Lambda|J}\,, \in \mathcal{F}(\gamma)$$

and the relation

$$\sum_{j \in L} (-1)^{\ell(L\setminus\Lambda|\Lambda)} f_{L\setminus\Lambda} f_{\Lambda|J} = 0\,, \text{ defining } \mathcal{F}(\gamma)$$

the same name, $(\mathcal{Y}_{I,J})$. It should be clear from context to which construct the name is attached.

Determinants

In the course of this note, we will be considering numerous (noncommutative) determinants and minors of square matrices $A = A_{R,C}$. For $K \subseteq R$ and $L \subseteq C$ with $|K| = |L| = m$, we write $[A_{K,L}]$ or $\big[K; L\big]$ for the (particular, evident by context) determinant of the matrix $A_{K,L}$. If it is furthermore evident from context that we are focusing on the rows (respectively, columns) of A, we may suppress the L (K) in this notation—typically taking the first m columns (rows) of A in this case.

2.2 Skew fields

2.2.1 Noncommutative localization

In this thesis, the ability to invert an element x of a ring R is of critical importance. To that end, one would like to work in a field of fractions for R or at least in a localization of R containing x^{-1}. In the case R is commutative, then the questions of existence, uniqueness, and construction of these objects are all elementary. Unfortunately, things get a great deal more complicated in the noncommutative case. For instance, Malcev [34] constructed a domain which cannot be embedded into any *skew field*—a not necessarily commutative ring in which every nonzero element has a two-sided inverse. Worse, even when R can be so embedded, a "minimal" such embedding may not be unique. One case where things work smoothly is the case $R = F\langle X\rangle$, the free F-algebra on a set X of noncommuting variables. When F is an infinite field of characteristic zero, there is a unique universal field of fractions, called the *free skew field* and denoted $F\lessdot X\gtrdot$, associated to R [8].

To ensure our ability to invert that which we wish, the quasideterminant calculations indicated in the next section are all carried out in $F\lessdot A\gtrdot$ where A is a collection of n^2

noncommuting indeterminants arranged in a matrix $A = (a_{ij})$.

2.2.2 Skew-field identities

In the present context, the quasideterminantal calculus may be simply stated as the study of free skew-field identities. As mentioned in the introduction, it may be the case that there are no genuinely new skew-field identities involving quasi-Plücker coordinates waiting to be discovered. The next result suggests that finding such an identity would not be an easy task, as skew-field identities involving quasideterminants are far from simple.

Theorem 1 (Reutenauer, [42]). *Say an element $f \in F\langle\!\langle A \rangle\!\rangle$ has* inversion-height m *if there is an expression for f in terms of A involving m nested inversions, and there is no such expression involving fewer nested inversions. Then the quasideterminants $|A|_{ij}$ of A have inversion height $n-1$*

Compare this with determinants (and all of the noncommutative determinants appearing in this thesis), which have inversion height zero.

2.3 Quasideterminants

2.3.1 Historical origins

The quasideterminant is a replacement for the determinant for matrices over noncommutative rings R. It was introduced in 1991 by Gelfand and Retakh [15], and extends the ideas of Heyting [24] from 1928. The quasideterminant—actually quasideterminants, there are n^2 of them, one for each position (i, j) in a matrix $A \in M_n(R)$—is a recursively constructed rational function that requires invertibility of certain submatrices.[1] Hence, it is not always defined; moreover, when it is defined it typically takes values in a localization of R, not R itself.

[1]This is a main factor explaining the 60 year gap in their study. Shortly after Heyting's paper, Ore [39] strongly criticized any candidate noncommutative determinant which was not a polynomial in the entries of the matrix.

Still, with all of these "faults," the quasideterminant is a nice replacement for the determinant: there is a Cramer's rule for solving linear systems; it is invariant under elementary row (column) operations; one can build A^{-1} using the quasideterminant, analogous to the adjoint/inverse construction in the commutative case. Finally, there is the *heredity property* which allows one to take the quasideterminant of a block matrix in stages. This property—which has no counterpart for the commutative determinant—is well-suited for induction and is essential for establishing many of the important quasideterminantal identities.

2.3.2 Current trends

Gelfand and Retakh argue that the quasideterminant should be a main organizing tool in noncommutative mathematics [17]. Support for this argument is steadily appearing in the literature [3, 5, 11, 19, 36, 37, 38, 43]. The results of this thesis provide further strong support for their argument.

The results cited above rely on one or both of the fundamental successes of the quasideterminantal calculus: an ability to easily and explicitly (i) factor noncommutative polynomials, and (ii) perform Gaussian elimination on matrices with noncommutative entries. This thesis is concerned with an application of item (ii).

2.3.3 Definition & first properties

As mentioned earlier, the computations in this section will be done in the free skew field $D = F\langle\!\langle a_{ij} \rangle\!\rangle$ built on a matrix A with distinct noncommuting indeterminant entries. As the definition will make clear, if we instead work with A over an arbitrary noncommutative ring R some quasideterminants may not be defined. The reader will find a more thorough treatment of the quasideterminant and its properties, including all of the statements below and more, in [18] and [30]. We include two proofs from the literature as they anticipate some of the new results appearing in this thesis.

Definition 1 (Quasideterminant, I). An $n \times n$ matrix A has in general n^2 quasideterminants—one for each position in A. The (ij)-quasideterminant is defined as follows:

$$|A|_{ij} = a_{ij} - \sum_{p \neq i, q \neq j} a_{iq} \left(|A^{ij}|_{pq} \right)^{-1} a_{pj}.$$

The quasideterminant is not a generalization of the determinant. Over a commutative field, the quasideterminant specializes to the *ratio* of two determinants:

$$|A|_{ij} = (-1)^{i+j} (\det A)/(\det A^{ij}).$$

Notation. It will be convenient to denote the (ij)-quasideterminant in another form:

$$\left| \begin{matrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{matrix} \right|_{ij} = \left| \begin{matrix} & \vdots & \\ \cdots & \boxed{a_{ij}} & \cdots \\ & \vdots & \end{matrix} \right|.$$

There is an alternate definition which we will also have occasion to use. Let ξ be the i-th row of A with the j-th coordinate deleted; and let ζ be the j-th column of A with the i-th coordinate deleted.

Definition 2 (Quasideterminant, II). For A, ξ, ζ as above, the (ij)-quasideterminant is defined as follows:

$$|A|_{ij} = a_{ij} - \xi (A^{ij})^{-1} \zeta.$$

In attempting to make these two definitions agree, one stumbles upon the first fundamental fact about quasideterminants,

Theorem 2 (Matrix Inverses). *If A^{-1} is defined over D and $(A^{-1})_{ji}$ is not equal to zero, then*

$$(|A|_{ij})^{-1} = (A^{-1})_{ji}, \tag{2.1}$$

The quasideterminant is extremely well-behaved for being a noncommutative determinant (or rather ratio of two). Consider its behavior under elementary transformations of columns.

Theorem 3 (Elementary Column Relations). *Let $A = (a_{ij})$ be a square matrix.*

- *(Column Permutations) Suppose* $\tau \in \mathfrak{S}_n$ *and* P_τ *is the associated (column) permutation matrix. Then* $|AP_\tau|_{p,\tau q} = |A|_{p,q}$.

- *(Rescaling Columns) Let* B *be the matrix obtained from* A *by multiplying its* q*th column by* ρ *on the right. Then*

$$|B|_{ij} = \begin{cases} |A|_{ij}\,\rho & \text{if } j = q \\ |A|_{ij} & \text{if } j \neq q \text{ and } \rho \text{ is invertible.} \end{cases}$$

- *(Adding to Columns) Let* B *be the matrix obtained from* A *by adding column* l *(multiplied on the right by a scalar* ρ*) to column* q*. Then* $|B|_{ij} = |A|_{ij}$ *if* $j \neq q$.

With these properties, we may easily deduce

Proposition 4. *If* A *is a square matrix and column* q *of* A *is a right-linear combination of the other columns, then* $|A|_{pq} = 0$ *(whenever it is defined).*

Proof. Through a sequence of steps $A = A(0), \ldots, A(t) = B$, column-reduce A to a matrix B: $\mathrm{col}_q(B) = 0$; $\mathrm{col}_j(B) = \mathrm{col}_j(A)$ $(j \neq q)$. The previous theorem indicates that at each stage

$$|A|_{pq} = |A(i)|_{pq} \quad (\forall 1 \leq i \leq t).$$

Finally, use the second definition of quasideterminant to conclude that $|B|_{pq}$ is indeed zero. □

Theorem 5 (Column Homological Relations). *Let* $A = (a_{ij})$ *be a square matrix. Then*

$$-|A^{kj}|_{il}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kl}^{-1} \cdot |A|_{kj} \quad (\forall l \neq j). \tag{2.2}$$

We will also find a use for the following identity of Krob and LeClerc, which gives a one-column *Laplace expansion* of the quasideterminant.

Proposition 6. *For* $A = (a_{ij})$*, the* (pq)*-quasideterminant has the following expansion:*

$$|A|_{pq} = a_{pq} - \sum_{i \neq p} |A^{iq}|_{pl} \cdot |A^{pq}|_{il}^{-1} \cdot a_{iq} \quad (\forall l \neq q). \tag{2.3}$$

Proof. Using (2.1) and (2.2) we have

$$\begin{aligned} 1 &= \sum_{i=1}^{n} |A|_{iq}^{-1} \cdot a_{iq} \\ |A|_{pq} &= a_{pq} + \sum_{i \neq p} |A|_{pq} \cdot |A|_{iq}^{-1} \cdot a_{iq} \\ |A|_{pq} &= a_{pq} - \sum_{i \neq p} |A^{iq}|_{pl} \cdot |A^{pq}|_{il}^{-1} \cdot a_{iq}. \end{aligned}$$

□

Theorem 7 (Muir's Law). *Let $A = A_{R,C}$ be a square matrix with row set R and column set C. Fix $R_0 \subsetneq R$ and $C_0 \subsetneq C$. Say an algebraic, rational expression $\mathcal{I} = \mathcal{I}(A, R_0, C_0)$ involving the quasi-minors $\{|A_{R',C'}|_{rc} : |R'| = |C'|, R' \subseteq R_0, C' \subseteq C_0\}$ is an* identity *if the equation $\mathcal{I} = 0$ is valid. Then for any $L \subseteq R \setminus R_0$ and $M \subseteq C \setminus C_0$ with $|L| = |M|$, the expression $\mathcal{I}'$ built from $\mathcal{I}$ by extending all minors $|A_{R',C'}|_{rc}$ to $|A_{L \cup R', M \cup C'}|_{rc}$ is also an identity.*

The following quasideterminantal version of Sylvester's Identity will also prove useful. It expresses the quasideterminant of an $n \times n$ matrix in terms of quasideterminants of $(n-1) \times (n-1)$ matrices.

Theorem 8 (Sylvester's Identity). *Let $A = A_{R,C}$ be a square matrix. Fix $r_0, r_1 \in R$ and $c_0, c_1 \in C$. Then the following identity holds (when all components are defined) among the quasi-minors of A:*

$$\begin{aligned} |A_{R,C}|_{r_0,c_0} &= \begin{vmatrix} \boxed{|A^{r_1,c_1}|_{r_0,c_0}} & |A^{r_1,c_0}|_{r_0,c_1} \\ |A^{r_0,c_1}|_{r_1,c_0} & |A^{r_0,c_0}|_{r_1,c_1} \end{vmatrix} \\ &= |A^{r_1,c_1}|_{r_0,c_0} - |A^{r_1,c_0}|_{r_0,c_1} \cdot |A^{r_0,c_0}|_{r_1,c_1}^{-1} \cdot |A^{r_0,c_1}|_{r_1,c_0}. \end{aligned}$$

In [17], the reader will find row versions of all the properties listed above, some of which are used in this thesis without further comment.

Chapter 3

Noncommutative Flags & Coordinates

3.1 Review of Classical Setting

In this section we work over a field $\mathbb{C}$ (cf. [49] for a treatment over any commutative ring of characteristic p not dividing $n!$).

3.1.1 Flags

We recall the classical notion of flags, whose generalization will be the main focus of this thesis. Fix a vector space $V \simeq \mathbb{C}^n$ and a composition $\gamma = (\gamma_1, \ldots, \gamma_r)$ of n.

Definition 3. A *flag* Φ of shape γ is an increasing chain of subspaces of V,

$$\Phi : (0) = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r = V\ ,$$

satisfying $\dim_{\mathbb{C}} (W_i/W_{i-1}) = \gamma_i$. For fixed V and γ, we let $F\ell(\gamma)$ denote the collection of all flags in V of shape γ.

Notation. Two important special cases are when $\gamma = (1^n)$ and $\gamma = (d, n-d)$. The former is the collection of *full flags*, $\dim W_i = i$, $1 \leq i \leq n$. We write $F\ell(\gamma)$ as $F\ell(n)$ in this case. The latter is the *Grassmannian,* i.e. the collection of d-dimensional subspaces of V We write $F\ell(\gamma)$ as $Gr(d,n)$ in this case.

If we fix a basis $\mathbf{B}^* = (f_1, \ldots, f_n)$ for V^*, we may represent a flag Φ as a matrix as follows. (i) Choose a basis $(w_1, \ldots, w_{\gamma_1})$ for W_1. (ii) Extend this to a basis $(w_1, \ldots, w_{\gamma_1}, w_{\gamma_1+1}, \ldots, w_{\gamma_1+\gamma_2})$ for W_2. (iii) Repeat until you have completed the sequence to a basis $\mathbf{w} = (w_1, \ldots, w_{|\gamma|})$ of V. (iv) define the matrix $A = A(\Phi, \mathbf{w}) = (a_{ij})$ by putting $a_{ij} = f_i(w_j)$. Then A is the collection of column vectors $[w_1|w_2|\cdots|w_n]$ with coordinatization provided by $\mathbf{B}$.

$$\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{array} \begin{bmatrix} \boxed{g_1} & * & * & * \\ 0 & \boxed{g_2} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \boxed{g_r} \end{bmatrix}$$

Figure 3.1: An upper block-triangular matrix, with $g_i \in \mathrm{GL}_{\gamma_i}(\mathbb{C})$ and "$*$" arbitrary.

The choice of basis for Φ was not unique, so neither is the matrix A. However, we do know exactly when two matrices A, B represent the same $\Phi \in F\ell(\gamma)$.

Lemma. Given $\Phi, \mathbf{w}$, and $A(\mathbf{w})$. $\mathbf{w}'$ is another basis for Φ if and only if $A(\mathbf{w}') = A(\mathbf{w}) \cdot g$ for some $g \in \mathrm{GL}_n(\mathbb{C})$ of the form

For fixed γ, the collection of such $g \in \mathrm{GL}_n$ is called a *parabolic subgroup.* We will denote this subgroup of GL_n by P_γ^+, the "+" standing for "*upper* block-triangular" matrices. We may now replace the above definition with a new one.

Definition 4. Given a composition $\gamma \models n$, we have $F\ell(\gamma) = \mathrm{GL}_n(\mathbb{C})/\sim$, where $A \sim A'$ if $\exists g \in \mathrm{P}_\gamma^+$ s.t. $A' = Ag$.

Next, we outline how to view $F\ell(\gamma)$ as a subvariety of some projective variety.

3.1.2 Determinants & Coordinates

The determinant of a square matrix X will be a main organizing tool in what follows. In addition to the well-known alternating property, the determinant has another property the reader should be familiar with:

Proposition 9 (Laplace's Expansion). *Let $X = (x_{ij})_{1 \le i,j \le m}$. Suppose that p, p' are fixed positive integers with $p + p' = m$, and that $J = (j_1, \ldots, j_m)$ is a fixed derangement of the columns of X. Then*

$$\left|X\right| = (-1)^{\ell(J)} \sum (-1)^{-\ell(i_1 \cdots i_p i'_1 \cdots i'_{p'})} \left|X_{\{i_1,\ldots,i_p\},\{j_1,\ldots,j_p\}}\right| \cdot \left|X_{\{i'_1,\ldots,i'_{p'}\},\{j_{p+1},\ldots,j_m\}}\right|$$

where the sum is over all partitions of $[m]$ into two increasing sets $i_1 < \cdots < i_p$ and $i'_1 < \cdots < i'_{p'}$.

Below we will take $(j_1, \ldots, j_m) = (1, \ldots, m)$, so what's written above is the expansion of the determinant down the first p columns of X. Alternatively, one may expand along the first p rows of X.

Recall the definition of $\|\gamma\|$ given in Chapter 2: letting $\gamma_{[i]}$ denote the truncated composition $(\gamma_1, \ldots, \gamma_i)$, we had $\|\gamma\|_i = |\gamma_{[i]}|$. Now consider the map $\eta_i : F\ell(\gamma) \to \mathbb{P}(\mathbb{C}^{\binom{n}{|\gamma_{[i]}|}})$ which sends $A(\Phi)$ to the $\binom{n}{|\gamma_{[i]}|}$-tuple of all minors one can possibly make from the first $\sum_{j \le i} \gamma_j$ columns of A. This tuple is rightly viewed as projective coordinates because (i) it misses 0, and (ii) it's only defined up to nonzero scalars:

(i) As A has full rank, there must exist one minor of size $\sum_{j \le i} \gamma_j$ which is nonzero.

(ii) We need $\eta_i(Ag) \equiv \eta_i(A)$ for $g \in \mathrm{P}_\gamma^+$, but the former equals $\eta_i(A) \cdot (\prod_{j \le i} \det g_j)$ (cf. the depiction of g in Figure 3.1).

We put all of these maps together to build a map $\eta : F\ell(\gamma) \to \mathbb{P}(\gamma) := \mathbb{P}^{\binom{n}{\gamma_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{|\gamma_{[r-1]}|}-1}$. This map is called the *Plücker embedding.*[1] Note that we stop at $r-1$. This is because there is nothing to gain by including the final factor ($\mathbb{P}^0$).

Represent a point $\pi \in \mathbb{P}(\gamma)$ by its coordinates $\pi = (p_I)_{I \in \binom{[n]}{\|\gamma\|}}$. When π belongs to the image of η —i.e. when $\exists A \in \mathrm{GL}_n(\mathbb{C})$ with (writing $|I| = d$) $p_I = \det A_{I,[d]}$ for all $I \in \binom{[n]}{\|\gamma\|}$—we say the $\{p_I\}$ are the *Plücker coordinates* of A.

The image of η is particularly nice, it is given by quadratic relations among the coordinates p_I.

Proposition 10. *Suppose $\gamma \models n$, and $A(\Phi) \in F\ell(\gamma)$. For all subsets $I = \{i_1, \ldots, i_{s+u}\}$ and $J = \{j_1, \ldots, j_{t-u}\}$ of $[n]$ and for all $1 \le u$ satisfying $s \ge t$ and $s, t \in \|\gamma\|$, we have the Young symmetry relations $(\mathcal{Y}_{I,J})_{(u)}$:*

$$0 = \sum_{\substack{\Lambda \subseteq I \\ |\Lambda| = u}} (-1)^{\ell(I \setminus \Lambda | \Lambda)} p_{I \setminus \Lambda} p_{\Lambda | J} \,. \tag{3.1}$$

Remark. Here, we have extended the definition of p_I from $I \in \binom{[n]}{d}$ to $I \in [n]^d$ at the

[1] For a geometric proof of this statement, see [21]; for an algebraic proof, see [14].

expense of adding the obvious alternating relations $(\mathcal{A}_K)$:

$$\forall K \in [n]^d \quad p_K = \begin{cases} 0 & \text{if K has repeated indices} \\ (\operatorname{sgn}\sigma) p_{\mathsf{set}(K)} & \text{otherwise, when } \sigma \text{ "straightens" } K \end{cases}$$

In its straightened form, we denote the right-hand side of (3.1) by $(\mathcal{Y}^*_{I,J})$. A similar formula holds among quantum- and quasi-minors as well. In all cases, the proof uses Laplace's expansion.

Proof. Consider the determinant presented below.

$$\begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,s} & a_{i_1,1} & \cdots & a_{i_1,t} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i_{s+u},1} & \cdots & a_{i_{s+u},s} & a_{i_{s+u},1} & \cdots & a_{i_{s+u},t} \\ 0 & \cdots & 0 & a_{j_1,1} & \cdots & a_{j_1,t} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{j_{t-u},1} & \cdots & a_{j_{t-u},t} \end{vmatrix}.$$

On the one hand, it's zero: clear the top-right block using the top-left block and discover a "hollow matrix" (one with a block of zeros which meets the diagonal). On the other hand, using Laplace's expansion, we that the determinant is exactly $(\mathcal{Y}_{I,J})$. $\square$

Equation (3.1) actually characterizes the image of η in $\mathbb{P}(\gamma)$: if $\pi \in \mathbb{P}(\gamma)$ satisfies (3.1*) for all allowable choices I, J, then $\pi \in \eta(F\ell(\gamma))$ (cf. [14]). This is stated in terms of coordinate functions f_I (with $f_I(\pi) = p_I$) in Hodge's "Basis Theorem" [26]:

Theorem 11. *In the homogeneous coordinate ring $\mathcal{O}_{\mathbb{P}(\gamma)} := \mathbb{C}[f_I \mid I \in \binom{[n]}{\|\gamma\|}]$, a homogeneous polynomial F is zero on the image of η if only if F belongs to the ideal of $\mathcal{O}_{\mathbb{P}(\gamma)}$ generated by $(\mathcal{Y}_{I,J^*})$ for all allowable choices $I, J \subseteq [n]$.*

3.1.3 Coordinate Algebra

Informed of the previous theorem, we may make the following definition.

Definition 5 (Flag Algebra). The *flag algebra* $\mathcal{F}(\gamma)$, i.e., the homogeneous coordinate ring of the flag variety $F\ell(\gamma)$, is the commutative $\mathbb{C}$-algebra with generators $\{f_I \mid I \in [n]^{\|\gamma\|}\}$ and relations $(\mathcal{A}_K)$ and $(\mathcal{Y}_{I,J})$ for allowable choices I, J, K.

3.2 Noncommutative Flags

Much of the preceding section may be generalized from $\mathbb{C}$ to division rings D with center $F \supseteq \mathbb{Q}$. We spell out this generalization in the present and the two subsequent sections.

Definition 6. A ring R is said to have (right) *invariant basis number* if $R^n \simeq R^m$ (as right R-modules) implies $n = m$.

In particular, any two minimal spanning sets of a finitely generated free module M_R have the same cardinality $\mathsf{rk}\, M$, which we call the (right, R-)*rank* of M. There is a characterization (cf. [31]) of IBN that is not left-right specific. In particular, a ring R has left IBN iff it has right IBN. So we may drop the modifier and speak of whether or not R has IBN.

The following key properties are easy to show.

Lemma. For any division ring D (not necessarily containing $\mathbb{Q}$), and the right D-module $V = V_D = D^n$, we have:

- D has IBN, and the traditional basis elements $\{e_i = 0_1 + \ldots + 1_i + \ldots + 0_n\}$ **form a basis** for $V = D^n$.

- Elements $v = \sum_j e_j v_j$ in V may be represented as **column vectors** $[v_1, \ldots, v_n]^T$, with D acting by multiplication from the right.

- Suppose $T \in \operatorname{End} V_D$, i.e. $\forall v, v' \in V, \forall d \in D, T(v + v'd) = T(v) + T(v')d$. Then the action of T may be given by matrix multiplication from the left. If $T(e_j) = \sum_{i=1}^n e_i t_{ij}$, then

$$T(v) = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

We return to the case where the center F of D contains $\mathbb{Q}$. Analogous results are obviously true for the left D-module $V = {}_DV = D^n$. We will develop notions of left and right flags simultaneously. As will soon be evident, the essential difference between the two theories is whether we consider rows or columns of a matrix.

Fix $V_D = D^n$ and a composition $\gamma = (\gamma_1, \ldots, \gamma_r)$ of n.

Definition 7. A (right) flag Φ of shape γ is an increasing chain of (right) D-submodules of V,

$$\Phi : (0) = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r = V\,,$$

satisfying $\mathsf{rk}\,(W_i/W_{i-1}) = \gamma_i$. For fixed γ, we let $F\ell(\gamma) = F\ell(V_D, \gamma)$ denote the collection of all flags in V_D of shape γ.

Notation. As in the commutative case, we write $F\ell(n)$ and $Gr(d,n)$ for $F\ell((1^n))$ and $F\ell(d, n-d)$ respectively.

If we fix the standard basis $\mathbf{B} = (e_1, \ldots, e_n)$ for D^n, we may represent a flag Φ as a matrix as follows. (i) Choose a basis $(w_1, \ldots, w_{\gamma_1})$ for W_1. (ii) Extend this to a basis $(w_1, \ldots, w_{\gamma_1}, w_{\gamma_1+1}, \ldots, w_{\gamma_1+\gamma_2})$ for W_2. (iii) Repeat until you have completed the sequence to a basis $\mathbf{w} = (w_1, \ldots, w_{|\gamma|})$ of V. (iv) Write $w_j = \sum_i e_i a_{i,j}$ for $1 \leq i, j \leq n$, $a_{ij} \in D$. (v) Build the matrix $A = A(\Phi, \mathbf{w}) = (a_{ij})$. Then A is the collection of column vectors $[w_1|w_2|\cdots|w_n]$ with coordinatization provided by $\mathbf{B}$.

The choice of basis for Φ was not unique, so neither is the matrix A. However, we do know exactly when two matrices A, B represent the same $\Phi \in F\ell(\gamma)$.

Lemma. $\mathbf{w}$ and $\mathbf{w}'$ represent the same flag Φ iff their associated matrices A, A' satisfy $A' = A \cdot g$ for some $g \in \mathrm{GL}_n(D)$ taking the form in Figure 3.1.

For fixed γ, we also call the collection of such $g \in \mathrm{GL}_n(D)$ a (right-) *parabolic subgroup,* and denote it by P_γ^+. Lower block-triangular matrices of analogous shape will play the role of parabolic subgroup for left D-modules; we denote this set by P_γ^-. After this lemma, we may replace the previous flag definition with a new one.

Definition 8. Given a composition $\gamma \models n$, we have $F\ell(V_D, \gamma) = \mathrm{GL}_n(D)/(\sim)$, where $A \sim A'$ iff $\exists g \in \mathrm{P}_\gamma^+$ s.t. $A' = Ag$.

Repeating the above discussion for ${}_DV = D^n$, we arrive at the analogous important

Definition 9. Given a composition $\gamma \models n$, $F\ell({}_DV, \gamma) = (\sim)\backslash\mathrm{GL}_n(D)$, where $A \sim A'$ iff $\exists g \in \mathrm{P}_\gamma^-$ s.t. $A' = gA$.

3.3 Quasi-Plücker Coordinates

Following the classic model, we would like to coordinatize our noncommutative $F\ell(\gamma)$. Obviously, the determinant is no longer available to us. In [17], Gelfand and Retakh give evidence that certain ratios of quasideterminants are the proper substitute.

As we have already mentioned, one difficulty encountered while working with quasideterminants is that they are not always defined. Another is that even when they are, they give seemingly undue weight to a specific row-column pair. The following proposition and definitions go a long way toward eliminating these problems.

Proposition 12. *Fix an $n \times n$ matrix A over a noncommutative ring, and fix $i, j \in [n]$, $M \in \binom{[n]}{m-1}$, and $L \in \binom{[n]}{m}$. As s ranges over L, those left and righ ratios appearing below which are defined share a common value.*

$$|A_{L,iM}|_{si}^{-1}|A_{L,jM}|_{sj} \qquad\qquad |A_{iM,L}|_{is}|A_{jM,L}|_{js}^{-1}$$

Definition 10 (Left/Column Coordinates). Fix two integers $1 \leq d < m$. Let B be a $d \times m$ matrix over D whose rows (columns) are indexed by R (C). Let $i, j \in C$ and $K \subseteq C$. Assume $|K| = d - 1$ and $i \notin K$. The *left quasi-Plücker coordinate* associated to (i, j, K) is given by

$$p_{ij}^K = p_{ij}^K(B) = |B_{R,iK}|_{si}^{-1}|B_{R,jK}|_{sj}\,.$$

Definition 11 (Right/Row Coordinates). Fix two integers $1 \leq d < m$. Let B' be an $m \times d$ matrix over D whose rows (columns) are indexed by R (C). Let $i, j \in R$ and $K \subseteq R$. Assume $|K| = d - 1$ and $j \notin K$. The *right quasi-Plücker coordinate* associated to (i, j, K) is given by

$$r_{ij}^K = r_{ij}^K(B') = |B'_{iK,C}|_{is}|B'_{jK,C}|_{js}^{-1}\,.$$

The coordinates are called "column" or "row" coordinates for obvious reasons. The labels "left" and "right" come from their invariance under an action of $\mathrm{GL}_d(D)$. The following important result was first formulated in [17]

Proposition 13. *Suppose $g \in \mathrm{GL}_d(D)$. Then, in the notation of the preceding definitions,*

$$p_{ij}^K(g \cdot B) = p_{ij}^K(B) \quad \textit{and} \quad r_{ij}^K(B' \cdot g) = r_{ij}^K(B')\,.$$

The proof amounts to showing that, between the two quasi-minors involved, the action of g cancels. Compare Theorem 3 for the essential ingredients of the proof.

We apply these constructions to our problem of coordinatizing flags by taking $m = n$, $d \in \|\gamma\|$, and viewing B (B') as the first d rows (columns) of $A(\Phi)$. We take left or right *quasi-Plücker coordinates* in accordance with whether we view A as a member of $F\ell({}_DV, \gamma)$ or $F\ell(V_D, \gamma)$.

As defined, the functions p_{ij}^K and r_{ij}^K, should receive rectangular-matrix inputs. When there are more rows (columns) in a given A than p_{ij}^K (r_{ij}^K) can naturally handle, we follow the implicit instruction to take only the first $|K| + 1$ rows (columns) of A. This will allow us to drastically simplify exposition in the sequel. For the remainder of the section, we discuss in detail only one member of the pair.

As a result of Theorem 13, we will have no embedding into a projective space, as our coordinates are not projective invariants of A, but true invariants. We may view our set of coordinates as a subset of D^N $(N = |\binom{[n]}{\|\gamma\|}|)$ only loosely: (i) for a given A, not all of the coordinates will be defined; (ii) the left and right D actions on D^N do not correspond to any well-defined action on the Φ which A represents. What remains true is that, in the following sense, $\{r_{ij}^K \mid |K| + 1 \in \|\gamma\|\}$ still characterizes $F\ell(\gamma)$: no greater collection of quasi-Plücker coordinates is invariant under P_γ^+; if f is a function on A which is P_γ^+ invariant, then f is a rational function in this collection of r_{ij}^K (cf. Theorem 16).

Working toward a statement analogous to Theorem 11, we start with the following results:

Proposition 14. *Let $A = (a_{ij})$ be an $n \times n$ matrix of formal, noncommuting variables. In the expressions $B = A_I$ below, interpret I as rows or columns as needed. Then the following identities hold in $F\langle\!\langle A \rangle\!\rangle$:*

- *Fix $M \in [n]^d$ $(d < n)$ with distinct entries. If $i, j \in [n]$ with $i \notin M$, then putting*

$B = A_{(i|j|M)}$, *we have*

$$p_{ij}^M(B) \text{ and } r_{ji}^M(B) \text{ do not depend on the ordering of } M.$$

- *Fix* $M \in \binom{[n]}{d}$ $(d < n-2)$. *If* $i, j, k \in [n] \setminus M$, *then putting* $B = A_{i \cup j \cup k \cup M}$, *we have*

$$p_{ij}^{k\cup M}(B)\, p_{jk}^{i\cup M}(B)\, p_{ki}^{j\cup M}(B) = -1$$

and

$$r_{ik}^{j\cup M}(B)\, r_{kj}^{i\cup M}(B)\, r_{ji}^{k\cup M}(B) = -1\,.$$

- *Fix* $M \in \binom{[n]}{d}$ $(d < n)$. *If* $i, j \in [n]$ *with* $i \notin M$, *then putting* $B = A_{i \cup j \cup M}$, *we have*

$$p_{ij}^M(B) = \begin{cases} 0 & \text{if } j \in M \\ 1 & \text{if } i = j \end{cases} \qquad \text{and} \qquad r_{ji}^M(B) = \begin{cases} 0 & \text{if } j \in M \\ 1 & \text{if } i = j \end{cases}$$

- *Fix* $M \in \binom{[n]}{d}$ $(d < n-1)$. *If* $i, j, k \in [n]$ *with* $i, j \notin M$, *then putting* $B = A_{i \cup j \cup k \cup M}$, *we have*

$$p_{ij}^M(B) p_{jk}^M(B) = p_{ik}^M(B) \qquad \text{and} \qquad r_{kj}^M(B) r_{ji}^M(B) = r_{ki}^M(B)$$

The fundamental identity holding among the quasi-Plücker coordinates appears below; it is the analog of (3.1). It was first observed (in the case of Grassmannians) in [16] under the name "quasi-Plücker relations."

Theorem 15 (Quasi-Plücker Relations). *Let* A *be an* $n \times n$ *matrix of formal, noncommuting variables. Fix* $L, M \in [n]$ *with* $s = |L| \geq |M| + 1 = t$ *and* $|L|, |M|+1 \in \|\gamma\|$. *Fix* $i \in [n] \setminus M$. *The following identities hold in* $F\langle\!\langle A \rangle\!\rangle$

$$\sum_{j \in L} p_{ij}^M(A) \cdot p_{ji}^{L\setminus j}(A) = 1\,. \tag{3.2}$$

$$\sum_{j \in L} r_{ij}^{L\setminus j}(A) \cdot r_{ji}^M(A) = 1\,. \tag{3.3}$$

We abbreviate these relations as $(\mathcal{P}_{i,L,M})$. When it is not clear from context whether we refer to the left or right version, we add an indicator, e.g. $({}^l\mathcal{P}_{i,L,M})$. As in the commutative case, they will be proven with a certain Laplace expansion.

Proof. We prove formula (3.3). If $I \subseteq [n]$ with $|I| = d$, we let A_I denote $A_{I,[d]}$. Also, we write $M = \{m_2, \ldots, m_t\}$ and $L = \{l_1, \ldots, l_s\}$ to simplify some coming indices. Fix $q \in [t]$. We compute the quasideterminant appearing below in two different ways.

$$\begin{vmatrix} \boxed{|A_{i\cup M}|_{iq}} & a_{i1} & a_{i2} & \cdots & a_{is} \\ |A_{l_1\cup M}|_{l_1 q} & a_{i1} & a_{l_1 2} & \cdots & a_{l_1 s} \\ \vdots & \vdots & & & \vdots \\ |A_{l_s\cup M}|_{l_s q} & a_{i1} & a_{l_s 2} & \cdots & a_{l_s s} \end{vmatrix}$$

First, let us name the pieces of the matrix above. Let $\xi = [\xi_0, \ldots, \xi_s]^T$ be the first column appearing above, let B denote the remaining columns, and let $C = [\xi|B]$ denote the entire matrix.

Method 1 (Using Proposition 4). This quasideterminant is zero, because the first column of C is a linear combination of its next t columns. For starters, notice that:

$$\begin{aligned} \xi_0 &= \begin{vmatrix} a_{i1} & \cdots & \boxed{a_{iq}} & \cdots & a_{it} \\ a_{m_2 1} & \cdots & a_{m_2 q} & \cdots & a_{m_2 t} \\ \vdots & & \vdots & & \vdots \\ a_{m_t 1} & \cdots & a_{m_t q} & \cdots & a_{m_t t} \end{vmatrix} \\ &= a_{iq} - \sum_{v\neq q} a_{iv} \sum_{k=2}^{t} \left|(A_{i\cup M})^{iq}\right|^{-1}_{m_k v} \cdot a_{m_k q}. \end{aligned}$$

Computing all of the coordinates of ξ at once, we have

$$\begin{aligned} \xi &= \operatorname{col}_q(B) - \operatorname{col}_1(B) \cdot \sum_{k=2}^{t} \left|(A_{i\cup M})^{iq}\right|^{-1}_{m_k 1} \cdot a_{m_k q} - \cdots \\ &\quad - \operatorname{col}_t(B) \cdot \sum_{k=2}^{t} \left|(A_{i\cup M})^{iq}\right|^{-1}_{m_k t} \cdot a_{m_k q} \\ &= \sum_{j=1}^{t} \operatorname{col}_j(B) \cdot \lambda_j \,, \end{aligned}$$

for some $\lambda_j \in F\langle\!\langle A \rangle\!\rangle$, what we meant to show.

Method 2 (Using Proposition 6). This quasideterminant has a Laplace expansion in terms of quasi-Plücker coordinates:

$$\begin{aligned}
0 &= \xi_0 - \sum_{j=1}^{s} |A_{i\cup(L\setminus l_j)}|_{ip} \cdot |A_{l_j\cup(L\setminus l_j)}|_{l_j p}^{-1} \cdot \xi_j \quad (\forall p) \\
\xi_0 &= \sum_{j=1}^{s} |A_{i\cup(L\setminus l_j)}|_{ip} \cdot |A_{l_j\cup(L\setminus l_j)}|_{l_j p}^{-1} \cdot \xi_j \\
1 &= \sum_{j=1}^{s} |A_{i\cup(L\setminus l_j)}|_{ip} \cdot |A_{l_j\cup(L\setminus l_j)}|_{l_j p}^{-1} \cdot |A_{l_j\cup M}|_{l_j q} \cdot |A_{i\cup M}|_{iq}^{-1} \\
1 &= \sum_{j\in L} r_{ij}^{L\setminus j} \cdot r_{ji}^{M} .
\end{aligned}$$

□

3.4 Toward a Coordinate Algebra

One would like a definition of the following sort: the (right-) flag algebra in the noncommutative setting is the algebra with generators r_{ij}^M and relations those described above in Propositions 14 and 15). The current state of the noncommutative theory does not contain an analog of the Basis Theorem. However, there does exist the following very compelling prelude:

Theorem 16. *Let $A = (a_{ij})$ be an $n \times n$ matrix with formal, noncommuting entries and suppose $f = f(a_{ij})$ is a rational function over the free skew-field $D = F\langle\!\langle A \rangle\!\rangle$. If $f(Ag) = f(A)$ for all $g \in \mathrm{P}_\gamma^+(D)$, then f is a rational function in the quasi-Plücker coordinates $r_{ij}^M(A)$, $|M| + 1 \in \|\gamma\|$.*

A Grassmannian version of this theorem appears in [17]. The proof is a consequence of noncommutative Gaussian Elimination and a simple application of the noncommutative Sylvester's Identity (Theorem 8) and induction. We illustrate the theorem with a 3×3 example, $\gamma = (2, 1)$.

Sketch of Proof. Using only elements of P_γ^+, we may transform A into

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21}a_{11}^{-1} & |A_{\{1,2\},\{1,2\}}|_{22} & |A_{\{1,2\},\{1,3\}}|_{23} \\ a_{31}a_{11}^{-1} & |A_{\{1,3\},\{1,2\}}|_{32} & |A_{\{1,3\},\{1,3\}}|_{33} \end{bmatrix},$$

and into

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21}a_{11}^{-1} & 1 & 0 \\ a_{31}a_{11}^{-1} & |A_{\{1,3\},\{1,2\}}|_{32}|A_{\{1,2\},\{1,2\}}|_{22}^{-1} & |A_{\{1,2,3\},\{1,2,3\}}|_{33} \end{bmatrix}.$$

Continuing Gaussian Elimination via elements of P_γ, we reach the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31}^{\emptyset} - r_{32}^{1}r_{21}^{\emptyset} & r_{32}^{1} & 1 \end{bmatrix}.$$

Consequently, f is a rational function in the Plücker coordinates r_{ij}^{M} of A. However, not all M appearing satisfy the hypotheses of the theorem; e.g. the symbol $r_{31}^{\emptyset}$ is not allowed because in this case, $|M|+1 \notin \|\gamma\| = \{2\}$. We have a little more work to do. From Proposition 14 and Theorem 15, we see that

$$\begin{aligned} r_{31}^{\emptyset} - r_{32}^{1}r_{21}^{\emptyset} &= r_{31}^{2}(r_{13}^{2}r_{31}^{\emptyset} - r_{13}^{2}r_{32}^{1}r_{21}^{\emptyset}) \\ &= r_{31}^{2}(r_{13}^{2}r_{31}^{\emptyset} + r_{12}^{3}r_{21}^{\emptyset}) \\ &= r_{31}^{2}\,, \end{aligned}$$

so we are left with the reduced form of A looking like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{31}^{2} & r_{32}^{1} & 1 \end{bmatrix}.$$

In short, if $\gamma = (\gamma_1, \ldots, \gamma_r)$, columns $|\gamma_{[i-1]}|+1$ through $|\gamma_{[i]}|$ of the reduced form of A will consist of an identity matrix (of size γ_i) atop a collection of right quasi-Plücker coordinates of size γ_i, $1 \leq i \leq r$. $\square$

Bearing in mind the absence of a Basis Theorem, we nevertheless state

Definition 12 (Ring of Quasi-Plücker Coordinates). The (right-) *noncommutative flag algebra* $\mathcal{Q}(\gamma)$, the ring of right quasi-Plücker coordinates, is the F-algebra with generators $\left\{ r_{ij}^{M} \mid M \subseteq [n] \text{ and } i,j \in [n] \text{ s.t. } j \notin M, |M|+1 \in \|\gamma\| \right\}$ and relations given in Proposition 14 and Theorem 15.

We return to the study of this interesting algebra in a later chapter.

Chapter 4

Amenable Determinants

Fix $D, n, V = D^n, \gamma \models n$, and $F\ell(\gamma)$ ($= F\ell(V_D, \gamma)$ or $= F\ell({}_DV, \gamma)$) as in the previous chapter. Here we spell out some minimal conditions one may impose on a determinant Det in order to build Grassmannians and flags via the quasideterminant. In the next chapter, we discuss where to find amenable determinants "in nature", e.g. associated to certain algebras of RTT type.

q-generic flags. First, consider an algebra $\mathcal{A}(n)$ on n^2 generators t_{ij} over a field F—ignoring the relations for now. Put all generators together in a matrix T. We view the t_{ij} as coordinate functions characterizing some set X inside D^{n^2}. Let us call X the set of *q-generic matrices over D for $\mathcal{A}(n)$*. Next, we define the *q-generic flags over D for $\mathcal{A}(n)$* as those points $\Phi \in F\ell(\gamma)$ s.t. the equivalence class $\big[A(\Phi)\big]$ has a representative in X.

q-generic flag algebras. Finally, we view the (row/column) Det minors of T in $\mathcal{A}(n)$ as coordinates for (right/left) q-generic flags. Sticking to our analogy with the classic case, we would like to define the homogeneous coordinate ring, the noncommutative "flag algebra," for the q-generic points of $F\ell(\gamma)$ **abstractly** in terms of generators and relations. The results of this chapter go a long way toward cataloging those relations.

4.1 Definition & First Properties

Definition 13. Let Det be a map from square sub-matrices of T to $\mathcal{A}(n)$. Write $\operatorname{Det} T_{R,C} = [T_{R,C}]$ for short. We will say Det is an *adequate determinant* if there are functions[1] $\mathfrak{I}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{c}}, \mathfrak{K}_{\mathfrak{r}}, \mathfrak{K}_{\mathfrak{c}} : \mathcal{P}[n] \times \mathcal{P}[n] \to F \setminus \{0\}$ associated to Det satisfying ($\forall R, C \in$

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[1] $\mathfrak{I}$ for $\mathfrak{I}$nverse, $\mathfrak{K}$ for $\mathfrak{K}$ommuting, $\mathfrak{r}$ for $\mathfrak{r}$ow, and $\mathfrak{c}$ for $\mathfrak{c}$olumn.

$\mathcal{P}[n]$, $|R| = |C|$):

1. $(\forall r \in R, c \in C) \quad [T_{r,c}] = t_{rc}$.

2. $(\forall r, r' \in R) \quad \sum_{c \in C} t_{rc} \frac{\mathfrak{I}_{\mathfrak{x}}(\{c\},C)}{\mathfrak{I}_{\mathfrak{r}}(\{r'\},R)} [(T_{R,C})^{r'c}] = [T_{R,C}] \cdot \delta_{rr'}$

   or

   $(\forall c, c' \in C) \quad \sum_{r \in R} \frac{\mathfrak{I}'_{\mathfrak{x}}(\{c\},C)}{\mathfrak{I}'_{\mathfrak{r}}(\{r\},R)} [(T_{R,C})^{rc}] t_{rc'} = [T_{R,C}] \cdot \delta_{cc'}$.

3. $(\forall r \subseteq R)(\forall c \subseteq C) \quad [T_{R,C}][T_{R \setminus r, C \setminus c}] = \frac{\mathfrak{K}_{\mathfrak{x}}(C \setminus c, C)}{\mathfrak{K}_{\mathfrak{r}}(R \setminus r, R)} [T_{R \setminus r, C \setminus c}][T_{R,C}]$.

Remark. Property 1 together with property 2 or $2'$ give a Laplace expansion for Det. Property 2 gives a way to move between 2 and $2'$, so we will make no use of $2'$ in the sequel. With our method, one may get partial results assuming properties 2 and $2'$ alone (without properties 1 and 2), but they are somewhat unsatisfactory. The reader may keep this idea in mind during the coming proofs to see what limited statements may be made in this case.

Definition 14. Suppose X, Y are two subsets of $\mathcal{P}[n]$. Let $X \times_\emptyset Y$ denote those pairs $(A, B) \in X \times Y$ satisfying $A \cap B = \emptyset$. Call a function $f : \binom{[n]}{1} \times_\emptyset \mathcal{P}[n] \to F \setminus \{0\}$ *measuring* if it satisfies: (i) $f(\{a\}, B \cup C) = f(\{a\}, B) f(\{a\}, C)$ for $\{a\}, B, C$ pairwise disjoint; (ii) $f(\{a\}, \emptyset) = 1$.

Remark. This notion abstracts the function ℓ, measuring the length of a permutation. We abuse notation and write $f(a, B)$ for $f(\{a\}, B)$. We may extend measuring functions to act on $\mathcal{P}[n] \times_\emptyset \mathcal{P}[n]$ by demanding $f(A, B) = \prod_{a \in A} f(a, B)$. Alternatively, we may extend f to act on $\binom{[n]}{1} \times \mathcal{P}[n]$ by taking $f(a, B) = f(a, B \setminus a)$. We put these two extensions together by letting $f(A, B) = \prod_{a \in A} f(a, B) = \prod_{a \in A} f(a, B \setminus a)$ for $(A, B) \in \mathcal{P}[n] \times \mathcal{P}[n]$.

Definition 15. Let Det be a map from square sub-matrices of T to $\mathcal{A}(n)$. Write $\operatorname{Det} T_{R,C} = [T_{R,C}]$ for short. We will say Det is an *amenable determinant* if there are measuring functions $\mathfrak{I}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{x}}, \mathfrak{K}_{\mathfrak{r}}, \mathfrak{K}_{\mathfrak{x}} : \mathcal{P}[n] \times \mathcal{P}[n] \to F \setminus \{0\}$ associated to Det satisfying:

1. $(\forall r, c \in [n]) \quad [T_{r,c}] = t_{rc}$.

2. $(\forall r, r' \in R) \quad \sum_{c \in C} t_{rc} \frac{\mathfrak{I}_{\mathfrak{x}}(c,C)}{\mathfrak{I}_{\mathfrak{r}}(r',R)} [(T_{R,C})^{r'c}] = [T_{R,C}] \cdot \delta_{rr'}$.

3. $(\forall R' \subseteq R)(\forall C' \subseteq C) \quad [T_{R,C}][T_{R',C'}] = \frac{\mathfrak{K}_{\mathfrak{r}}(C',C)}{\mathfrak{K}_{\mathfrak{r}}(R',R)}[T_{R',C'}][T_{R,C}]$.

Beginning from the generic noncommutative flag $F\ell(\gamma)$, and its quasi-Plücker coordinates, we build flag algebras $\mathcal{F}(\gamma)$ for noncommutative algebras $\mathcal{A}(n)$ possessing amenable determinants. Taken together, we call $(\mathcal{A}(n), \mathrm{Det})$ an *amenable pair.*

Several remarks are in order.

- We will need to be able to invert $T_{R,C}$ for many different $R, C \subseteq [n]$. From here on out, we pass to a larger ring $\mathcal{T}(n)$, some noncommutative localization of $\mathcal{A}(n)$, if necessary, to assume that $(\forall\, R, C)$ $(\forall\, 1 \leq k, l, \leq |R|)$ $\quad ((T_{R,C})^{-1})_{kl}$ is defined in $\mathcal{T}(n)$ and nonzero.

- We could make do with less; in the next two sections, we work with adequate determinants. Amenable determinants are adequate, so the results proven there hold in this more restrictive setting. We will need the amenable property only in Section 4.4, where a "q-commuting" property is proven (compare Theorems 19 and 21).

- The amenable property isn't necessary to define homogeneous coordinate rings for flags and Grassmannians. Noncommutative settings with adequate determinants also have such coordinate algebras; the coordinate functions there simply won't satisfy the strong version of the q-Commuting property. Indeed something even weaker than adequate is necessary. We only need $[T_{R,C}]$ to satisfy the conditions of Definition 13 for those $R, C \in \binom{[n]}{\|\gamma\|}$ in order to build flag coordinates for $F\ell(\gamma)$ over $\mathcal{T}(n)$. In the sequel we spend no further effort in this direction.

- All the examples the authors knows of determinants which are adequate are also amenable.[2]

- Measuring or not, it is immediate that the functions $\mathfrak{K}_*$ for amenable determinants

[2]Actually, the Yangians don't fit into this notion of amenable at all. However, an analogous notion, call it *spectral-parameter (SP) amenable* works in this setting, and indeed the Yangian determinant is SP-amenable, not merely "SP-adequate." We'll see this in due time.

must satisfy, for all $A, B \in \binom{[n]}{d}$ and for all $1 \leq d \leq n$,

$$\mathfrak{K}_{\mathfrak{r}}(A, A) = \mathfrak{K}_{\mathfrak{x}}(B, B)\,. \tag{4.1}$$

For property 3 of the definition reads $[T_{A,B}][T_{A',B'}] = 1 \cdot [T_{A',B'}][T_{A,B}]$ when $A' = A, B' = B$.

Proposition 17. *If T has an adequate determinant* $\operatorname{Det} T = [T]$*, then* $[T]$ *has an expansion in terms of quasideterminants.*

Proof. Indeed, any $[T_{R,C}]$ has such an expansion for any $R, C \subseteq [n]\,(|R| = |C|)$. We employ Theorem 2, which says $|X|_{ij}^{-1}$ and $(X^{-1})_{ji}$ are equal when both exist. Let $S_{C,R}$ be the $|R| \times |R|$ matrix (s_{cr}) given by $s_{cr} = \frac{\mathfrak{I}_{\mathfrak{x}}(c,C)}{\mathfrak{I}_{\mathfrak{r}}(r,R)}[(T_{R,C})^{rc}]$. Obviously, condition (2) of Definition 13 implies that

$$|T_{R,C}|_{rc}^{-1} = (S_{C,R})_{cr}[T_{R,C}]^{-1}.$$

Consequently, using condition (2), we have

$$\begin{aligned} |T_{R,C}|_{rc} &= \frac{\mathfrak{I}_{\mathfrak{r}}(r,R)}{\mathfrak{I}_{\mathfrak{x}}(c,C)}[T_{R,C}][(T_{R,C})^{rc}]^{-1} \\ &= \frac{\mathfrak{I}_{\mathfrak{r}}(r,R)}{\mathfrak{I}_{\mathfrak{x}}(c,C)}\frac{\mathfrak{K}_{\mathfrak{r}}(R^r,R)}{\mathfrak{K}_{\mathfrak{x}}(C^c,C)}[(T_{R,C})^{rc}]^{-1}[T_{R,C}]\,, \end{aligned}$$

or

$$\begin{aligned} [T_{R,C}] &= \frac{\mathfrak{I}_{\mathfrak{x}}(c,C)}{\mathfrak{I}_{\mathfrak{r}}(r,R)}|T_{R,C}|_{r,c}[T_{R^r,C^c}] && (4.2) \\ &= \frac{\mathfrak{I}_{\mathfrak{x}}(c,C)}{\mathfrak{I}_{\mathfrak{r}}(r,R)}\frac{\mathfrak{K}_{\mathfrak{x}}(C^c,C)}{\mathfrak{K}_{\mathfrak{r}}(R^r,R)}[T_{R^r,C^c}]|T_{R,C}|_{rc}\,. && (4.3) \end{aligned}$$

Let $R = [n]$ be all the rows of T, and let $I = (i_1, i_2, \ldots, i_n)$ be a derangement of $[n]$. Put $R^{(k)} = R \setminus \{i_1, \ldots, i_{k-1}\}$. Define C, J, and $C^{(k)}$ similarly. Repeatedly applying the above identity to $T_{R^{(k)},C^{(k)}}$ we may deduce

$$[T] = \left(\prod_{k=1}^{n}\frac{\mathfrak{I}_{\mathfrak{x}}(j_k, C^{(k)})}{\mathfrak{I}_{\mathfrak{r}}(i_k, R^{(k)})}\right) \times |T|_{i_1,j_1}|T^{i_1,j_1}|_{i_2,j_2}\cdots|t_{i_n,j_n}|_{i_n,j_n}$$

and

$$\begin{aligned} [T] \;=\; & \left(\prod_{k=1}^{n}\frac{\mathfrak{I}_{\mathfrak{x}}(j_k, C^{(k)})}{\mathfrak{I}_{\mathfrak{r}}(i_k, R^{(k)})}\right)\left(\prod_{k=1}^{n}\frac{\mathfrak{K}_{\mathfrak{x}}(C^{(k+1)}, C^{(k)})}{\mathfrak{K}_{\mathfrak{r}}(R^{(k+1)}, R^{(k)})}\right) \times \\ & |t_{i_n,j_n}|_{i_n,j_n}\cdots|T^{i_1,j_1}|_{i_2,j_2}|T|_{i_1,j_1} \end{aligned}$$

and any number of identities in between the two. $\square$

What we really care about is not a full factorization of Det but rather just (4.2) and (4.3). These allow us to replace quasi-Plücker coordinates with ratios of Det minors. Specifically, (4.2) demonstrates a passage from r_{ij}^M to $[T_{iM,[p]}][T_{jM,[p]}]^{-1}$, while (4.3) demonstrates a passage from p_{ij}^M to $[T_{[p],iM}]^{-1}[T_{[p],jM}]$. We record these key identities now for future reference.

Proposition 18. *Fix* $M \in \binom{[n]}{d-1}$ *and* $i, j \in [n] \setminus M$. *Then for all* $L \in \binom{[n]}{d}$

$$p_{ij}^M(T_{L,ijM}) = \frac{\mathfrak{I}_{\mathfrak{r}}(i, iM)\mathfrak{K}_{\mathfrak{r}}(M, iM)}{\mathfrak{I}_{\mathfrak{r}}(j, jM)\mathfrak{K}_{\mathfrak{r}}(M, jM)}[T_{L,iM}]^{-1}[T_{L,jM}] \tag{4.4}$$

$$r_{ij}^M(T_{ijM,L}) = \frac{\mathfrak{I}_{\mathfrak{r}}(i, iM)}{\mathfrak{I}_{\mathfrak{r}}(j, jM)}[T_{iM,L}][T_{jM,L}]^{-1} \tag{4.5}$$

The row and column situations mirror each other. In the coming sections, when faced with a proposition containing statements about both, we'll demonstrate only one.

4.2 Weak q-Commuting Relations

We are now ready to prove the first important result concerning adequate determinants.

Theorem 19. *Suppose* R, C *index the rows and columns of* T. *The following identities hold among the indicated* Det *minors of* T.

- *Suppose* $K \subseteq R$ *and* $L \subseteq C$ *satisfy* $|K| = |L| + 1 = m$. *Then for all* $a, b \in C \setminus L$,

$$[T_{K,bL}][T_{K,aL}] = -\frac{\mathfrak{I}_{\mathfrak{r}}(a, abL)\mathfrak{I}_{\mathfrak{r}}(b, bL)\mathfrak{K}_{\mathfrak{r}}(L, bL)}{\mathfrak{I}_{\mathfrak{r}}(b, abL)\mathfrak{I}_{\mathfrak{r}}(a, aL)\mathfrak{K}_{\mathfrak{r}}(L, aL)} \cdot [T_{K,aL}][T_{K,bL}]\,. \tag{4.6}$$

- *Suppose* $K \subseteq R$ *and* $L \subseteq C$ *satisfy* $|K| + 1 = |L| = m$. *Then for all* $i, j \in R \setminus K$,

$$[T_{jK,L}][T_{iK,L}] = -\frac{\mathfrak{I}_{\mathfrak{r}}(j, jK)\mathfrak{I}_{\mathfrak{r}}(i, ijK)\mathfrak{K}_{\mathfrak{r}}(jK, ijK)}{\mathfrak{I}_{\mathfrak{r}}(i, iK)\mathfrak{I}_{\mathfrak{r}}(j, ijK)\mathfrak{K}_{\mathfrak{r}}(iK, ijK)} \cdot [T_{iK,L}][T_{jK,L}]\,. \tag{4.7}$$

Remark. In the statement of the theorem, m is not specified. The proof below suggests that $m < n$, but if there is an adequate determinant for $\mathcal{T}(n+1)$, as well as for $\mathcal{T}(n)$, then we recover the case $m = n$. This possibility should cause no concern. We say "weak" because we'll show a more elaborate version later. We say "q" because the current discussion is an attempt to generalize the quantum determinant setting, where the q-commuting theorem says certain minors commute up to a power of q, cf. (6.5).

Proof. (column-minors proof) We must work with slightly larger matrices than those indicated in the statement of the theorem. To this end, we let $K^+ = K \cup k_+$ for $k_+ \in R \setminus K$, and let $K^- = K \setminus k_-$ for $k_- \in K$.

We use (4.3) and (2.2) to demonstrate (4.6):

$$|T_{K^+,abL}|_{k_+,a}^{-1} \cdot |T_{K^+,abL}|_{k_+,b} = -|T_{K,aL}|_{k_-,a}^{-1} \cdot |T_{K,bL}|_{k_-,b}$$

$$\begin{aligned}
\text{(lhs)} &= \left(\frac{\mathfrak{I}_{\mathfrak{r}}(a, abL)}{\mathfrak{I}_{\mathfrak{r}}(k_+, K^+)} \cdot [T_{K,bL}][T_{K^+,abL}]^{-1} \right) \times \\
&\quad \left(\frac{\mathfrak{I}_{\mathfrak{r}}(k_+, K^+)}{\mathfrak{I}_{\mathfrak{r}}(b, abL)} \cdot [T_{K^+,abL}][T_{K,aL}]^{-1} \right) \\
&= \frac{\mathfrak{I}_{\mathfrak{r}}(a, abL)}{\mathfrak{I}_{\mathfrak{r}}(b, abL)} \cdot [T_{K,bL}][T_{K,aL}]^{-1}
\end{aligned}$$

$$\begin{aligned}
\text{(rhs)} &= -\left(\frac{\mathfrak{I}_{\mathfrak{r}}(a, aL)}{\mathfrak{I}_{\mathfrak{r}}(k_-, K)} \frac{\mathfrak{K}_{\mathfrak{r}}(L, aL)}{\mathfrak{K}_{\mathfrak{r}}(K^-, K)} \cdot [T_{K,aL}]^{-1}[T_{K^-,L}] \right) \times \\
&\quad \left(\frac{\mathfrak{I}_{\mathfrak{r}}(k_-, K)}{\mathfrak{I}_{\mathfrak{r}}(b, bL)} \frac{\mathfrak{K}_{\mathfrak{r}}(K^-, K)}{\mathfrak{K}_{\mathfrak{r}}(L, bL)} \cdot [T_{K^-,L}]^{-1}[T_{K,bL}] \right) \\
&= \frac{\mathfrak{I}_{\mathfrak{r}}(a, aL)\mathfrak{K}_{\mathfrak{r}}(L, aL)}{\mathfrak{I}_{\mathfrak{r}}(b, bL)\mathfrak{K}_{\mathfrak{r}}(L, bL)} \cdot [T_{K,aL}]^{-1}[T_{K,bL}] \, .
\end{aligned}$$

Equating the two sides and clearing denominators completes the proof. □

In the case of amenable determinants, the coefficients in the statement of the theorem take on a simpler form. For example, $\frac{\mathfrak{I}_{\mathfrak{r}}(a,abL)}{\mathfrak{I}_{\mathfrak{r}}(a,aL)} = \mathfrak{I}_{\mathfrak{r}}(a, b)$. When we extend this weak q-commuting property in Section 4.4, the measuring property of $\mathfrak{I}_*$ and $\mathfrak{K}_*$ will be essential.

4.3 Young Symmetry Relations

Our next result is equally important toward the goal of building flags and Grassmannians for $\mathcal{A}(n)$. Before we reach the statement, another word about the alternating property of Det. When $\{i\}, \{j\}, M$ are pairwise disjoint subsets of $[n]$, the statement

$$r_{ji}^M = \frac{\mathfrak{I}_{\mathfrak{r}}(j, jM)}{\mathfrak{I}_{\mathfrak{r}}(i, iM)} [T_{jM}][T_{iM}]^{-1}$$

and the corresponding statement involving left flag coordinates are *true* statements. That is both sides are defined and (by what has come before) equal. However, when

$j \in M$, the left-hand side above is zero, while the right-hand side may not be. This will not present a problem in this section, because (by the alternating property for quasideterminants) we may simply drop the zero terms before making the translation from quasi-Plücker coordinates to quantum Plücker coordinates. We pick up the alternating thread again in Chapter 6.

Theorem 20. *Fix two integers $1 \le s \le r < n$. The following relations hold among the indicated* Det *minors of T.*

- *(Row Relations) For all $K, M \subseteq [n]$ with $|K| = r+1$, $|M| = s-1$ we have:*

$$0 = \sum_{k \in K \setminus M} \frac{\mathfrak{I}_{\mathfrak{r}}(k, kM)}{\mathfrak{I}_{\mathfrak{r}}(k, K)\mathfrak{K}_{\mathfrak{r}}(K \setminus k, K)} \cdot [T_{K\setminus k,[r]}][T_{kM,[s]}]\,. \tag{4.8}$$

- *(Column Relations) For all $K, M \subseteq [n]$ with $|K| = r+1$, $|M| = s-1$ we have:*

$$0 = \sum_{k \in K \setminus M} \frac{\mathfrak{I}_{\mathfrak{x}}(k, K)}{\mathfrak{I}_{\mathfrak{x}}(k, kM)\mathfrak{K}_{\mathfrak{x}}(M, kM)} \cdot [T_{[s],kM}][T_{[r],K\setminus k}]\,. \tag{4.9}$$

Proof. (row-minors proof) We begin with (3.3). From our data (K, M), we build data (i, L, M) to use $({}^{r}\mathcal{P}_{i,L,M})$—i.e. (3.3). Let $i = \min K$, $L = K \setminus i$, and let M be the same across the two instances.

$$\begin{aligned} 1 &= \sum_{k \in L} r_{ik}^{L\setminus k} r_{ki}^{M} \\ &= \sum_{k \in L \setminus M} r_{ik}^{L\setminus k} r_{ki}^{M} \\ &= \sum_{k \in L \setminus M} |T_{iL^k,[r]}|_{ic} |T_{L,[r]}|_{kc}^{-1} \cdot |T_{kM,[s]}|_{kc'} |T_{iM,[s]}|_{ic'}^{-1}\,. \end{aligned}$$

Applying (4.5), we may rewrite this last equality as

$$1 = \sum_{k \in L \setminus M} \frac{\mathfrak{I}_{\mathfrak{r}}(i, iL^k)}{\mathfrak{I}_{\mathfrak{r}}(k, kL^k)} \frac{\mathfrak{I}_{\mathfrak{r}}(k, kM)}{\mathfrak{I}_{\mathfrak{r}}(i, iM)} [T_{iL^k,[r]}][T_{kL^k,[r]}]^{-1} \cdot [T_{kM,[s]}][T_{iM,[s]}]^{-1}\,.$$

Using (4.7), this becomes

$$\begin{aligned}
1 &= \sum_{k\in L\setminus M}\left(-\frac{\mathfrak{I}_{\mathfrak{r}}(k,kL^k)\mathfrak{I}_{\mathfrak{r}}(i,ikL^k)}{\mathfrak{I}_{\mathfrak{r}}(i,iL^k)\mathfrak{I}_{\mathfrak{r}}(k,ikL^k)}\frac{\mathfrak{K}_{\mathfrak{r}}(kL^k,ikL^k)}{\mathfrak{K}_{\mathfrak{r}}(iL^k,ikL^k)}\right)\times \\
&\quad \frac{\mathfrak{I}_{\mathfrak{r}}(i,iL^k)}{\mathfrak{I}_{\mathfrak{r}}(k,kL^k)}\frac{\mathfrak{I}_{\mathfrak{r}}(k,kM)}{\mathfrak{I}_{\mathfrak{r}}(i,iM)}[T_{kL^k,[r]}]^{-1}[T_{iL^k,[r]}][T_{kM,[s]}][T_{iM,[s]}]^{-1} \\
&= -\sum_{k\in L\setminus M}\left(\frac{\mathfrak{I}_{\mathfrak{r}}(k,kM)}{\mathfrak{I}_{\mathfrak{r}}(i,iM)}\frac{\mathfrak{I}_{\mathfrak{r}}(i,ikL^k)}{\mathfrak{I}_{\mathfrak{r}}(k,ikL^k)}\frac{\mathfrak{K}_{\mathfrak{r}}(kL^k,ikL^k)}{\mathfrak{K}_{\mathfrak{r}}(iL^k,ikL^k)}\right)\times \\
&\quad [T_{kL^k,[r]}]^{-1}[T_{iL^k,[r]}][T_{kM,[s]}][T_{iM,[s]}]^{-1}
\end{aligned}$$

Now move to the left-hand side all things independent of k and get

$$\frac{\mathfrak{I}_{\mathfrak{r}}(i,iM)}{\mathfrak{I}_{\mathfrak{r}}(i,K)\mathfrak{K}_{\mathfrak{r}}(K\setminus i,K)}\cdot[T_{K\setminus i,[r]}][T_{iM,[s]}] = -\sum_{k\in K\setminus i\setminus M}\frac{\mathfrak{I}_{\mathfrak{r}}(k,kM)}{\mathfrak{I}_{\mathfrak{r}}(k,K)\mathfrak{K}_{\mathfrak{r}}(K\setminus k,K)}\cdot[T_{K\setminus k,[r]}][T_{kM,[s]}]\,,$$

or

$$0 = \sum_{k\in K\setminus M}\frac{\mathfrak{I}_{\mathfrak{r}}(k,kM)}{\mathfrak{I}_{\mathfrak{r}}(k,K)\mathfrak{K}_{\mathfrak{r}}(K\setminus k,K)}\cdot[T_{K\setminus k,[r]}][T_{kM,[s]}]\,.$$

□

4.4 q-Commuting Relations

Here we make our first use of the measuring properties of the functions $\mathfrak{I}_*$ and $\mathfrak{K}_*$ associated to Det.

Definition 16. Given $i,j\in[n]$, consider the expressions

$$\lambda_j = \lambda_j(i) = -\frac{\mathfrak{I}_{\mathfrak{r}}(i,j)\mathfrak{K}_{\mathfrak{r}}(j,i)}{\mathfrak{I}_{\mathfrak{r}}(j,i)}.$$

$$\rho_j = \rho_j(i) = -\frac{\mathfrak{I}_{\mathfrak{r}}(i,j)}{\mathfrak{I}_{\mathfrak{r}}(j,i)\mathfrak{K}_{\mathfrak{r}}(i,j)}.$$

As indicated in the notation, we consider these as functions of one variable (namely, i) with one parameter (j). Given $J,I\subseteq[n]$ with $J\cap I=\emptyset$ and $|J|\le|I|$, we say J *can't distinguish* I *as columns (as rows)* if λ_j (ρ_j) is constant on I for each $j\in J$. We extend this definition to pairs (I,J) with $I\cap J\neq\emptyset$ by saying J can't distinguish I if λ_j (ρ_j) is constant for all $j\in J\setminus I$ as a function on $I\setminus J$.

Remark. This definition becomes much more transparent when applied to the specific determinants introduced in the next chapter. It will amount to the existence of a certain partition $J'\dot\cup J''$ of J so that $J' \prec I \prec J''$ (cf. Chapter 2 for notation). For now, write λ_J for the product $\prod_{j\in J}\lambda_j$ evaluated at some $i \in I$ (defining ρ_J similarly).

The following theorem is the main result of this section. For a cleaner statement, we collect all the notation here before we begin. Fix $J \in \binom{[n]}{s}$ and $I, K \in \binom{[n]}{r}$ with $s \leq r$. Let $\bar{K}$ be the first s elements of K, and let $\hat{K} = K \setminus \bar{K}$. Fix $M, L \in \binom{[n]}{t}$. Suppose additionally that I, J, M are pairwise disjoint, and that $L \cap K = \emptyset$.

Theorem 21 (q-Commuting Relations). *Let* Det *be an amenable determinant with associated measuring functions $\mathfrak{I}_*$ and $\mathfrak{K}_*$. For all $I, J, K, L, M \in \mathcal{P}[n]$ as above with $1 \leq s \leq r \leq n$ and $0 \leq t \leq n-r$ we have*

- *If J can't distinguish I as columns, then*

$$[T_{\bar{K}L,JM}][T_{KL,IM}] = \frac{\lambda_J \mathfrak{K}_{\mathfrak{r}}(\bar{K},K)}{\mathfrak{K}_{\mathfrak{x}}(J,I)} \cdot \frac{\mathfrak{K}_{\mathfrak{x}}(M,J)\mathfrak{K}_{\mathfrak{r}}(L,\hat{K})}{\mathfrak{K}_{\mathfrak{x}}(M,I)}[T_{KL,IM}][T_{\bar{K}L,JM}] \tag{4.10}$$

- *If J can't distinguish I as rows, then*

$$[T_{JM,\bar{K}L}][T_{IM,KL}] = \frac{\rho_J \mathfrak{K}_{\mathfrak{r}}(J,I)}{\mathfrak{K}_{\mathfrak{x}}(\bar{K},K)} \cdot \frac{\mathfrak{K}_{\mathfrak{r}}(M,I)}{\mathfrak{K}_{\mathfrak{r}}(M,J)\mathfrak{K}_{\mathfrak{x}}(L,\hat{K})}[T_{IM,KL}][T_{JM,\bar{K}L}] \tag{4.11}$$

We begin by investigating a property of $\mathfrak{K}_*$ that will prove essential. Next, we introduce two key propositions that will serve as the base case for an induction proof. Finally, we prove the q-commuting property for amenable determinants.

Proposition 22 (Key Properties of $\mathfrak{K}_*$). *If* Det *is an amenable determinant then there exists a constant $\theta \in F \setminus \{0\}$ such that for all $i, j \in [n]$, $i \neq j$*

$$\mathfrak{K}_{\mathfrak{x}}(i,j)\mathfrak{K}_{\mathfrak{x}}(j,i) = \theta \quad \text{and} \quad \mathfrak{K}_{\mathfrak{r}}(i,j)\mathfrak{K}_{\mathfrak{r}}(j,i) = \theta\,. \tag{4.12}$$

In particular, for all $A \in \mathcal{P}[n]$,

$$\mathfrak{K}_{\mathfrak{x}}(A,A) = \theta^{\binom{|A|}{2}} \quad \text{and} \quad \mathfrak{K}_{\mathfrak{r}}(A,A) = \theta^{\binom{|A|}{2}}\,. \tag{4.13}$$

Proof. We recall the observation that, if Det is an amenable determinant, then—measuring or no—the functions $\mathfrak{K}_*$ satisfy $\mathfrak{K}_{\mathfrak{r}}(A,A) = \mathfrak{K}_{\mathfrak{x}}(B,B)$ for all $A, B \in \binom{[n]}{d}$ and for all $1 \leq d \leq n$.

Consider the case when $A = \{i, j\}$. The measuring property implies $\mathfrak{K}_{\mathfrak{r}}(A, A) = \mathfrak{K}_{\mathfrak{r}}(i, ij)\mathfrak{K}_{\mathfrak{r}}(j, ij) = \mathfrak{K}_{\mathfrak{r}}(i, j)\mathfrak{K}_{\mathfrak{r}}(j, i)$; and that this last expression is independent of i and j. Call this constant θ. One similarly concludes that $\mathfrak{K}_{\mathfrak{x}}(i, j)\mathfrak{K}_{\mathfrak{x}}(j, i) = \theta$.

Finally, for $A = \{a_1, a_2, \ldots, a_d\}$, we have

$$\begin{aligned}
\mathfrak{K}_{\mathfrak{x}}(A, A) &= \mathfrak{K}_{\mathfrak{r}}(A, A) \\
&= \prod_{i=1}^{d} \mathfrak{K}_{\mathfrak{r}}(a_i, A) = \prod_{i=1}^{d} \mathfrak{K}_{\mathfrak{r}}(a_i, A^i) \\
&= \prod_{i \neq j} \mathfrak{K}_{\mathfrak{r}}(a_i, a_j)\mathfrak{K}_{\mathfrak{r}}(a_j, a_i) \\
&= \theta^{\binom{d}{2}}, \text{ as needed.}
\end{aligned}$$

□

Proposition 23. *Let* Det *be an amenable determinant. Suppose* $I \subseteq [n]$ *and* $j \in [n] \setminus I$ *are such that* $\{j\}$ *can't distinguish* I*. Then—writing* $|I| = r$*—for all* $K \in \binom{[n]}{r}$ *and for all* $k \in K$ *we have:*

- $[T_{j,k}][T_{I,K}] = \rho_j \cdot \frac{\mathfrak{K}_{\mathfrak{r}}(j,I)}{\mathfrak{K}_{\mathfrak{x}}(k,K)} [T_{I,K}][T_{j,k}]\,,$
- $[T_{k,j}][T_{K,I}] = \lambda_j \cdot \frac{\mathfrak{K}_{\mathfrak{r}}(k,K)}{\mathfrak{K}_{\mathfrak{x}}(j,I)} [T_{K,I}][T_{k,j}]\,.$

Proof. (row-minors proof) Writing the quasi-Plücker relation ${}^{r}\mathcal{P}_{j,I,\emptyset}$ for $T_{jI,K}$ in terms of Det minors, we have

$$1 = \sum_{i \in I} \frac{\mathfrak{I}_{\mathfrak{r}}(j, jI^i)}{\mathfrak{I}_{\mathfrak{r}}(i, iI^i)} \frac{\mathfrak{I}_{\mathfrak{r}}(i, i)}{\mathfrak{I}_{\mathfrak{r}}(j, j)} [T_{jI^i,K}][T_{I,K}]^{-1}[T_{i,k}][T_{j,k}]^{-1}\,,$$

or

$$[T_{j,k}] = \sum_{i \in I} \frac{\mathfrak{I}_{\mathfrak{r}}(j, I^i)}{\mathfrak{I}_{\mathfrak{r}}(i, I^i)} [T_{jI^i,K}][T_{I,K}]^{-1}[T_{i,k}]\,,$$

using the measuring property of $\mathfrak{I}_{\mathfrak{r}}$.

Remark. If $K = \{k_1, \ldots, k_r\}$ then as stated (and proven, cf. Theorem 15) the identity ${}^{r}\mathcal{P}_{j,I,\emptyset}$ above involves coordinates $r_{ji}^{I^i}(T_{jI,K})$ and $r_{ji}^{\emptyset}(T_{jI,k_1})$, while we have used an arbitrary k in the second factor. This modified identity is also true, with the same proof.

We have already established that $[T_{j\cup I\setminus i,K}]$ and $[T_{I,K}]$ q-commute (Theorem 19). Clearing the denominator to the left, we get

$$\begin{aligned}
[T_{I,K}][T_{j,k}] &= \sum_{i\in I}\left(-\frac{\mathfrak{I}_{\mathfrak{r}}(i,I^i)\mathfrak{I}_{\mathfrak{r}}(j,I)\mathfrak{K}_{\mathfrak{r}}(I,jI)}{\mathfrak{I}_{\mathfrak{r}}(j,I^i)\mathfrak{I}_{\mathfrak{r}}(i,jI^i)\mathfrak{K}_{\mathfrak{r}}(jI^i,jI)}\right)\times \\
&\quad \frac{\mathfrak{I}_{\mathfrak{r}}(j,I^i)}{\mathfrak{I}_{\mathfrak{r}}(i,I^i)}[T_{j\cup I\setminus i,K}][T_{i,k}] \\
&= \frac{1}{\rho_j\mathfrak{K}_{\mathfrak{r}}(j,I)}\left\{\sum_{i\in I}\mathfrak{K}_{\mathfrak{r}}(i,I^i)\cdot\frac{\mathfrak{I}_{\mathfrak{r}}(j,I^i)}{\mathfrak{I}_{\mathfrak{r}}(i,I^i)}[T_{jI^i,K}][T_{i,k}]\right\}. \qquad (4.14)
\end{aligned}$$

If instead we use property (2) of Det to clear denominators to the right, we get

$$\begin{aligned}
[T_{j,k}][T_{I,K}] &= \sum_{i\in I}\left(\frac{\mathfrak{K}_{\mathfrak{r}}(i,I^i)}{\mathfrak{K}_{\mathfrak{r}}(k,K)}\right)\cdot\frac{\mathfrak{I}_{\mathfrak{r}}(j,I^i)}{\mathfrak{I}_{\mathfrak{r}}(i,I^i)}[T_{jI^i,K}][T_{i,k}] \\
&= \frac{1}{\mathfrak{K}_{\mathfrak{r}}(k,K)}\left\{\sum_{i\in I}\mathfrak{K}_{\mathfrak{r}}(i,I^i)\frac{\mathfrak{I}_{\mathfrak{r}}(j,I^i)}{\mathfrak{I}_{\mathfrak{r}}(i,I^i)}[T_{jI^i,K}][T_{i,k}]\right\}. \qquad (4.15)
\end{aligned}$$

Comparing (4.14) and (4.15), we conclude that $[T_{j,k}]$ and $[T_{I,K}]$ q-commute as desired.

□

Proposition 24. *Let* Det *be an amenable determinant. Suppose* $J\in\binom{[n]}{s}$ *and* $I\in\binom{[n]}{r}$ *satisfy* $I\cap J\neq\emptyset$ *and* $s\le r$. *Then for all* $\bar{K}=\{k_1,\ldots,k_s\}$ *and* $K=\bar{K}\cup\{k_{s+1},\ldots,k_r\}$, *and for all* $M\in[n]\setminus(I\cup J)$ *and* $L\in[n]\setminus K$ *with* $1\le|L|=|M|=t\le n-r$, *we have:*

- *If* $[T_{J,\bar{K}}][T_{I,K}]=X\cdot[T_{I,K}][T_{J,\bar{K}}]$ *for some* $X=X_{I,J}^{\bar{K},K}$ *in* $F\setminus\{0\}$, *then*

$$[T_{JM,\bar{K}L}][T_{IM,KL}]=\frac{\mathfrak{K}_{\mathfrak{r}}(M,I)}{\mathfrak{K}_{\mathfrak{r}}(M,J)\mathfrak{K}_{\mathfrak{r}}(L,K\setminus\bar{K})}\cdot X\cdot[T_{IM,KL}][T_{JM,\bar{K}L}]\,. \qquad (4.16)$$

- *If* $[T_{\bar{K},J}][T_{K,I}]=Y\cdot[T_{K,I}][T_{\bar{K},J}]$ *for some* $Y=Y_{\bar{K},K}^{I,J}$ *in* $F\setminus\{0\}$, *then*

$$[T_{\bar{K},J}][T_{K,I}]=\frac{\mathfrak{K}_{\mathfrak{r}}(M,J)\mathfrak{K}_{\mathfrak{r}}(L,K\setminus\bar{K})}{\mathfrak{K}_{\mathfrak{r}}(M,I)}\cdot Y\cdot[T_{K,I}][T_{\bar{K},J}]\,. \qquad (4.17)$$

Proof. (row-minor proof) The statement is a consequence of Muir's Law (Theorem 7). It will be convenient to begin from the modified equation

$$[T_{I,K}]^{-1}[T_{J,\bar{K}}]=X\cdot[T_{J,\bar{K}}][T_{I,K}]^{-1}\,. \qquad (4.18)$$

To ease notation, let $\hat{K}=K\setminus\bar{K}$. Also, for any $A=\{a_1,\ldots,a_p\}\subseteq[n]$ let $A^{(k)}=\{a_k,\ldots,a_p\}$ and $A^{((k))}$ denote $\{a_1,\ldots,a_{p-k+1}\}$ (i.e. delete the first $k-1$ or last $k-1$ elements of A respectively). When $k>p$ understand $A^{(k)}$ and $A^{((k))}$ to be empty.

We use Proposition 17 to write $[T_{J,\bar{K}}]$ and $[T_{I,K}]$ in terms of quasideterminants. We peel off row-indices from the head of the list, and column-indices from the tail of the list. Now, (4.18) becomes:

$$\begin{aligned}(lhs) \quad &= \quad \left(\left\{ \frac{\prod_{\ell=1}^{r} \mathfrak{I}_{\mathfrak{r}}(k_{r+1-\ell}, K^{((\ell))}) \mathfrak{K}_{\mathfrak{r}}(K^{((\ell+1))}, K^{((\ell))})}{\prod_{\ell=1}^{r} \mathfrak{I}_{\mathfrak{r}}(i_\ell, I^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(I^{(\ell+1)}, I^{(\ell)})} \right\} \times \right. \\ &\qquad \left. |T_{i_r,k_r}|_{i_r,k_r} \cdots |T_{I^{i_1},K^{k_1}}|_{i_2,k_2} |T_{I,K}|_{i_1,k_1} \right)^{-1} \times \\ &\qquad \left(\left\{ \frac{\prod_{\ell=1}^{s} \mathfrak{I}_{\mathfrak{r}}(k_{s+1-\ell}, \bar{K}^{((\ell))}) \mathfrak{K}_{\mathfrak{r}}(\bar{K}^{((\ell+1))}, \bar{K}^{((\ell))})}{\prod_{\ell=1}^{s} \mathfrak{I}_{\mathfrak{r}}(j_\ell, J^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)}, J^{(\ell)})} \right\} \times \right. \\ &\qquad \left. |T_{j_s,k_s}|_{j_s,k_s} \cdots |T_{J^{j_1},\bar{K}^{k_1}}|_{j_2,k_2} |T_{J,\bar{K}}|_{j_1,k_1} \right).\end{aligned}$$

$$\begin{aligned}(rhs) \quad &= \quad X \cdot \left(\left\{ \frac{\prod_{\ell=1}^{s} \mathfrak{I}_{\mathfrak{r}}(k_{s+1-\ell}, \bar{K}^{((\ell))})}{\prod_{\ell=1}^{s} \mathfrak{I}_{\mathfrak{r}}(j_\ell, J^{(\ell)})} \right\} \times \right. \\ &\qquad \left. |T_{J,\bar{K}}|_{j_1,k_1} |T_{J^{j_1},\bar{K}^{k_1}}|_{j_2,k_2} \cdots |T_{j_s,k_s}|_{j_s,k_s} \right) \times \\ &\qquad \left(\left\{ \frac{\prod_{\ell=1}^{r} \mathfrak{I}_{\mathfrak{r}}(k_{r+1-\ell}, K^{((\ell))})}{\prod_{\ell=1}^{r} \mathfrak{I}_{\mathfrak{r}}(i_\ell, I^{(\ell)})} \right\} \times \right. \\ &\qquad \left. |T_{I,K}|_{i_1,k_1} |T_{I^{i_1},K^{k_1}}|_{i_2,k_2} \cdots |T_{i_r,k_r}|_{i_r,k_r} \right)^{-1}.\end{aligned}$$

Take $M = \{m_1, \ldots, m_t\}$, and apply Muir's Law to get

$$\begin{aligned}(lhs) \quad &= \quad \frac{\beta_J^{\bar{K}}}{\beta_I^{K}} \left(|T_{IM,KL}|_{i_1,k_1}^{-1} \cdots |T_{i_rM,k_rL}|_{i_r,k_r}^{-1} \right) \times \\ &\qquad \left(|T_{j_sM,k_s}|_{j_s,k_s} \cdots |T_{JM,\bar{K}L}|_{j_1,k_1} \right) \\ (rhs) \quad &= \quad X \cdot \frac{\tilde{\beta}_J^{\bar{K}}}{\tilde{\beta}_I^{K}} \left(|T_{JM,\bar{K}L}|_{j_1,k_1} \cdots |T_{j_sM,k_sL}|_{j_s,k_s} \right) \times \\ &\qquad \left(|T_{i_rM,k_rL}|_{i_r,k_r}^{-1} \cdots |T_{IM,KL}|_{i_1,k_1}^{-1} \right).\end{aligned}$$

Here $\beta, \tilde{\beta}$ just replace the products detailed above. Focusing on the left-hand side for a moment, we may multiply and divide by quasi-minors of $T_{M,L}$ to get

$$\begin{aligned}&\frac{\beta_J^{\bar{K}}}{\beta_I^{K}} \left(|T_{IM,KL}|_{i_1,k_1}^{-1} \cdots |T_{i_rM,k_rL}|_{i_r,k_r}^{-1} \cdot \left\{ |T_{M,L}|_{m_1,l_1}^{-1} \cdots |T_{m_t,l_t}|_{m_t,l_t}^{-1} \right\} \right) \times \\ &\qquad \left(\left\{ |T_{m_t,l_t}|_{m_t,l_t} \cdots |T_{M,L}|_{m_1,l_1} \right\} \cdot |T_{j_sM,k_sL}|_{j_s,k_s} \cdots |T_{JM,\bar{K}L}|_{j_1,k_1} \right).\end{aligned}$$

Now, multiplying and dividing by the β corresponding to $(J \cup M, \bar{K} \cup L)$ and $(I \cup$

$M, K \cup L)$ we may reinterpret the left-hand side as Det minors:

$$(lhs) = \frac{\beta_J^{\bar{K}}}{\beta_I^K} \frac{\beta_{I\cup M}^{K\cup L}}{\beta_{J\cup M}^{\bar{K}\cup L}} [T_{IM,KL}]^{-1} [T_{JM,\bar{K}L}] \,.$$

Similarly, the right-hand side becomes

$$(rhs) = X \cdot \frac{\tilde{\beta}_J^{\bar{K}}}{\tilde{\beta}_I^K} \frac{\tilde{\beta}_{I\cup M}^{K\cup L}}{\tilde{\beta}_{J\cup M}^{\bar{K}\cup L}} [T_{JM,\bar{K}L}][T_{IM,KL}]^{-1} \,.$$

It is left to consider the expression

$$Y_0 = \left(\frac{\beta_J^{\bar{K}}}{\beta_I^K} \frac{\beta_{I\cup M}^{K\cup L}}{\beta_{J\cup M}^{\bar{K}\cup L}} \right)^{-1} \cdot \frac{\tilde{\beta}_J^{\bar{K}}}{\tilde{\beta}_I^K} \frac{\tilde{\beta}_{I\cup M}^{K\cup L}}{\tilde{\beta}_{J\cup M}^{\bar{K}\cup L}}$$

and hope that Y_0 simplifies to the advertised Y.

We focus first on the $\beta, \tilde{\beta}$ pieces involving J, call this $Y_{0,J}$, then we move on to $Y_0 = Y_{0,J} Y_{0,I}$. Writing out $Y_{0,J}$ in terms of $\mathfrak{I}_*, \mathfrak{K}_*$, we have

$$\begin{aligned} Y_{0,J} &= \frac{\tilde{\beta}_J^{\bar{K}}}{\beta_J^{\bar{K}}} \cdot \frac{\beta_{J\cup M}^{\bar{K}\cup L}}{\tilde{\beta}_{J\cup M}^{\bar{K}\cup L}} \\ &= \frac{\frac{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{x}}(k_{s+1-\ell}, \bar{K}^{((\ell))})}{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, J^{(\ell)})}}{\frac{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{x}}(k_{s+1-\ell}, \bar{K}^{((\ell))}) \mathfrak{K}_{\mathfrak{x}}(\bar{K}^{((\ell+1))}, \bar{K}^{((\ell))})}{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, J^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)}, J^{(\ell)})}} \times \\ &\quad \frac{\frac{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{x}}(k_{s+1-\ell}, L\bar{K}^{((\ell))}) \mathfrak{K}_{\mathfrak{x}}(L\bar{K}^{((\ell+1))}, L\bar{K}^{((\ell))}) \prod_{\ell=1}^t \mathfrak{I}_{\mathfrak{x}}(l_{t+1-\ell}, L^{(\ell)}) \mathfrak{K}_{\mathfrak{x}}(L^{((\ell+1))}, L^{((\ell))})}{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, MJ^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(MJ^{(\ell+1)}, MJ^{(\ell)}) \prod_{\ell=1}^t \mathfrak{I}_{\mathfrak{r}}(m_\ell, M^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(M^{(\ell+1)}, M^{(\ell)})}}{\frac{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{x}}(k_{s+1-\ell}, L\bar{K}^{((\ell))}) \prod_{\ell=1}^t \mathfrak{I}_{\mathfrak{x}}(l_{t+1-\ell}, L^{((\ell))})}{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, MJ^{(\ell)}) \prod_{\ell=1}^t \mathfrak{I}_{\mathfrak{r}}(m_\ell, M^{(\ell)})}} \,. \end{aligned}$$

First, note that the factors involving only M, L will also appear in $Y_{0,I}$ (and with opposite numerator-denominator parity!). Also $\bar{K}^{((\ell))} = K^{((\ell+r-s))}$, so some of the $\mathfrak{K}_{\mathfrak{x}}$ factors appearing here also appear in $Y_{0,I}$ (again with opposite parity). Let us write $\tilde{Y}_{0,J}$ for the quantity $Y_{0,J}$ with these factors suppressed. Next, we use the measuring property to arrive at a simpler expression:

$$\begin{aligned} \tilde{Y}_{0,J} &= \frac{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, J^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)}, J^{(\ell)})}{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, J^{(\ell)})} \cdot \frac{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, MJ^{(\ell)})}{\prod_{\ell=1}^s \mathfrak{I}_{\mathfrak{r}}(j_\ell, MJ^{(\ell)}) \mathfrak{K}_{\mathfrak{r}}(MJ^{(\ell+1)}, MJ^{(\ell)})} \\ &= \frac{\prod_{\ell=1}^s \mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)}, J^{(\ell)})}{\prod_{\ell=1}^s \mathfrak{K}_{\mathfrak{r}}(MJ^{(\ell+1)}, MJ^{(\ell)})} \\ &= \frac{\mathfrak{K}_{\mathfrak{r}}(M, M)^{-s}}{\prod_{\ell=1}^s \mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)}, M) \mathfrak{K}_{\mathfrak{r}}(M, J^{(\ell)})} \,. \end{aligned}$$

Repeating these simplifications for $Y_{0,I}$ we arrive at

$$\begin{aligned} Y_0 &= Y_{0,J}\cdot Y_{0,I} = \tilde{Y}_{0,J}\cdot \tilde{Y}_{0,I} \\ &= \frac{\mathfrak{K}_{\mathfrak{r}}(M,M)^{-s}}{\prod_{\ell=1}^{s}\mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)},M)\mathfrak{K}_{\mathfrak{r}}(M,J^{(\ell)})}\frac{\mathfrak{K}_{\mathfrak{r}}(M,M)^{r}\prod_{\ell=1}^{r}\mathfrak{K}_{\mathfrak{r}}(I^{(\ell+1)},M)\mathfrak{K}_{\mathfrak{r}}(M,I^{(\ell)})}{\mathfrak{K}_{\mathfrak{r}}(L,L)^{r-s}\prod_{\ell=1}^{r-s}\mathfrak{K}_{\mathfrak{r}}(K^{(\!(\ell+1)\!)},L)\mathfrak{K}_{\mathfrak{r}}(L,K^{(\!(\ell)\!)})} \\ &= \frac{\prod_{\ell=1}^{r}\mathfrak{K}_{\mathfrak{r}}(I^{(\ell+1)},M)\mathfrak{K}_{\mathfrak{r}}(M,I^{(\ell)})}{\prod_{\ell=1}^{s}\mathfrak{K}_{\mathfrak{r}}(J^{(\ell+1)},M)\mathfrak{K}_{\mathfrak{r}}(M,J^{(\ell)})\prod_{\ell=1}^{r-s}\mathfrak{K}_{\mathfrak{r}}(K^{(\!(\ell+1)\!)},L)\mathfrak{K}_{\mathfrak{r}}(L,K^{(\!(\ell)\!)})}\,. \end{aligned}$$

In the last step, we used the key property of $\mathfrak{K}_*$ noted in (4.1). Finally, we can radically simplify this expression for Y_0 by appealing to Proposition 22. Note, e.g., that $\mathfrak{K}_{\mathfrak{r}}(I^{(\ell+1)},M)\mathfrak{K}_{\mathfrak{r}}(M,I^{(\ell)}) = \mathfrak{K}_{\mathfrak{r}}(I^{(\ell+1)},M)\mathfrak{K}_{\mathfrak{r}}(M,i_\ell)\mathfrak{K}_{\mathfrak{r}}(M,I^{(\ell+1)})$. In terms of θ, this equals $\theta^{(r-\ell)t}\mathfrak{K}_{\mathfrak{r}}(M,i_\ell)$. Repeating this calculation for all products above, we see that

$$Y_0 = \frac{\mathfrak{K}_{\mathfrak{r}}(M,I)}{\mathfrak{K}_{\mathfrak{r}}(M,J)\mathfrak{K}_{\mathfrak{r}}(L,\hat{K})}.$$

And so we conclude that $Y_0 = Y$, as desired. □

We are now ready for the proof of the main theorem.

Proof of Theorem 21. Proposition 24 allows us to first consider the case $M = L = \emptyset$, and pass to the general case afterward. We proceed by induction on s, the base case having been handled in Proposition 23.

(row-minors proof) Given $J = \{j_1,\dots,j_s\}$, $I = \{i_1,\dots,i_r\}$, $K = \{k_1,\dots,k_r\}$, and $\bar{K} = \{k_1,\dots,k_s\}$ as in the statement of the theorem, we introduce some convenient notation. Let $\hat{J} = J\setminus j_1$ and $\hat{K} = K\setminus\bar{K}$. Also, we write $\bar{k} = k_s$ and let $\hat{\bar{K}} = \bar{K}\setminus\bar{k}$. Finally, we introduce an abuse of this "hat" notation: we let $\hat{I} = I\setminus i$ when the particular i on which the notation depends is clear from context. Now consider the quasi-Plücker coordinate identity ${}^{r}\mathcal{P}_{j_1,I,\hat{J}}$ applied to the matrix $T_{I\cup J,K}$. In terms of Det minors, it reads

$$1 = \sum_{i\in I}\frac{\mathfrak{I}_{\mathfrak{r}}(j_1,j_1I^i)}{\mathfrak{I}_{\mathfrak{r}}(i,iI^i)}\frac{\mathfrak{I}_{\mathfrak{r}}(i,i\hat{J})}{\mathfrak{I}_{\mathfrak{r}}(j_1,j_1\hat{J})}[T_{j_1I^i,K}][T_{I,K}]^{-1}[T_{i\hat{J},\bar{K}}][T_{j_1\hat{J},\bar{K}}]^{-1}\,,$$

or

$$[T_{J,\bar{K}}] = \sum_{i\in I}\frac{\mathfrak{I}_{\mathfrak{r}}(j_1,\hat{I})}{\mathfrak{I}_{\mathfrak{r}}(i,\hat{I})}\frac{\mathfrak{I}_{\mathfrak{r}}(i,\hat{J})}{\mathfrak{I}_{\mathfrak{r}}(j_1,\hat{J})}[T_{j_1\hat{I},K}][T_{i\hat{I},K}]^{-1}[T_{i\hat{J},\bar{K}}]\,. \tag{4.19}$$

By the weak q-commuting property of adequate determinants, we may write

$$\begin{aligned}
[T_{j_1\hat I,K}][T_{i\hat I,K}]^{-1} &= -\frac{\mathfrak{I}_\mathfrak{r}(j_1,i)\mathfrak{K}_\mathfrak{r}(i,j_1\hat I)}{\mathfrak{I}_\mathfrak{r}(i,j_1)\mathfrak{K}_\mathfrak{r}(j_1,i\hat I)}[T_{i\hat I,K}]^{-1}[T_{j_1\hat I,K}] \\
&= \frac{1}{\rho_{j_1}\mathfrak{K}_\mathfrak{r}(j_1,I)}\cdot\mathfrak{K}_\mathfrak{r}(i,\hat I)[T_{i\hat I,K}]^{-1}[T_{j_1\hat I,K}]\,. \qquad (4.20)
\end{aligned}$$

Alternatively, induction and Proposition 24 allow us to write

$$\begin{aligned}
[T_{i\hat I,K}]^{-1}[T_{i\hat J,\bar K}] &= [T_{i\hat I,\bar k K^{\bar k}}]^{-1}[T_{i\hat J,\bar k\hat{\bar K}}] \\
&= \frac{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,\hat I)}{\mathfrak{K}_\mathfrak{x}(\hat{\bar K},K^{\bar k})}\frac{\mathfrak{K}_\mathfrak{r}(i,\hat I)}{\mathfrak{K}_\mathfrak{r}(i,\hat J)\mathfrak{K}_\mathfrak{x}(\bar k,K^{\bar k}\setminus\hat{\bar K})}\cdot[T_{i\hat J,\bar k\hat{\bar K}}][T_{i\hat I,\bar k K^{\bar k}}]^{-1}\,.
\end{aligned}$$

Focusing on the coefficient, we have

$$\begin{aligned}
(\text{coeff}) &= \frac{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,\hat I)}{\mathfrak{K}_\mathfrak{x}(\hat{\bar K},K^{\bar k})}\cdot\frac{\mathfrak{K}_\mathfrak{r}(\hat J,i)}{\theta^{s-1}}\cdot\frac{\mathfrak{K}_\mathfrak{r}(i,\hat I)}{\mathfrak{K}_\mathfrak{x}(\bar k,\hat K)} \\
&= \frac{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,I)}{\mathfrak{K}_\mathfrak{x}(\hat{\bar K},K^{\bar k})}\cdot\frac{1}{\mathfrak{K}_\mathfrak{x}(\hat{\bar K},\bar k)\mathfrak{K}_\mathfrak{x}(\bar k,\hat{\bar K})}\cdot\frac{\mathfrak{K}_\mathfrak{r}(i,\hat I)}{\mathfrak{K}_\mathfrak{x}(\bar k,\hat K)} \\
&= \frac{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,I)}{\mathfrak{K}_\mathfrak{x}(\hat{\bar K},K)}\frac{\mathfrak{K}_\mathfrak{r}(i,\hat I)}{\mathfrak{K}_\mathfrak{x}(\bar k,K)} \\
&= \frac{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,I)}{\mathfrak{K}_\mathfrak{x}(\bar K,K)}\cdot\mathfrak{K}_\mathfrak{r}(i,\hat I)\,,
\end{aligned}$$

or

$$[T_{i\hat I,K}]^{-1}[T_{i\hat J,\bar K}] = \frac{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,I)}{\mathfrak{K}_\mathfrak{x}(\bar K,K)}\cdot\mathfrak{K}_\mathfrak{r}(i,\hat I)[T_{i\hat J,\bar K}][T_{i\hat I,K}]^{-1}\,. \qquad (4.21)$$

Using (4.20) and (4.21) to simplify (4.19), we see that

$$\begin{aligned}
\rho_{j_1}\mathfrak{K}_\mathfrak{r}(j_1,I)[T_{I,K}][T_{J,\bar K}] &= \sum_{i\in I}\mathfrak{K}_\mathfrak{r}(i,\hat I)[T_{j_1\hat I,K}][T_{i\hat J,\bar K}] \\
\frac{\mathfrak{K}_\mathfrak{x}(\bar K,K)}{\rho_{\hat J}\mathfrak{K}_\mathfrak{r}(\hat J,I)}[T_{J,\bar K}][T_{I,K}] &= \sum_{i\in I}\mathfrak{K}_\mathfrak{r}(i,\hat I)[T_{j_1\hat I,K}][T_{i\hat J,\bar K}]\,.
\end{aligned}$$

Equating the left-hand sides above completes the proof. □

4.5 Pre–Flag Algebras

After the preceding sections, we may make the following definition

Definition 17 (Pre–Flag Algebra). Given a composition $\gamma \models n$, and a noncommutative algebra $\mathcal{A}(n)$ with amenable determinant Det, the *left pre–flag algebra* $\tilde{\mathcal{F}}(\gamma)$

associated to $\mathcal{A}(n)$ is the F-algebra with generators $\left\{\tilde{f}_I \mid I \in \binom{[n]}{d},\, d \in \|\gamma\|\right\}$ and relations given by (4.9) and (4.10). The *right pre–flag algebra* is denoted by the same symbol, given the same generators, and given relations (4.8) and (4.11).

Remark. More should be said. The equations alluded to in the definition involve minors of the form $[T_{R,C}]$. When we are considering left (column) flags, we write $\tilde{f}_I$ for the "coordinate function" $[T_{[d],I}]$ (assuming $|I| = d$); when we are considering right (row) flags, we write $\tilde{f}_I$ for the coordinate function $[T_{I,[d]}]$.

Remark. Still more should be said. In Theorem 21, we deal with two triples of indices. . . the important sets (J, I, M) and some behind-the-scenes sets $(\bar{K}, K, L)$. Take $|J| = r$, $|I| = s$, and $|M| = t$. The only choice for the behind-the-scenes sets which agrees with the convention "take the first d rows (columns)" is to put $L = [t]$, $\bar{K} \cup L = [r + t]$, and $L \cup K = [s + t]$. Unfortunately, it is necessary to make a choice because the behind-the-scenes coefficients $\mathfrak{K}_*$ really don't disappear (though they may be made simpler up to a power of θ):

$$\begin{aligned}
\mathfrak{K}_*(\bar{K}, L)\mathfrak{K}_*(L, \hat{K}) &= \mathfrak{K}_*(\bar{K}, \bar{K})\mathfrak{K}_*(\bar{K}, \hat{K})\mathfrak{K}_*(L, \hat{K}) \\
&= q^{\binom{|\bar{K}|}{2}}\mathfrak{K}_*(\bar{K}L, \hat{K}) \\
&= q^{\binom{|\bar{K}|}{2}-\binom{|\bar{K}L|}{2}}\mathfrak{K}_*(\bar{K}L, \hat{K})\mathfrak{K}_*(\bar{K}L, \bar{K}L) \\
&= q^{-rt-\binom{t}{2}}\mathfrak{K}_*([r+t], [s+t]).
\end{aligned}$$

In any particular (amenable) noncommutative setting, these relations may not exhaust the identities that the minors of T satisfy. For instance, most determinants have some kind of (row or column) "alternating" property which was not quite assumed in the definition of amenable determinant. Indeed, the existence of such a property is typically the source of the *adjoint* property which we do assume for amenable determinants (Definition 13.(2 & $2'$)).

If the determinant is alternating, one may make the Young symmetry identity look much cleaner by: (i) letting the generators be indexed by $[n]^{\|\gamma\|}$ instead of by $\binom{[n]}{\|\gamma\|}$; (ii) rewriting $\mathfrak{I}_*(i, J)\tilde{f}_{i\cup J}$ as $\tilde{f}_{i|J}$. This is a minor change, producing an isomorphic algebra. However, more significant gaps may exist. As we will see in Chapter 6, the quantum flag of Taft and Towber has some relations of a novel character. It is an open question whether or not there are quasideterminantal identities which explain these extra relations.

We will pick up this discussion again in a later chapter. For now, we turn our attention to finding amenable determinants.

Chapter 5

Sources of Amenable Determinants

The majority of this chapter amounts to a cataloging of amenable determinants which arise via the R-matrix formalism. The balance, Sections 5.1 and 5.8, comprises an overview of the R-matrix formalism and a new example of an amenable determinant that *does not* come from an R-matrix. The reader will forgive the terse explanations and lack of motivations in the coming sections, as giving even one of these algebras a just treatment could double the length of this thesis. Excellent sources for more information include the books by Chari and Pressley [7] and Kassel [28]. The former stresses the physics point-of-view alluded to in the introduction.

5.1 R-Matrices and Determinants

5.1.1 The FRT construction

Fix a field F and a finite dimensional vector space $V = F^N$. Let $\tau \in \operatorname{End} V \otimes V$ be the "twist" map sending $e_i \otimes e_j$ to $e_j \otimes e_i\, (\forall i, j)$. The following theorem of Faddeev, Reshetikhin, and Takhtadzhyan is fundamental for the results of this chapter.

Theorem 25 (F-R-T, [41]). *Let F, V be as above, and fix $C \in End\, V \otimes V$. There exists a bialgebra $\mathcal{A}(C, N) = \mathcal{A}$ together with a linear map $\Delta_V : V \to \mathcal{A} \otimes V$ such that*

(i) the map Δ_V equips V with the structure of left-comodule over $\mathcal{A}$,

(ii) the map $\tau \circ C$ becomes a comodule map with respect to this structure,

(iii) if $\mathcal{A}'$ is another bialgebra coacting on V via a linear map Δ'_V such that condition 25 is satisfied, then there exists a unique bialgebra morphism $f : \mathcal{A} \to \mathcal{A}'$ such

that

$$\Delta'_V = (f \otimes \mathbb{I}_V) \circ \Delta_V.$$

The bialgebra $\mathcal{A}(C, N)$ is unique up to isomorphism.

Let $\{e_i \mid 1 \le i \le N\}$ be the standard basis for V. If C is defined by

$$C(e_i \otimes e_j) = \sum_{1 \le m,n \le N} c_{ij}^{mn} e_m \otimes e_n$$

(think multiplication on the right by the matrix $(c_{\text{row}}^{\text{col}})$), then the bialgebra of the theorem is as follows.

(*Co- Structures*) The coalgebra map on $\mathcal{A}(C, N)$ and the comodule map on V are the standard structures placed on the ring of matrix functionals $\mathcal{M}(n)$:

$$\Delta(t_i^j) = \sum_{1 \le k \le N} t_i^k \otimes t_k^j \qquad \text{and} \qquad \varepsilon(t_i^j) = \delta_{ij}\,,$$

while

$$\Delta_V e_i = \sum_{1 \le k \le N} t_i^j \otimes e_j\,.$$

(*Algebra Structure*) The algebra structure is defined so as to make $(\Delta_{V \otimes V} \circ (\tau C))$ and $((\tau C) \circ \Delta_{V \otimes V})$ agree on $V \otimes V$.

Notation. Suppose $X, Y, \{Z_k \mid 1 \le k \le m\}$ are F-modules. Then $Z = Z_1 \otimes \cdots \otimes Z_m$ is another one, as is the set $\tilde{Z}$ built by replacing Z_i with X and Z_j with Y. Given any $f \in \operatorname{End} X$ and $g \in \operatorname{End} Y$, we may extend f and g to be endomorphisms of $\tilde{Z}$ by concatenating with the identity map:

$$f_i := (\mathbb{I}_{Z_1} \otimes \cdots \otimes \mathbb{I}_{Z_{i-1}} \otimes f \otimes \mathbb{I}_{Z_{i+1}} \otimes \cdots \otimes \mathbb{I}_{Z_m}) \in \operatorname{End} Z.$$

Define g_j similarly. Also, if $h \in \operatorname{End} X \otimes Y$, we write h_{ij} for the obvious endomorphism of Z constructed analogously.

Now, consider the free algebra $\mathcal{A}_0 = F\langle t_i^j \mid 1 \le i, j \le N\rangle$. We will build $\mathcal{A}(C, N)$ from $\mathcal{A}$ by equating two endomorphisms of the F-module $V' = V \otimes V \otimes \mathcal{A}_0$. Consider the matrix

$$T = \begin{pmatrix} t_1^1 & \cdots & t_1^n \\ \vdots & & \vdots \\ t_n^1 & \cdots & t_n^n \end{pmatrix}$$

over $\mathcal{A}_0$, and define $T \in \mathrm{End}_F V \otimes \mathcal{A}_0$ by $T(e_i \otimes a) := \sum_{1\leq m\leq N} e_m \otimes t_i^m a$ (think right-multiplication of e_i by T). We demand

$$C_{12}T_{13}T_{23} = T_{23}T_{13}C_{12} \tag{5.1}$$

as endomorphisms of V'. In terms of the generators t_i^j, the relations take the form:

$$(\forall i,j)(\forall m,n) \qquad \sum_{1\leq k,l\leq N} c_{kl}^{mn} t_i^k t_j^l = \sum_{1\leq k,l\leq N} c_{ij}^{kl} t_l^n t_k^m \,.$$

5.1.2 R-matrices

Definition 18. An endomorphism $C \in \mathrm{End}\, V \otimes V$ is said to be an *R-matrix* if it is invertible, and moreover satisfies the *quantum Yang-Baxter equation:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{5.2}$$

in $\mathrm{End}\, V \otimes V \otimes V$.

In terms of matrix coefficients (and Einstein summation notation), this identity reads

$$(\forall a,b,c)(\forall u,v,w) \quad r_{ab}^{k_1k_2} r_{k_1c}^{uk_3} r_{k_2k_3}^{vw} = r_{bc}^{l_1l_2} r_{al_2}^{l_3w} r_{l_3l_1}^{uv}.$$

By now the R-matrix is ubiquitous in the study of noncommutative structures, especially those coming from physics. Indeed, it would not be controversial to define a quantum group as a Hopf algebra with an R-matrix. Which brings us to our next point. By the FRT construction one is given a bialgebra. To get a Hopf algebra, we need to define the antipode S on T. If T may be formally inverted (perhaps after extending the algebra $\mathcal{A}(C,N)$ to a larger algebra $\mathcal{T}(N)$), then putting $S(T) = T^{-1}$ is a good start. Aside from these motivational remarks (continued in the next subsection), we'll have no further use for the notions of bialgebras and Hopf algebras; so we make no effort to be more precise.

5.1.3 Determinants from R-matrices

Notation. We change notation slightly. Let F be as before, and put $V = F^n$. When $\mathcal{A}(C,n)$ is the bialgebra of the FRT construction, and C is an R-matrix, we say that

$\mathcal{A}(n)$ is an RTT-algebra—dropping the reference to C in the notation. Also, many of the established RTT-algebras use generators t_{ij} instead of t_i^j so we will change notation eventually to be consistent with later sections. One unchanging convention throughout the rest of the chapter: we write expressions for R with the understanding that it acts on $V \otimes V$ by right-multiplication.

Consider the tensor algebra $\mathbf{T}(V)$: the F-module with basis $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}$ (for $k \in \mathbb{N}$, $(i_1, \ldots, i_k) \in [n]^k$). It is graded by length (the k above), with graded piece denoted $\mathbf{T}(V)_k$ We may sometimes write $e_{i_1} e_{i_2} \cdots e_{i_k}$ or even $e_{i_1 i_2 \cdots i_k}$ to simplify notation. We may extend the $\mathcal{A}$-comodule action on V to $\mathbf{T}(V)_k$ by using $\binom{k}{2}$ twists and multiplications (i.e. letting all t_i^j commute past all e_k):

$$\Delta_{\mathbf{T}(V)}\left(e_{i_1} \otimes \cdots \otimes e_{i_k}\right) = \sum_{j_1, \ldots, j_k} t_{i_1}^{j_1} \cdots t_{i_k}^{j_k} \otimes \left(e_{j_1} \otimes \cdots \otimes e_{j_k}\right).$$

Obviously, R remains a comodule map on $\mathbf{T}(V)_k$ ($k \geq 2$) when its action is restricted to any two, fixed factors (i.e. $R = R_{ab}$, $1 \leq a < b \leq k$). Under certain conditions, one can build a nice one-dimensional $\mathcal{A}$-comodule by taking the quotient of $\mathbf{T}(V)$ by the two-sided ideal generated by $(\alpha_{ij}\mathbb{I} + (\tau R))(e_i e_j)$ (a graded ideal!) and then focusing on the highest nonzero graded piece. Here, the α_{ij} are appropriately chosen constants in F.

Example. Let $R = \mathbb{I}_n \otimes \mathbb{I}_n = \mathbb{I} \in \operatorname{End} V \otimes V$, then the exterior algebra $\Lambda(n)$ is the quotient of $\mathbf{T}(V)$ by the relation $(1 \cdot \mathbb{I} + (\tau R))v = 0$, $(\forall v \in V^{\otimes 2})$. On inspection, this simply reads $e_i e_j = -e_j e_i$.

While there are certainly many choices one could make for the coefficients α_{ij}, not all of them respect the $\mathcal{A}$-comodule structure on $\mathbf{T}(V)$. When a coherent choice is made—for example, when all α_{ij} are the same scalar α—we get a "determinant" by letting T coact on the quotient. Call the quotient Λ_R, and fix a generator $v \in \Lambda_R$, then

$$\operatorname{Det} T \text{ is the element } D \in \mathcal{A} \text{ satisfying } \Delta_{\Lambda_R}(v) = D \otimes v.$$

Determinants for sub-matrices $T_{I,J}$ ($|I| = |J| = d$) of T are built by beginning with an RTT-algebra of lesser dimension (d^2). The element D is clearly group-like in $\mathcal{A}$, but this is not the end of the similarities between det and Det.

In [22], Gurevich outlines a set of sufficient conditions on R to guarantee the existence of a nice Λ_R. For these "closed Hecke symmetries," he proves that determinants arising as above will always have a Laplace-type expansion, and moreover, they often satisfy

$$Dt_i^j = \beta_{ij} t_i^j D\,, \tag{5.3}$$

i.e., the determinant is not only group-like, but "β-central" in $\mathcal{A}$.

His conditions do not quite guarantee that Det is amenable (or even adequate). However, if every monomial $t_{i_1}^{j_1} t_{i_2}^{j_2} \cdots t_{i_m}^{j_m}$ appearing in the expression $\mathrm{Det}\, T_{I,J}$ satisfied $\prod_{1\leq k\leq m} \beta_{i_k j_k} =$ constant, then (5.3) gives Det the adequate property.

5.1.4 What's coming next

In the coming sections, we present several known determinants fitting into the R-matrix formalism. The main result each time is just a verification that these determinants are amenable and a display of the Young symmetry and q-commuting relations. The reader may feel free to skip to Section 5.8 at any time.

5.2 Commutative Determinant

Let $\mathcal{M}(n)$ be the free commutative $\mathbb{C}$-algebra generated by t_{ij}—the ring of polynomials on the $\mathbb{C}$-space $M_n(\mathbb{C})$. If $I, J \in \binom{[n]}{m}$, define $\det T_{I,J}$ by

$$\det T_{I,J} = [T_{I,J}] := \sum_{\pi\in\mathfrak{S}_C} (-1)^{\ell(\pi)} x_{i_1,\pi j_1} x_{i_2,\pi j_2} \cdots x_{i_m,\pi j_m}\,.$$

Letting $V = \mathbb{C}^n$, it is easy to see that $\mathcal{M}$ is an RTT-algebra with $R = \mathbb{I}_n \otimes \mathbb{I}_n$. Also, det is reproduced by the coaction of $\mathcal{M}$ on the n-th graded piece of $\mathbf{T}(V)$ modulo the ideal generated by $\{(\mathbb{I} + (\tau R))v \mid v \in V^{\otimes 2}\}$.

By the well-known alternating and Laplace-expansion properties of det, it is easy to check that det is an amenable determinant. Moreover, it is well-known that $\mathcal{M}(n)$ has a field of fractions $\mathcal{T}(n)$ in which all $T_{I,J}$ may be inverted. In short,

Proposition 26. *The pair* $(\mathcal{M}(n), \det)$ *is an amenable pair* $(\mathcal{A}(n), \mathrm{Det})$ *with associated measuring functions given by*

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{I}_{\mathfrak{r}}(a, A) = \mathfrak{I}_{\mathfrak{x}}(a, A) = (-1)^{\ell(a|A^a)}$

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{K}_{\mathfrak{r}}(a, A) = \mathfrak{K}_{\mathfrak{x}}(a, A) = 1.$

In $\mathcal{M}(n)$, the row-minors $\left\{\det T_{A,[d]} \mid A \in \binom{[n]}{d}\right\}$ satisfy

- *$(\forall 1 \leq s \leq r < n)$ If $K, M \subseteq [n]$ are subsets satisfying $|K| = r+1, |M| = s-1$, then*

$$0 = \sum_{k \in K \setminus M} (-1)^{\ell(k|M) - \ell(k|K^k)} [T_{K \setminus k,[r]}][T_{kM,[s]}] \tag{5.4}$$

- *If $I, J \subseteq [n]$ $(|J| = s \leq r = |I|)$ are such that J can't distinguish I as rows, then*

$$[T_{J,[s]}][T_{I,[r]}] = 1 \cdot [T_{I,[r]}][T_{J,[s]}] \tag{5.5}$$

5.3 Quantum Determinant

Fix a field F containing $\mathbb{Q}$. For the remainder of the section, fix a distinguished element $q \in F \setminus \{-1, 0, 1\}$, and fix $V = F^n$.

5.3.1 Definitions & R-matrix

Definition 19. A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called *q-generic* (over F) if

$$\begin{aligned} ba &= qab && (5.6) \\ dc &= qcd && (5.7) \\ ca &= qac && (5.8) \\ db &= qbd && (5.9) \\ cb &= bc && (5.10) \\ da &= ad + (q - q^{-1})bc && (5.11) \end{aligned}$$

An $n \times m$ matrix X is said to be *q-generic* if every 2×2 sub-matrix $X_{\{i,j\},\{k,l\}}$ is q-generic.

We are ready for our first important, noncommutative example [41, 28, 35].

Definition 20. Let $\mathcal{M}_q(n)$ denote the F-algebra with n^2 generators t_{ij} and relations given by demanding $T = (t_{ij})$ be a q-generic matrix over F. Let $\det_q$ by defined by

$$\det{}_q T_{I,J} = [T_{I,J}] := \sum_{\pi \in \mathfrak{S}_C} (-q)^{-\ell(\pi)} t_{i_1,\pi j_1} t_{i_2,\pi j_2} \dots t_{i_m,\pi j_m}$$

for all $1 \leq m \leq n$ and all $I, J \in \binom{[n]}{m}$.

$\mathcal{M}_q(n)$ is without question the most widely studied "quantization" of the algebra of matrix functionals presented in the previous section. Put $D = \det_q T$, and introduce a formal (central in $\mathcal{M}_q$) inverse S of D. The resulting quantum group $\mathrm{GL}_q(n) := \mathcal{M}_q(n)\big[S\big]/(SD-1)$ is the quantum analog of the ring of regular functions $K[G]$ on $G = GL_n(\mathbb{C})$. Like its classic counterpart, $\mathrm{GL}_q(n)$ is a Hopf algebra. We will not focus on this property in the sequel, indeed we will not focus on $\mathrm{GL}_q(n)$ at all.

Let $R \in \operatorname{End} V \otimes V$ be given by

$$R = q^{-1} \sum_i \mathbb{E}_{ii} \otimes \mathbb{E}_{ii} + \sum_{i \neq j} \mathbb{E}_{ii} \otimes \mathbb{E}_{jj} + (q^{-1} - q) \sum_{i<j} \mathbb{E}_{ij} \otimes \mathbb{E}_{ji}, \tag{5.12}$$

again, thought of as acting on the right. For example, when $n = 2$—and in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ of $V \otimes V$—we have

$$R = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1} - q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

It is a tedious but straightforward exercise to show that $\mathcal{M}_q(n)$ is an RTT-algebra for this R-matrix; and moreover, $\det_q$ is reproduced by the coaction of $\mathcal{M}_q(n)$ on the n-th graded piece of $\mathbf{T}(V)$ modulo the ideal generated by $\{(q + (\tau R))v \mid v \in V^{\otimes 2}\}$. Compare Takeuchi's article [48] for more details[1].

5.3.2 First properties

An important consequence of the relations (5.6–5.11) is that $\mathcal{M}_q(n)$ is a noetherian domain (cf. Proposition 45), and hence has an Ore field of fractions. Taking $\mathcal{T}(n)$ to

[1] A caveat for the reader. In [48], the twist τ is incorporated into the definition of R (i.e. into the Yang-Baxter equation). The resulting formulas are essentially the same, but some care should be taken in the translation from that setting to the present one.

be this field of fractions, one finds the existence of all $(T_{I,J})^{-1}_{ij}$ needed in Chapter 4.

Following the standard proof in the commutative setting (cf. [45]), one can show that $\det_q$ is "q-alternating" in rows:

Theorem 27. *Suppose $X = X_{[n],[n]}$ is an $n \times n$ q-generic matrix, and $I \in [n]^n$. Then*

$$\det_q X_{I,[n]} = \begin{cases} 0 & \textit{if } I \textit{ contains repeated indices,} \\ (-q)^{-\ell(I)} \det_q X & \text{otherwise.} \end{cases} \tag{5.13}$$

Remark. As it turns out, $\det_q$ is not column-alternating. If $I \in [n]^n$ contains distinct entries, then $\det_q X_{[n],I} = (-q)^{-\ell(I)} \det_q X$, but repeated columns don't result in zero. For example,

$$\begin{vmatrix} x_{11} & x_{11} \\ x_{21} & x_{21} \end{vmatrix}_q = x_{11}x_{21} - q^{-1}x_{11}x_{21} \neq 0.$$

This column-defect will make it most convenient to talk about right quantum flags in the sequel.

One may also follow the commutative proofs to give a q-Laplace expansion for the quantum determinant. Combining the alternating and Laplace-expansion properties, one readily deduces that

$$\sum_{j \in J} t_{ij} \frac{(-q)^{-\ell(i'|I)}}{(-q)^{-\ell(j|J)}} [T_{I \setminus i', J \setminus j}] = \delta_{ii'} [T_{I,J}] \,.$$

This result first appeared in [16], see also [30]. Finally, one has:

$$(\forall i \in I, \forall j \in J) \quad [T_{I,J}] t_{ij} = t_{ij} [T_{I,J}] \,.$$

One can show this directly, but a more clever argument uses the fact that the adjoint matrix $S(T) = (s_{ji})$ with $s_{ji} = \frac{(-q)^{-\ell(i'|I)}}{(-q)^{-\ell(j|J)}} [T_{I \setminus i', J \setminus j}]$ not only satisfies $TS = [T]\mathbb{I}_n$ but also $ST = [T]\mathbb{I}_n$ (cf. [48] for details).

5.3.3 Main result

In summation, $\det_q$ is an amenable determinant for $\mathcal{M}_q(n)$. For later use we catalog the key identities.

Proposition 28. *The pair* $(\mathcal{M}_q(n), \det_q)$ *is an amenable pair* $(\mathcal{A}(n), \mathrm{Det})$ *with associated measuring functions given by*

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{I}_{\mathfrak{r}}(a, A) = \mathfrak{I}_{\mathfrak{x}}(a, A) = (-q)^{-\ell(a|A^a)}$
- $(\forall a \in A \subseteq [n]) \quad \mathfrak{K}_{\mathfrak{r}}(a, A) = \mathfrak{K}_{\mathfrak{x}}(a, A) = 1$

In $\mathcal{M}_q(n)$, *the row-minors* $\left\{\det_q T_{A,[d]} \mid A \in \binom{[n]}{d}\right\}$ *satisfy*

- $(\forall 1 \leq s \leq r < n)$ *If* $K, M \subseteq [n]$ *are subsets satisfying* $|K| = r+1, |M| = s-1$, *then*
$$0 = \sum_{k \in K \setminus M} (-q)^{-\ell(k|M) - \ell(K^k|k)} [T_{K \setminus k,[r]}][T_{kM,[s]}] \tag{5.14}$$
- *If* $I, J \subseteq [n]$ $(|J| = s \leq r = |I|)$ *are such that* J *can't distinguish* I *as rows, then for any* $i \in I$
$$[T_{J,[s]}][T_{I,[r]}] = q^{\ell(J|i) - \ell(i|J)} \cdot [T_{I,[r]}][T_{J,[s]}] \tag{5.15}$$

5.4 Multi-Parameter Determinant

In their joint paper [1], Artin, Schelter, and Tate introduce a vast generalization of the algebra $\mathcal{M}_q(n)$ from the previous section—replacing one q with $\binom{n}{2}$ q's and their reciprocals. Fix a field F containing $\mathbb{Q}$. Fix a distinguished element $\lambda \in F \setminus \{0, -1\}$ and distinguished elements $q_{ij} \in F \setminus \{0, 1\}$ satisfying $q_{ii} = 1$, $q_{ij} q_{ji} = 1$. Fix $V = F^n$.

5.4.1 Definitions & R-matrix

Definition 21. Define $\mathcal{M}_{\vec{q}}(n)$ to be the F-algebra with generators $\{t_{ij} \mid 1 \leq i, j \leq n\}$ and relations as follows:

$$t_{jb} t_{ia} = \begin{cases} \frac{q_{ji}}{q_{ba}} t_{ia} t_{jb} + (\lambda - 1) q_{ji} t_{ib} t_{ja} & \text{if } j > i \wedge b > a \\ \lambda \frac{q_{ji}}{q_{ba}} t_{ia} t_{jb} & \text{if } j > i \wedge b \leq a \\ \frac{1}{q_{ba}} t_{ia} t_{jb} & \text{if } j = i \wedge b > a \,. \end{cases} \tag{5.16}$$

Notation. Given $J \in \binom{[n]}{m}$, we define the generalized sign of $\pi \in \mathfrak{S}_J$, $\mathrm{sgn}_{\vec{q}}(\pi)$, as follows:

$$\mathrm{sgn}_{\vec{q}}(\pi) := \prod_{\substack{j<j' \in J \\ \pi j > \pi j'}} \left(-q_{\pi j, \pi j'}\right).$$

Definition 22. Let $\det_{\vec{q}} T = [T_{[n],[n]}]$ denote the quantum determinant for T defined by

$$\det{}_{\vec{q}} T = [T] := \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}_{\vec{q}}(\pi) t_{1,\pi 1} t_{2,\pi 2} \cdots t_{n,\pi n}.$$

For $0 < m < n$ and row-indices $I = \{i_1 < \ldots < i_m\}$ and column-indices $J = \{j_1 < \ldots < j_m\}$, define quantum minors in a similar fashion:

$$[T_{I,J}] = \sum_{\pi \in \mathfrak{S}_J} \mathrm{sgn}_{\vec{q}}(\pi) t_{i_1, \pi j_1} \cdots t_{i_m, \pi j_m}.$$

As we mentioned in the introduction, the Artin-Schelter approach to noncommutative geometry is generally distinct from the quantum groups approach. Perhaps not surprisingly, there is some overlap between the two. See [1] for more details and motivation. In [23], M. Hazewinkel shows that the AST quantum algebra $\mathcal{M}_{\vec{q}}(n)$ is an RTT-algebra, with associated R-matrix given by

$$\begin{aligned} R \;=\; & \lambda^{-1} \sum_i \mathbb{E}_{ii} \otimes \mathbb{E}_{ii} + \sum_{i<j} q_{ji} \mathbb{E}_{jj} \otimes \mathbb{E}_{ii} + \\ & \sum_{i<j} \left((\lambda^{-1} q_{ij}) \mathbb{E}_{ii} \otimes \mathbb{E}_{jj} + (\lambda^{-1} - 1) \mathbb{E}_{ij} \otimes \mathbb{E}_{ji} \right). \end{aligned} \tag{5.17}$$

For example, when $n = 3$—and in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, \ldots, e_3 \otimes e_3)$ of $V \otimes V$—we have

$$R = \begin{pmatrix} \lambda^{-1} & & & & & & & & \\ & \lambda^{-1} q_{12} & & \lambda^{-1} - 1 & & & & & \\ & & \lambda^{-1} q_{13} & & & & \lambda^{-1} - 1 & & \\ & & & q_{21} & & & & & \\ & & & & \lambda^{-1} & & & & \\ & & & & & \lambda^{-1} q_{23} & & \lambda^{-1} - 1 & \\ & & & & & & q_{31} & & \\ & & & & & & & q_{32} & \\ & & & & & & & & \lambda^{-1} \end{pmatrix}.$$

It is not difficult to show that the definition of $\det_{\vec{q}}$ also comes from this R-matrix, via the coaction of $\mathcal{M}_{\vec{q}}(n)$ on the n-th graded piece of $\mathbf{T}(V)$ modulo the ideal generated by $\{(\mathbb{I}_n + (\tau R))v \mid v \in V^{\otimes 2}\}$.

5.4.2 First properties

A-S-T prove all of the properties necessary to conclude that $\det_{\vec{q}}$ is amenable. The next two theorems appear in [1]. The corollaries are easy consequences of the proofs of the theorems appearing there.

Notation. Let us extend the usual definition of $\ell(\,\cdot\,)$ as follows. If $a \in A = \{a_1 < \ldots < a_p\}$, say $a = a_i$, then let $\ell(a|A) := \ell(a|A \setminus a) = \ell(a_i, a_1, \ldots, \widehat{a_i}, \ldots, a_p) = i - 1$.

Theorem 29. *Let Q_j denote the product $\lambda^j \prod_{m=1}^n q_{jm}$. Then for all $j, k \in [n]$, we have*

$$[T]t_{jk} = \frac{Q_k}{Q_j} t_{jk}[T]\,.$$

Corollary 30. *Given a set $A \subseteq [n]$ and an element $a \in A$, let $Q_{a,A}$ denote the product $\lambda^{\ell(a|A)} \prod_{a' \in A} q_{aa'}$. Then for all row-indices I and column indices J with $|I| = |J|$, and $i \in I, j \in J$, we have*

$$[T_{I,J}]t_{ij} = \frac{Q_{j,J}}{Q_{i,I}} t_{ij}[T_{I,J}]\,.$$

Corollary 31. *In the notation of the previous corollary, the quantum minors $[T_{I\setminus i,J\setminus j}]$ q-commute with $[T_{I,J}]$ by the formula*

$$[T_{I,J}][(T_{I,J})^{ij}] = \frac{Q_{j,J}}{Q_{i,I}}[(T_{I,J})^{ij}][T_{I,J}]\,.$$

Theorem 32. *Let γ_j denote the product $\prod_{m=1}^{j-1}(-q_{jm})$ and β_j denote the product $\prod_{m=j+1}^{n}(-\lambda q_{jm})$. The matrix T of generators has a right- (and left-) inverse $S = (s_{jk})$ given by the formula*

$$s_{jk} = \frac{\gamma_k}{\gamma_j}[T^{kj}][T]^{-1} = \frac{\beta_k}{\beta_j}[T]^{-1}[T^{kj}]\,.$$

Remark. In particular, $[T]$ is not a zero divisor, and hence can be inverted in a suitable noncommutative localization of $\mathcal{M}_{\vec{q}}(n)$. The same goes for all $[T_{I,J}]$; indeed, one can show, cf. [1], that $\mathcal{M}_{\vec{q}}(n)$ is an Ore domain. Call the associated field of fractions $\mathcal{T}(n)$. This is the setting in which the calculations of Chapter 4 should be performed.

Corollary 33. *Given $a \in A \subseteq [n]$, define $\gamma_{a,A} = \prod_{a' \in A, \ell(a'|a)=0}(-q_{aa'})$. Then the following identities hold for any $i, i' \in I \subseteq [n]$ and $J \subseteq [n]$ with $|I| = |J|$:*

$$\sum_{j \in J}^{p} t_{ij} \left(\frac{\gamma_{i',I}}{\gamma_{j,J}} [T_{I^{i'},J^j}] \right) = \delta_{ii'} [T_{I,J}] \,,$$

where δ is the Kronecker delta. In particular, the sub-matrix of generators $T_{I,J}$ has an inverse $S(T_{I,J}) = (s_{ji})$ given by

$$s_{ji} = \frac{\gamma_{i,I}}{\gamma_{j,J}} [T_{I^i,J^j}][T_{I,J}]^{-1} \,.$$

5.4.3 Main result

We summarize the key properties of $\det_{\vec{q}}$. For a change of pace, we list the column-minor relations.

Proposition 34. *The pair $(\mathcal{M}_{\vec{q}}(n), \det_{\vec{q}})$ is an amenable pair $(\mathcal{A}(n), \mathrm{Det})$ with associated measuring functions given by:*

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{I}_{\mathfrak{r}}(a, A) = \mathfrak{I}_{\mathfrak{r}}(a, A) = \gamma_{a,A}$
- $(\forall a \in A \subseteq [n]) \quad \mathfrak{K}_{\mathfrak{r}}(a, A) = \mathfrak{K}_{\mathfrak{r}}(a, A) = Q_{a,A}$

In $\mathcal{M}_{\vec{q}}(n)$, the column-minors $\left\{[T_A] = \det T_{[d],A} \mid A \in \binom{[n]}{d}\right\}$ satisfy

- *$(\forall 1 \leq s \leq r < n)$ If $K, M \subseteq [n]$ are subsets satisfying $|K| = r+1, |M| = s-1$, then*
$$0 = \sum_{k \in K \setminus M} \frac{(-1)^{-\ell(k|K^k)} \prod_{k' \in K : k' < k} q_{kk'}}{(-\lambda)^{-\ell(M|k)} \prod_{m' \in M : k < m'} q_{m'k}} [T_{k \cup M}][T_{K \setminus k}] \tag{5.18}$$
- *If $I, J, M \subseteq [n]$ $(|J| = s \leq r = |I|, |M| = u)$ are pairwise disjoint, and if J can't distinguish I as rows, then for any $i \in I$*
$$[T_{J \cup M}][T_{I \cup M}] = \frac{\lambda^{\binom{s}{2}+\binom{u}{2}}}{\lambda^{-\ell(J|i)-su}} \frac{Q_{[s+u],[r+u]\setminus[s+u]}}{Q_{J \cup M, I \cup M}} \cdot [T_{I \cup M}][T_{J \cup M}] \tag{5.19}$$

5.5 Two-Parameter Determinant

Suppose the constants λ, q_{ij} in the field F of the previous section are transcendental over a subfield, say $F = F'(\lambda, q_{ij})$. Suppose moreover that we let $q_{ij} \to \alpha \, (i < j)$ and $\lambda \to \beta\alpha$. Denote this new field extension of F' again by F. Again let $V = F^n$.

5.5.1 Definitions & background

Under the transformation indicated above, the A-S-T algebra $\mathcal{M}_{\vec{q}}(n)$ becomes a two-parameter deformation of the commutative setting. This special deformation was independently introduced by Takeuchi in [47], and is a more transparent generalization of the famous quantization of Section 5.3: one parameter for rows, one for columns.

Definition 23. A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called (α, β)-*generic* (over F) if

$$ba = \alpha ab \tag{5.20}$$

$$dc = \alpha cd \tag{5.21}$$

$$ca = \beta ac \tag{5.22}$$

$$db = \beta bd \tag{5.23}$$

$$cb = \beta\alpha^{-1} bc \tag{5.24}$$

$$da = ad + (\beta - \alpha^{-1})bc \tag{5.25}$$

An $n \times m$ matrix X is said to be (α, β)-*generic* if every 2×2 sub-matrix $X_{\{i,j\},\{k,l\}}$ is (α, β)-generic.

Definition 24. Let $\mathcal{M}_{\alpha,\beta}(n)$ denote the F-algebra with n^2 generators t_{ij} and relations given by demanding $T = (t_{ij})$ be an (α, β)-generic matrix over F. Let $\det_{\alpha\beta}$ by defined by

$$\det_{\alpha\beta} T_{I,J} = [T_{I,J}] := \sum_{\pi \in \mathfrak{S}_C} (-\alpha)^{-\ell(\pi)} t_{i_1,\pi j_1} t_{i_2,\pi j_2} \dots t_{i_m,\pi j_m}$$

for all $1 \leq m \leq n$ and all $I, J \in \binom{[n]}{m}$.

This algebra and the determinant are again given by an R-matrix:

$$\begin{aligned} R \quad = \quad & (\alpha\beta)^{-1} \sum_{i=1}^{n} \mathbb{E}_{ii} \otimes \mathbb{E}_{ii} + \sum_{i<j} \alpha^{-1} \mathbb{E}_{jj} \otimes \mathbb{E}_{ii} + \\ & \sum_{i<j} \left(\beta^{-1} \mathbb{E}_{ii} \otimes \mathbb{E}_{jj} + ((\alpha\beta)^{-1} - 1) \mathbb{E}_{ij} \otimes \mathbb{E}_{ji} \right). \end{aligned} \tag{5.26}$$

Not surprisingly, this R-matrix is the result of applying the transformation on constants indicated above to the R-matrix of A-S-T. Similarly, the determinant of A-S-T becomes the present two-parameter determinant. Recall the generalized sign of Section 5.4: given $\pi \in \mathfrak{S}_J$, $\mathrm{sgn}_{\vec{q}}(\pi) := \prod_{j<j' \in J,\, \pi j > \pi j'} q_{\pi j, \pi j'}$. Under the transformation $q_{ij} \to \alpha\,(i<j)$, all of the terms become α^{-1} and the product has total length equal to $\ell(\pi)$.

One might expect all properties of the pair A-S-T $(\mathcal{M}_{\vec{q}}(n), \det_{\vec{q}})$ to pass through the limit and hold in the Takeuchi pair $(\mathcal{M}_{\alpha,\beta}(n), \det_{\alpha\beta})$. While this is generally true, for instance there exists an Ore field of fractions $\mathcal{T}(n)$ and $\det_{\alpha\beta}$ is again amenable, it is not universally so. Namely, the Takeuchi determinant is only row alternating, while the A-S-T determinant is both row and column alternating. The existence of a row alternating property for each is proven as in the classical case. Below, we show what happens when we try to take determinants of matrices with repeated columns.[2]

$$\begin{aligned} \begin{vmatrix} t_{11} & t_{11} \\ t_{21} & t_{21} \end{vmatrix}_{\alpha,\beta} &= \sum_{\pi \in \mathfrak{S}_2} (-\alpha)^{-\ell(\pi)} t_{i_1, j_{\pi 1}} t_{i_2, j_{\pi 2}} \\ &= t_{11}t_{21} - \alpha^{-1} t_{11} t_{21} \end{aligned}$$

while

$$\begin{aligned} \begin{vmatrix} t_{11} & t_{11} \\ t_{21} & t_{21} \end{vmatrix}_{\vec{q}} &= \sum_{\pi \in \mathfrak{S}_2} (\mathrm{sgn}_{\vec{q}}\pi) t_{i_1, j_{\pi 1}} t_{i_2, j_{\pi 2}} \\ &= t_{11}t_{21} - q_{11} t_{11} t_{21} \end{aligned}$$

Here we have extended the definition of $\det_{\vec{q}}$ to allow J to be a tuple, not a subset. The definition agrees with the old one when $J = \mathsf{rect}(J)$. These calculations indicate that the left and right pre–flag algebras are both nice objects for the A-S-T setting, while the Takeuchi (and one-parameter) deformation of $\mathcal{M}(n)$ favors the right pre–flag algebra.

5.5.2 Main result

One may readily verify the results of this section by appealing to the results of the previous section or by consulting the survey article [48].

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[2] It should be noted that the Takeuchi determinant (and the usual quantum determinant) are *almost* column alternating. If the columns are merely out of order, then one recovers the usual determinant up to a power of $(-\alpha)^{-1}$ (respectively, $(-q)^{-1}$); the alternating property fails only when there are repeated columns.

Proposition 35. *The pair* $(\mathcal{M}_{\alpha,\beta}(n), \det_{\alpha\beta})$ *is an amenable pair* $(\mathcal{A}(n), \mathrm{Det})$ *with associated measuring functions given by*

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{I}_{\mathfrak{r}}(a, A) = \mathfrak{I}_{\mathfrak{x}}(a, A) = (-\alpha)^{-\ell(a|A)}$
- $(\forall a \in A \subseteq [n]) \quad \mathfrak{K}_{\mathfrak{r}}(a, A) = \mathfrak{K}_{\mathfrak{x}}(a, A) = \beta^{\ell(a|A)}\alpha^{\ell(A|a)}$

In $\mathcal{M}_{\alpha,\beta}(n)$*, the row-minors* $\left\{[T_A] = \det_{\alpha\beta} T_{A,[d]} \mid A \in \binom{[n]}{d}\right\}$ *satisfy*

- $(\forall 1 \le s \le r < n)$ *If* $K, M \subseteq [n]$ *are subsets satisfying* $|K| = r+1, |M| = s-1$*, then*
$$0 = \sum_{k \in K \setminus M} \frac{(-\alpha)^{-\ell(k|M) - \ell(K^k|k)}}{(\beta\alpha^{-1})^{\ell(K^k|k)}} \cdot [T_{K\setminus k}][T_{kM}] \tag{5.27}$$
- *If* $I, J, M \subseteq [n]$ $(|J| = s \le r = |I|, |M| = u)$ *are pairwise disjoint, and if* J *can't distinguish* I *as rows, then for any* $i \in I$
$$[T_{J\cup M}][T_{I\cup M}] = \frac{(\beta\alpha)^{-\ell(i|J)-su}}{(\beta\alpha)^{\binom{s}{2}+\binom{u}{2}}} \frac{\mathfrak{K}_{\mathfrak{r}}(JM, IM)}{\mathfrak{K}_{\mathfrak{r}}([s+u], [r+u]\setminus[s+u])} \cdot [T_{I\cup M}][T_{J\cup M}] \tag{5.28}$$

5.6 Another Quantum Determinant

Another specialization of the A-S-T algebra $\mathcal{A}(n)$ will be useful later. We pass from $F = F'(\lambda, q_{ij})$ to $F = F'(q)$.

Definition 25. Define $\mathcal{A}^{\mathrm{I}}(n)$ to be the F-algebra with generators $\{t_{ij} \mid 1 \le i, j \le n\}$ and relations as follows:

$$t_{jb}t_{ia} = \begin{cases} t_{ia}t_{jb} + (q^2-1)t_{ib}t_{ja} & \text{if } j > i \wedge b > a \\ q^2 t_{ia}t_{jb} & \text{if } j > i \wedge b \le a \\ t_{ia}t_{jb} & \text{if } j = i \wedge b > a\,. \end{cases}$$

Definition 26. For $0 < m \le n$ and row-indices $I = \{i_1 < \ldots < i_m\}$ and column-indices $J = \{j_1 < \ldots < j_m\}$, let $\det_{\mathrm{I}}$ be the determinant for $\mathcal{A}^{\mathrm{I}}(n)$ defined by

$$\det_{\mathrm{I}} T = [T_{I,J}] := \sum_{\pi \in \mathfrak{S}_m} (-1)^{\ell(\pi)} t_{i_1, j_{\pi 1}} \cdots t_{i_m, j_{\pi m}}\,.$$

This algebra and its determinant are again given by an R-matrix:

$$R = q^{-2}\sum_{i=1}^{n} \mathbb{E}_{ii} \otimes \mathbb{E}_{ii} + \sum_{i<j} \left(\mathbb{E}_{jj} \otimes \mathbb{E}_{ii} + q^{-2}\mathbb{E}_{ii} \otimes \mathbb{E}_{jj} + (q^{-2}-1)\mathbb{E}_{ij} \otimes \mathbb{E}_{ji}\right). \tag{5.29}$$

All of the comments in the previous section hold, except that this determinant actually retains the column-alternating property that the A-S-T determinant possesses. We summarize the important identities below. We choose to display the column-minor identities because they will be useful in Section 7.2.

Proposition 36. *The pair* $(\mathcal{A}^{\mathrm{I}}(n), \det_{\mathrm{I}})$ *is an amenable pair* $(\mathcal{A}(n), \mathrm{Det})$ *with associated measuring functions given by*

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{I}_{\mathfrak{r}}(a, A) = \mathfrak{I}_{\mathfrak{x}}(a, A) = (-1)^{\ell(a|A)}$
- $(\forall a \in A \subseteq [n]) \quad \mathfrak{K}_{\mathfrak{r}}(a, A) = \mathfrak{K}_{\mathfrak{x}}(a, A) = (q^2)^{\ell(a|A)}$

In $\mathcal{A}^{\mathrm{I}}(n)$, *the column-minors* $\left\{[T_A] = \det_{\mathrm{I}} T_{[d],A} \mid A \in \binom{[n]}{d}\right\}$ *satisfy*

- $(\forall 1 \leq s \leq r < n)$ *If* $K, M \subseteq [n]$ *are subsets satisfying* $|K| = r+1, |M| = s-1$, *then*
$$0 = \sum_{k \in K \setminus M} (-1)^{\ell(M|k)+\ell(k|K^k)} (q^2)^{-\ell(M|k)} \cdot [T_{M \cup k}][T_{K \setminus k}] \tag{5.30}$$
- *If* $I, J, M \subseteq [n]$ $(|J| = s \leq r = |I|, |M| = u)$ *are pairwise disjoint, and if* J *can't distinguish* I *as columns, then for any* $i \in I$
$$[T_{J \cup M}][T_{I \cup M}] = \frac{(q^2)^{-\ell(J|i)}}{(q^2)^{-\ell(J|I)}} \frac{(q^2)^{\binom{s}{2}+\ell([s+u]|[r+u] \setminus [s+u])}}{(q^2)^{\ell(M|I)-\ell(M|J)}} \cdot [T_{I \cup M}][T_{J \cup M}] \tag{5.31}$$

5.7 Yangians

Here we summarize a *spectral parameter* notion of the R-matrix formalism. Things work essentially the same way. If the reader is already familiar with the Yangians, he may skip to subsection 5.7.3.

5.7.1 Spectral parameter R-matrices and determinants

Write $V = \mathbb{C}^n$, and let $\mathcal{U} = \{u_1, u_2, \ldots\}$ be formal parameters. Write $\mathbb{C}(\mathcal{U})$ for the field extension $\mathbb{C}(u_1, u_2, \ldots)$. For $u \in \mathcal{U}$, we consider endomorphisms $C(u) \in \mathrm{End}_{\mathbb{C}} V^{\otimes m} \otimes \mathbb{C}(\mathcal{U})$.

Write the map $C(u) \in \operatorname{End} V \otimes V \otimes \mathbb{C}(u)$ as $\sum_{i,j,k,l} \mathbb{E}_{ik} \otimes \mathbb{E}_{jl} \otimes c_{ij}^{kl}(u)$ (again, acting on the right). As in Section 5.1, we may extend $C(u)$ to a map of $V^{\otimes m} \otimes C(u)$ by indicating on which factors $C(u)$ should act:

$$C(u)_{ab} := \mathbb{I}^{\otimes(a-1)} \otimes \mathbb{E}_{ik} \otimes \mathbb{I}^{\otimes(b-a-1)} \mathbb{E}_{jl} \otimes \mathbb{I}^{\otimes(m-b-1)} \otimes c_{ij}^{kl}(u).$$

Definition 27. An endomorphism $C(u) \in \operatorname{End}_{\mathbb{C}} V \otimes V \otimes \mathbb{C}(u,v)^{\otimes m}$ is called a *spectral parameter R-matrix* if it satisfies the *quantum Yang Baxter equation with spectral parameter*,

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{5.32}$$

as an element of $\operatorname{End} V \otimes V \otimes V \otimes \mathbb{C}(u,v)$. We drop the modifier "spectral parameter" when it is clear from context.

Example (Yangian R-matrix). Let τ be the twist map $e_i \otimes e_j \mapsto e_j \otimes e_i$. Then the map $R(u) \in \operatorname{End} V \otimes V \otimes \mathbb{C}(u)$ given by

$$R = (\mathbb{I} \otimes 1 - \tau \otimes u^{-1}) \tag{5.33}$$

is an R-matrix with spectral parameter.

Write $\mathcal{A}_0(\mathcal{U})$ for the free noncommutative $\mathbb{C}$-algebra built on the symbols $\{t_i^j(u) \mid 1 \leq i,j \leq n,\, u \in \mathcal{U}\}$. Let $T(u) = (t_i^j(u))$ be an $n \times n$ matrix of formal noncommuting variables (with parameter). We may let $T(u)$ act on $V \otimes \mathcal{A}_0(\mathcal{U})$ by writing

$$(\forall i\, \forall a) \quad T(u)(e_i \otimes a) := \sum_{1 \leq k \leq n} e_k \otimes t_i^k(u)a.$$

As in Section 5.1, we define a quotient of $\mathcal{A}_0$ by demanding equality of two maps on $V' = V \otimes V \otimes \mathcal{A}_0$.

Definition 28. An algebra $\mathcal{A}(R,n)$ with spectral parameters is an RTT-algebra if, together with R, its n^2 generators $t_j^i(u)$ satisfy the relation

$$R_{12}(u-v)T_{13}(u)T_{23}(v) = T_{23}(v)T_{13}(u)R_{12}(u-v), \tag{5.34}$$

viewed as an equation in $\operatorname{End} V \otimes V \otimes \mathcal{A}_0$.

The construction of determinants for spectral parameter RTT-algebras follows roughly the steps outlined above, e.g. looking for a one-dimensional object on which to (co-) act. One difference is worth a few words. In the construction, we again extend the action of R to $V^{\otimes m}$. However, we let R_{ij} act with spectral parameter $u_i - u_j$—more natural in light of (5.32) & (5.34). Only in the last step do we apply a reduction to one parameter, arguing that we may take $u_i - u_{i+1} = 1$. For the *Yangian R-matrix* in the example above, the details are available in Molev's survey article [36].

Definition 29. Say a spectral parameter determinant Det is *amenable* if there are measuring functions $\mathfrak{I}_{\mathfrak{r}}$, $\mathfrak{I}_{\mathfrak{x}}$, $\mathfrak{K}_{\mathfrak{r}}$, $\mathfrak{K}_{\mathfrak{x}} : \mathcal{P}[n] \times \mathcal{P}[n] \to F$ associated to Det satisfying:

- $(\forall i \in I)(\forall j \in J) \quad [T_{i,j}(u)] = t_{ij}(u)$.
- $(\forall i, i' \in R) \quad \sum_{j \in J} t_{ij}(u') \left\{ \frac{\mathfrak{I}_{\mathfrak{x}}(j,J)}{\mathfrak{I}_{\mathfrak{r}}(i',I)} [T_{I^{i'},J^j}(u'')] \right\} = [T_{I,J}(u)]\delta_{ii'}$.
- $(\forall I' \subseteq I)(\forall J' \subseteq J) \quad [T_{I,J}(u)][T_{I',J'}(v)] = \frac{\mathfrak{K}_{\mathfrak{x}}(J',J)}{\mathfrak{K}_{\mathfrak{r}}(I',I)} [T_{I',J'}(v')][T_{I,J}(u')]$.

5.7.2 Review of Yangian for $\mathfrak{gl}_n$

For a Lie group/algebra pair $(G, \mathfrak{g})$, we have outlined above several ways to deform the ring of global regular functions $K[G]$.[3] The algebra we define below is another example of a quantum group. It is different from those preceding it, namely it is a deformation of the universal enveloping algebra $U(\mathfrak{g})$. The Yangian $Y(\mathfrak{gl}_n)$ was introduced by Drinfeld [10] at roughly the same time as the quantum group $\mathrm{GL}_q(n)$ of Section 5.3. For more information on its Lie and representation theoretic background, and for current trends in the study of Yangians, the reader is urged to consult [36], [27], and [5]. The latter uses the quasideterminant and noncommutative Gaussian elimination to give several presentations of $Y(\mathfrak{gl}_n)$.

Definition 30. The *Yangian* for $\mathfrak{gl}_n$ is the $\mathbb{C}$-algebra $Y(n)$ with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots$ where $1 \le i, j \le n$, and defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \tag{5.35}$$

[3] This is not entirely accurate. We have given an overview of several deformations of $\mathcal{M}(n)$. It is traditional to reserve the term "quantum group" for a slightly different object, a certain extension+quotient of the deformed $\mathcal{M}(n)$, outlined in the discussion following Definition 20

where $r, s = 0, 1, 2, \ldots$ and $t_{ij}^{(0)} := \delta_{ij} \cdot 1$.

If we collect the generators $t_{ij}^{(r)}$ $(r = 0, 1, \ldots)$ together in the generating series,

$$t_i^j(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in Y(n)[[u^{-1}]], \tag{5.36}$$

and then collect these generating series together in a matrix $T(u) = (t_i^j(u))$, we may express the relations more compactly.

Theorem 37. *The algebra $Y(n)$ is a spectral parameter RTT-algebra with R matrix given by the Yangian R-matrix in (5.33).*

We conclude this section with a few more important results the reader may find in [36].

Definition 31. Fix $I, J \in \binom{[n]}{m}$. In $Y(n)[[u^{-1}]]$, the quantum determinant $\mathrm{qdet} T_{I,J}(u)$ is defined by

$$\mathrm{qdet} T_{I,J}(u) = [T_{I,J}(u)] := \sum_{\pi \in \mathfrak{S}_J} t_{i_1}^{\pi j_1}(u - m + 1) \cdots t_{i_{m-1}}^{\pi j_{m-1}}(u - 1) t_{i_m}^{\pi j_m}(u)\,.$$

Theorem 38. *Fix $I, J \in \binom{[n]}{m}$. The Yangian determinant* qdet *has a cofactor matrix $S(T_{I,J}(u))$ and is central:*

$$\sum_{j \in J} t_i^j(u - m + 1) \left\{ (-1)^{j - i'} [T_{I \setminus i', J \setminus j}(u)] \right\} = \delta_{ii'} [T_{I,J}(u)] \mathbb{I}, \tag{5.37}$$

$$(\forall I' \subseteq I)(\forall J' \subseteq J) \quad [T_{I,J}(u)][T_{I',J'}(v)] = [T_{I',J'}(v)][T_{I,J}(u)]. \tag{5.38}$$

5.7.3 Main result

There is one important thing to notice about $T(u)$ as defined above. It is invertible in $Y(n)[[u^{-1}]]$ because it may be viewed as a formal power series $T(u) = \sum_{i \geq 0} a_i u^{-i}$ with each a_i a matrix over $Y(n)$ and a_0 a unit (indeed equal to 1, or $\mathbb{I}$). The same may be said for all sub-matrices $T_{I,J}(u)$. In short, $\mathcal{T}(n) := Y(n)[[u^{-1}]]$ is a suitable setting to carry out the calculations in Chapter 4. In doing so, one finds a spectral parameter version of weak q-commuting, Young symmetry, and Muir's Law identities. However, the proof of the key Proposition 23 fails to carry over to this setting. There is likely a patch to the proof—which would give us a strong q-commuting identity—however, I have not found one yet.

Proposition 39. *The pair* $(Y(n)[[u^{-1}]], \mathrm{qdet})$ *is a spectral parameter amenable pair* $(\mathcal{A}(n), \mathrm{Det})$ *with associated measuring functions given by*

- $(\forall a \in A \subseteq [n]) \quad \mathfrak{I}_{\mathfrak{r}}(a, A) = \mathfrak{I}_{\mathfrak{r}}(a, A) = (-1)^{\ell(a|A^a)}$
- $(\forall a \in A \subseteq [n]) \quad \mathfrak{K}_{\mathfrak{r}}(a, A) = \mathfrak{K}_{\mathfrak{r}}(a, A) = 1$
- *In the adjoint property for* $[T_{I,J}(u)]$, $u' = u - |I| + 1$, $u'' = u$.
- *In the commuting property for* $[T_{I,J}(u)]$, $u' = u$ *and* $v' = v$.

In $Y(n)[[u^{-1}]]$, *the row-minors* $\left\{[T_A(u)] = \mathrm{qdet}\, T_{A,[d]}(u) \mid A \in \binom{[n]}{d}\right\}$ *satisfy*

- $(\forall 1 \leq s \leq r < n)$ *If* $K, M \subseteq [n]$ *are subsets satisfying* $|K| = r+1, |M| = s-1$, *then*

$$0 = \sum_{k \in K \setminus M} (-1)^{\ell(K^k|k)+\ell(k|M)} [T_{K \setminus k}(u + r - s + 1)][T_{k \cup M}(u)]. \tag{5.39}$$

- *If* $M \in \binom{[n]}{m}$ *and* $i, j \in [n] \setminus M\ )i \neq j)$ *then*

$$[T_{M \cup i}(u)][T_{M \cup j}(u-1)] = [T_{M \cup j}(u)][T_{M \cup i}(u1)]. \tag{5.40}$$

- *If* $I, J, M \in \mathcal{P}[n]$ *are pairwise disjoint* $(|M| = m)$ *and*

$$[T_J(u)][T_I(u-p)] = X \cdot [T_I(u)][T_J(u-p)]$$

  for some $X \in F$ *and some* $p \in \mathbb{Z}$, *then*

$$[T_{J \cup M}(u+m)][T_{I \cup M}(u+m-p)] = [T_{I \cup M}(u+m)][T_{J \cup M}(u+m-p)]. \tag{5.41}$$

5.8 A New Example

Clearly, the R-matrix formalism is a rich source of amenable determinants. However, there are examples of amenable determinants NOT coming from R-matrix constructions. Here is one that we will see again later. As usual, fix a field F containing a distinguished element $q \notin \{-1, 0, 1\}$, and a vector space $V = F^n$.

Definition 32. Let $\mathcal{A}^{\mathrm{II}}(n)$ be the F-algebra with n^2 generators t_{ij} and four classes of relations given by

$$\begin{aligned} t_{jk}t_{ik} &= qt_{ik}t_{jk} && (i<j) \\ t_{kj}t_{ki} &= t_{ki}t_{kj} && (i<j) \\ t_{jk}t_{il} &= qt_{il}t_{jk} && (i<j;k<l) \\ t_{jl}t_{ik} &= q^{-1}t_{ik}t_{jl} + \left(q-q^{-1}\right)t_{il}t_{jk} && (i<j;k<l). \end{aligned} \tag{5.42}$$

Remark. These relations only involve 2×2 square sub-matrices of $T=(t_{ij})$, so we may conclude that the subalgebras of $\mathcal{A}^{\mathrm{II}}(n)$ generated by $(T_{I,J})$ will all be isomorphic—and isomorphic to $\mathcal{A}^{\mathrm{II}}(m)$, for $|I|=|J|=m$.

Definition 33. Given a square $m\times m$ sub-matrix $T_{I,J}$ of T (including $I=J=[n]$), we define a determinant $\det_{\mathrm{II}}T_{I,J}$ by

$$\det\nolimits_{\mathrm{II}}T_{I,J} = [T_{I,J}] := \sum_{\pi\in\mathfrak{S}_J}(-1)^{\ell(\pi)}t_{i_1,\pi j_1}t_{i_2,\pi j_2}\cdots t_{i_m,\pi j_m}. \tag{5.43}$$

This function will prove to be an amenable determinant for $\mathcal{A}^{\mathrm{II}}(n)$. Before we set about demonstrating this, we should settle an outstanding claim from the introductory remarks.

Proposition 40. *There exists no endomorphism $C\in End\, V\otimes V$ which, under the FRT construction, produces the relations for $\mathcal{A}^{\mathrm{II}}(n)$ as presented above.*

Remark. In particular, $\mathcal{A}^{\mathrm{II}}(n)$ is not an RTT-algebra—again, with the presentation given above.

Proof. Focusing on the case $n=2$, we may pose the question as a linear algebra problem:

- Work in the vector space $W=\mathrm{span}\{w_{ij,kl}\mid 1\le i,j,k,l\le 2\}$.
- Let c_{ij}^{mn} be 16 unknown variables over F.
- Consider the vectors $\mathfrak{w}_{ij}^{mn}=\sum_{k_1,k_2}c_{k_1k_2}^{mn}w_{ik_1,jk_2}-\sum_{l_1,l_2}c_{ij}^{l_1l_2}w_{l_2n,l_1m}$ in W, and the subspace $\mathfrak{W}$ which they span, cf. (5.1)

- If, e.g., the vector $w_{22,11} - q^{-1}w_{11,22} - (q - q^{-1})w_{12,21}$ belongs to $\mathfrak{W}$, then there is hope that we can find an endomorphism C.

Alas, this vector, is not in $\mathfrak{W}$. □

5.8.1 First properties

Proposition 41 (q-Alternating in Rows). *Let $A, B \in [n]^m$ be tuples of row and column indices respectively. Suppose moreover that B is "straightened", with distinct entries—that is, $B = (b_1 < b_2 < \ldots < b_m)$. Finally, write $A' = (a'_1 \leq \ldots \leq a'_m)$ for the straightened form of A (fixing a $\sigma \in \mathfrak{S}_m$ of minimal length so that $A' = (a_{\sigma 1}, \ldots, a_{\sigma m})$). Then*

$$[T_{A,B}] = \begin{cases} 0 & \textit{if } A \textit{ contains repeated indices.} \\ (-q)^{-\ell(\sigma)}[T_{A',B}] & \textit{otherwise.} \end{cases}$$

Proof. We first consider the effect of the simple transposition $s_i = (i, i+1)$ on A (and on $[T_{A,B}]$) when $a_{i+1} < a_i$. We begin by breaking the elements of $\mathfrak{S}_m$ into two disjoint, equinumerous sets: $S' = \{\pi \in \mathfrak{S}_m \mid \pi(i) < \pi(i+1)\}$ and $S'' = \mathfrak{S}_m \setminus S'$. Without loss of generality, we may assume $B = (1, 2, \ldots, m)$. Also, let us suppress the B in the notation, writing $[T_{A,B}]$ as $\big[A\big]$. Now we may write

$$\begin{aligned}
\big[A\big] &= \big[a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_m\big] \\
&= \sum_{\pi \in \mathfrak{S}_m} (\operatorname{sgn} \pi) t_{a_1 \pi(1)} \cdots t_{a_m \pi(m)} \\
&= \sum_{\pi \in S'} (\operatorname{sgn} \pi) t_{a_1 \pi(1)} \cdots (t_{a_i \pi(i)} t_{a_{i+1} \pi(i+1)} - t_{a_i \pi(i+1)} t_{a_{i+1} \pi(i)}) \cdots t_{a_m \pi(m)} \\
&= \sum_{\pi \in S'} \cdots \Big(\{q t_{a_{i+1} \pi(i+1)} t_{a_i \pi(i)}\} - \\
&\qquad \{q^{-1} t_{a_{i+1} \pi(i)} t_{a_i \pi(i+1)} + (q - q^{-1}) t_{a_{i+1} \pi(i+1)} t_{a_i \pi(i)}\} \Big) \cdots \\
&= \sum_{\pi \in S'} (-q)^{-1} \cdots (t_{a_{i+1} \pi(i)} t_{a_i \pi(i+1)} - t_{a_{i+1} \pi(i+1)} t_{a_i \pi(i)}) \cdots \\
&= (-q)^{-1} \big[a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_m\big] .
\end{aligned}$$

Induction on the length of the permutation σ straightening A completes the proof when no two indices of A are alike. When there are two like indices, we may use the above procedure to successively straighten A until we come upon a new A and a new i with $a_i = a_{i+1}$. Consider the definition of $\big[A\big]$ in this final case:

$$
\begin{aligned}
\big[A\big] &= \big[a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_m\big] \\
&= \sum_{\pi \in \mathfrak{S}_m} (\operatorname{sgn} \pi) t_{a_1\pi(1)} \cdots t_{a_m\pi(m)} \\
&= \sum_{\pi \in S'} (\operatorname{sgn} \pi) t_{a_1\pi(1)} \cdots (t_{a_i\pi(i)} t_{a_i\pi(i+1)} - t_{a_i\pi(i+1)} t_{a_i\pi(i)}) \cdots t_{a_m\pi(m)} \\
&= \sum_{\pi \in S'} (\operatorname{sgn} \pi) t_{a_1\pi(1)} \cdots (t_{a_i\pi(i)} t_{a_i\pi(i+1)} - [t_{a_i\pi(i)} t_{a_i\pi(i+1)}]) \cdots t_{a_m\pi(m)} \\
&= 0\,.
\end{aligned}
$$

□

Proposition 42 (Alternating in Columns). *Let $A, B \in [n]^m$ be tuples of row and column indices respectively. Suppose moreover that A is "straightened", with distinct entries—that is, $A = (a_1 < a_2 < \ldots < a_m)$. Finally, write $B' = (b'_1 \leq \ldots \leq b'_m)$ for the straightened form of B (fixing a $\sigma \in \mathfrak{S}_m$ of minimal length so that $B' = (b_{\sigma 1}, \ldots, b_{\sigma m})$). Then Then*

$$
[T_{A,B}] = \begin{cases} 0 & \text{if } B \text{ contains repeated indices.} \\ (-1)^{-\ell(\sigma)}[T_{A,B'}] & \text{otherwise.} \end{cases}
$$

Proof. We first consider the effect of the simple transposition $s = s_i = (i, i+1)$ on B (and on $[T_{A,B}]$) when $b_{i+1} < b_i$. Put $B' = s_i(B)$. Without loss of generality, we may assume $A = (1, 2, \ldots, m)$. Again, we simplify notation by putting $[T_{A,B}] = \big[B\big]$. Now we may write

$$
\begin{aligned}
\big[B\big] &= \sum_{\pi \in \mathfrak{S}_m} (\operatorname{sgn} \pi) t_{1,b_{\pi(1)}} \cdots t_{i,b_{\pi(i)}} t_{i+1,b_{\pi(i+1)}} \cdots t_{m,b_{\pi(m)}} \\
&= \sum_{\pi \in \mathfrak{S}_m} (\operatorname{sgn} \pi \circ s) t_{1,b_{\pi s(1)}} \cdots t_{i,b_{\pi s(i)}} t_{i+1,b_{\pi s(i+1)}} \cdots t_{m,b_{\pi s(m)}} \\
&= -\sum_{\pi \in \mathfrak{S}_m} (\operatorname{sgn} \pi) t_{1,b_{\pi(1)}} \cdots t_{i-1,b_{\pi(i-1)}} t_{i,b_{\pi(i+1)}} t_{i+1,b_{\pi(i)}} t_{i+2,b_{\pi(i+2)}} \cdots \\
&= -\big[B'\big]\,.
\end{aligned}
$$

Induction on the length of the permutation σ straightening B completes the proof when no two indices of B are alike. When there are two like indices, we may use the above procedure to successively straighten B until we come upon a new B and a new i with $b_i = b_{i+1}$. In this case, the above calculations show that for this B, $\big[B\big] = -\big[B\big]$. □

Definition 34 (Quantum Cofactor Matrix). From any square sub-matrix $T_{I,J}$ of the matrix of generators, we define a new matrix $S(T_{I,J}) = (s_{ji})$ by

$$s_{ji} = (-1)^{\ell(j|J^j j)}(-q)^{\ell(i|I^i i)}\big[T_{I^i,J^j}\big] .$$

Proposition 43. *For any $I, J \in \binom{[n]}{m}$ and $S(T_{I,J})$ as defined above, we have*

$$(T_{I,J}) \cdot S(T_{I,J}) = [T_{I,J}]\mathbb{I}_m .$$

Proof. The proof is the traditional proof in the commutative setting. Below, we show that the diagonal entries are correct, i.e. $(\forall k)\ \sum_{j\in J} t_{i_k j} s_{j i_k} = [T_{I,J}]$.

$$\begin{aligned}
[T_{I,J}] &= \sum_{\pi\in\mathfrak{S}_m} (\operatorname{sgn}\pi) t_{i_1,j_{\pi 1}} \cdots t_{i_k,j_{\pi k}} \cdots t_{i_m,j_{\pi m}} \\
&= (-q)^{k-1} \sum_{\pi\in\mathfrak{S}_m} (\operatorname{sgn}\pi) t_{i_k,j_{\pi 1}} t_{i_1,j_{\pi 2}} \cdots \widehat{t}_{i_k} \cdots t_{i_m,j_{\pi m}} ,
\end{aligned}$$

by the row q-alternating property. Now collect together those π with $\pi(1) = p$ to complete the proof:

$$\begin{aligned}
[T_{I,J}] &= (-q)^{k-1} \sum_{p=1}^{m} \sum_{\pi(1)=p} (\operatorname{sgn}\pi) t_{i_k,j_p} t_{i_1,j_{\pi 2}} \cdots \widehat{t}_{i_k} \cdots t_{i_m,j_{\pi m}} \\
&= \sum_{p=1}^{m} t_{i_k,j_p} (-q)^{k-1} (-1)^{p-1} \times \\
&\qquad \sum_{\pi'\in\mathfrak{S}_{[m]\setminus p}} (\operatorname{sgn}\pi') t_{i_1,j_{\pi' 1}} \cdots \widehat{t}_{i_k,j_{\pi' p}} \cdots t_{i_m,j_{\pi' m}} \\
&= \sum_{p=1}^{m} t_{i_k,j_p} (-q)^{k-1} (-1)^{p-1} \big[T_{I\setminus i_k, J\setminus j_p}\big] \\
&= \sum_{j\in J} t_{i_k,j} s_{j i_k} .
\end{aligned}$$

□

Before presenting our next result about $\mathcal{A}^{\mathrm{II}}(n)$, we remind the reader of an important noncommutative generalization of Hilbert's Basis Theorem.

Theorem 44 (Ore Extensions). *Fix a field F. If R is a (not necessarily commutative) noetherian F-algebra without zero divisors, and if $\alpha \in \mathrm{Aut}_F R$ and $\delta \in \mathrm{End}_F R$ satisfy $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$, then the* skew-polynomial ring $R[t; \alpha, \delta]$ *generated by R and t together with*

$$(\forall a \in R) \quad ta = \alpha(a)t + \delta(a)$$

is again a noetherian F-algebra without zero divisors.

Ore studied these extensions of R when trying to build a large class of rings which could be embedded in skew-fields. The next result shows that $\mathcal{A}^{\mathrm{II}}(n)$ may be so embedded, giving us some hope that $(\mathcal{A}^{\mathrm{II}}(n), \det_{\mathrm{II}})$ will be an amenable pair.

Proposition 45. *The algebra $\mathcal{A}^{\mathrm{II}}(n)$ is a noetherian domain, and as such has a well-defined Ore field of fractions $\mathcal{T}(n)$ in which every $[T_{I,J}]$ is invertible.*

Sketch of Proof. Let $F\langle X\rangle$ denote the subalgebra of $\mathcal{A}^{\mathrm{II}}(n)$ generated by the set X. Consider the following tower of subalgebras (adding one generator at a time, in lexicographic order):

$$\begin{aligned} F \quad \subsetneq \quad & F\langle x_{11}\rangle \subsetneq F\langle x_{11}, x_{12}\rangle \subsetneq \cdots \subsetneq F\langle x_{11}, \ldots, x_{1n}\rangle \subsetneq \\ & F\langle x_{11}, \ldots, x_{1n}, x_{21}\rangle \subsetneq \cdots \subsetneq F\langle x_{11}, \ldots, x_{nn}\rangle = \mathcal{A}^{\mathrm{II}}(n) \end{aligned}$$

Compare this to the chain of Ore extensions $(R_{ij} \mid 1 \le i, j \le n)$ defined as follows (putting $R_{0n} = F$):

$$R_{i,j} = \begin{cases} R_{i,j-1}[x_{ij}; \alpha_{ij-1}, \delta_{ij-1}] & \text{if } 1 \le i \le n \text{ and } 1 < j \le n \\ R_{i-1,n}[x_{i1}; \alpha_{i-1n}, \delta_{i-1n}] & \text{if } 1 \le i \le n \text{ and } j = 1 \end{cases}$$

Here the α_{ij} (respectively, δ_{ij}) are arbitrary automorphisms (endomorphisms) of R_{ij}. If we can find α_{ij} and δ_{ij} so that $F\langle x_{11}, \ldots, x_{ij}\rangle \simeq R_{ij}$, we will be done.

This is fairly straightforward, and works just as in the standard quantum case $(\mathcal{M}_q(n))$, cf. [28] for details on that argument. Below, we show how to add x_{22} to the chain.

Observation: If an algebra A si given by generators and relations, one may define a derivation δ by defining its action on generators, demanding it have the appropriate linear and multiplicative properties, and checking it respects the relations.

That said, we need only look at how x_{22} moves past linear terms in X:

$$x_{22}\Big(a_0 + a_1x_{11} + \sum_{1<i\leq n} a_ix_{1i} + b_1x_{21}\Big) = \\ \Big(a_0 + q^{-1}a_1x_{11} + \sum_{1<i\leq n} qa_ix_{1i} + b_1x_{21}\Big)x_{22} + a_1(q-q^{-1})x_{12}x_{21}$$

Evidently we should define $\alpha = \alpha_{21}$ by $\alpha|_F = \mathbb{I}$, $\alpha(x_{11}) = q^{-1}x_{11}$, $\alpha(x_{1j}) = qx_{1j}(j > 1)$, and $\alpha(x_{21}) = x_{21}$. This is clearly an automorphism of R_{21}. Also, we should define $\delta = \delta_{21}$ by $\delta(F) = 0$, $\delta(x_{21}) = \delta(x_{1j}) = 0(j > 1)$, and $\delta(x_{11}) = (q-q^{-1})x_{12}x_{21}$. Appealing to the observation above, we need only check that

$$\begin{aligned} \delta(x_{1j}x_{1i} - x_{1i}x_{1j}) &= 0 \ (1 < i < j), \\ \delta(x_{1j}x_{11} - x_{11}x_{1j}) &= 0 \ (1 < j), \\ \delta(x_{21}x_{1j} - qx_{1j}x_{21}) &= 0 \ (1 < j), \\ \delta(x_{21}x_{11} - qx_{11}x_{21}) &= 0. \end{aligned}$$

All are routine, we check the last one.

$$\begin{aligned} \delta(x_{21}x_{11}) &= \alpha(x_{21})\delta(x_{11}) + 0 \cdot x_{11} \\ &= x_{21}(q-q^{-1})x_{12}x_{21} \\ &= q(q-q^{-1})x_{12}x_{21}x_{21}, \end{aligned}$$

while

$$\begin{aligned} q\delta(x_{11}x_{21}) &= q\alpha(x_{11}) \cdot 0 + q\delta(x_{11})x_{21} \\ &= q(q-q^{-1})x_{12}x_{21}x_{21}. \end{aligned}$$

□

The results we have shown thus far are already enough to enable us to write $\det_{\mathrm{II}}$ in terms of quasideterminants.

Proposition 46. *Given subsets $R, C \in \binom{[n]}{m}$ (row and column indices),*

$$|T_{I,J}|_{ij} = (-1)^{-\ell(j|J^j)}(-q)^{-\ell(i|I^i)} \cdot \left[T_{I,J}\right] \cdot \left[T_{I\backslash i,J\backslash j}\right]^{-1} \tag{5.44}$$

With this result, we may deduce some commuting relations for matrix minors.

Theorem 47. *For all $I, J \in \binom{[n]}{m}$, and all $i \in I$, $j \in J$,*

$$\left[T_{I,J}\right]t_{ij} = q^{\ell(j|J^j)-\ell(J^j|j)} \cdot t_{ij}\left[T_{I,J}\right]. \tag{5.45}$$

Proof. By the remark preceding Definition 33, we need only consider the case $m = n$, i.e. all possibilities $I, J \in \binom{[n]}{m}$ and $i \in I, j \in J$, are equivalent to the case $I = J = [m]$, $i, j \in [m]$. We begin in the case $n = 2$ and proceed by induction. The base case is not $n = 1$ because, as we will soon see, we need two rows which are distinct from i, and two columns which are distinct from j. Putting $I = J = \{1, 2\}$, we check that (5.45) is valid in all possible instances (putting $p' = \ell(j|J^j)$ and $p'' = \ell(J^j|j)$):

$$\begin{aligned}
\left[T_{\{12\},\{12\}}\right] \cdot t_{11} &= \left[12;12\right] \cdot t_{11} \\
&= (t_{11}t_{22} - t_{12}t_{21})t_{11} \\
&= t_{11}(q^{-1}t_{11}t_{22} + (q - q^{-1})t_{12}t_{21}) - qt_{12}t_{11}t_{21} \\
&= q^{-1}t_{11}(t_{11}t_{22} - t_{12}t_{21}) \\
&= q^{0-1}t_{11}[12;12] = q^{p'-p''}t_{11}[12;12].
\end{aligned}$$

$$\begin{aligned}
\left[12;12\right] \cdot t_{22} &= (t_{11}t_{22} - t_{12}t_{21})t_{22} \\
&= (qt_{22}t_{11} - q(q - q^{-1})t_{12}t_{21})t_{22} - q^{-1}t_{22}t_{12}t_{21} \\
&= t_{22}(qt_{11}t_{22} - (q - q^{-1})t_{12}t_{21} - q^{-1}t_{12}t_{21}) \\
&= q^{1-0}t_{22}[12;12] = q^{p'-p''}t_{22}[12;12].
\end{aligned}$$

$$\begin{aligned}
\left[12;12\right] \cdot t_{12} &= (t_{11}t_{22} - t_{12}t_{21})t_{12} \\
&= t_{12}(qt_{11}t_{22} - qt_{12}t_{21}) \\
&= q^{p'-p''}t_{12}[12;12].
\end{aligned}$$

$$\begin{aligned}
\big[12;12\big]\cdot t_{21} &= (t_{11}t_{22}-t_{12}t_{21})t_{21}\\
&= t_{21}(q^{-1}t_{11}t_{22}-q^{-1}t_{12}t_{21})\\
&= q^{p'-p''}t_{21}\big[12;12\big]\,.
\end{aligned}$$

Now suppose $(T_{I',J'},t_{ij})$ satisfies the theorem $\forall I' \subsetneq I=[n]$, $\forall J' \subsetneq J=[n]$, and $\forall i\in I'$, $j\in J'$. Fix two rows $r_0,r_1\in I\setminus i$ and two columns $c_0,c_1\in J\setminus j$. Using Sylvester's identity (Theorem 8) and (5.44),

$$\begin{aligned}
\big[T\big]\cdot t_{ij} &= \Big(|T|_{r_0,c_0}\cdot(-1)^{\ell(c_0|J\setminus c_0)}(-q)^{\ell(r_0|I\setminus r_0)}\cdot\big[T^{r_0,c_0}\big]\Big)\cdot t_{ij}\\
&= \big(|T^{r_1,c_1}|_{r_0,c_0}-|T^{r_1,c_0}|_{r_0,c_1}\cdot|T^{r_0,c_0}|^{-1}_{r_1,c_1}\cdot|T^{r_0,c_1}|_{r_1,c_0}\big)\times\\
&\quad (-1)^{\ell(c_0|J)}(-q)^{\ell(r_0|I)}\cdot\big[T^{r_0,c_0}\big]t_{ij}\\
&= \Big(f_{11}\big[T^{r_1,c_1}\big]\big[T^{r_1r_0,c_1c_0}\big]^{-1}-f_{12}\big[T^{r_1,c_0}\big]\big[T^{r_1r_0,c_0c_1}\big]^{-1}\times\\
&\quad \Big(f_{22}\big[T^{r_0,c_0}\big]\big[T^{r_0r_1,c_0c_1}\big]^{-1}\Big)^{-1}\cdot f_{21}\big[T^{r_0,c_1}\big]\big[T^{r_0r_1,c_1c_0}\big]^{-1}\Big)\times\\
&\quad t_{ij}q^{\ell(j|J\setminus c_0,j)-\ell(J\setminus c_0,j|j)}(-1)^{\ell(c_0|J\setminus c_0)}(-q)^{\ell(r_0|I\setminus r_0)}\big[T^{r_0,c_0}\big]
\end{aligned}$$

Here f_{ab} are some constants depending on (q,r_0,r_1,c_0,c_1) which we could compute if we wished (cf. (5.44)), but we'll be reversing our steps in a moment, so it's not important. Put $J^0=J\setminus c_0$ and define J^1, J^{01}, and I^0 similarly. Also, write $\langle\!\langle a,A\rangle\!\rangle$ as shorthand for $\ell(a|A\setminus a)-\ell(A\setminus a|a)$. Continuing, we have

$$\begin{aligned}
\big[T\big]\cdot t_{ij} &= \Big(t_{ij}\cdot f_{11}q^{\langle\!\langle j,J^1\rangle\!\rangle}\big[T^{r_1,c_1}\big]q^{-\langle\!\langle j,J^{01}\rangle\!\rangle}\big[T^{r_1r_0,c_1c_0}\big]^{-1}-\\
&\quad t_{ij}\cdot f_{12}q^{\langle\!\langle j,J^0\rangle\!\rangle}\big[T^{r_1,c_0}\big]q^{-\langle\!\langle j,J^{01}\rangle\!\rangle}\big[T^{r_1r_0,c_0c_1}\big]^{-1}\times\\
&\quad q^{\langle\!\langle j,J^{01}\rangle\!\rangle}\big[T^{r_0r_1,c_0c_1}\big]q^{-\langle\!\langle j,J^0\rangle\!\rangle}\big[T^{r_0,c_0}\big]^{-1}f_{22}^{-1}\times\\
&\quad f_{21}q^{\langle\!\langle j,J^1\rangle\!\rangle}\big[T^{r_0,c_1}\big]q^{-\langle\!\langle j,J^{01}\rangle\!\rangle}\big[T^{r_0r_1,c_1c_0}\big]^{-1}\Big)\times\\
&\quad q^{\langle\!\langle j,J^0\rangle\!\rangle}(-1)^{\ell(c_0|J^0)}(-q)^{\ell(r_0|I^0)}\big[T^{r_0,c_0}\big]
\end{aligned}$$

$$\begin{aligned}
&= t_{ij} q^{\langle\!\langle j, J^0 \rangle\!\rangle + \cdots} \times \\
&\quad \Big(f_{11} \big[T^{r_1,c_1}\big] \big[T^{r_1 r_0, c_1 c_0}\big]^{-1} - f_{12} \big[T^{r_1,c_0}\big] \big[T^{r_1 r_0, c_0 c_1}\big]^{-1} \times \\
&\quad \Big(f_{22} \big[T^{r_0,c_0}\big] \big[T^{r_0 r_1, c_0 c_1}\big]^{-1} \Big)^{-1} \cdot f_{21} \big[T^{r_0,c_1}\big] \big[T^{r_0 r_1, c_1 c_0}\big]^{-1} \Big) \times \\
&\quad (-1)^{\ell(c_0|J^0)} (-q)^{\ell(r_0|I^0)} \big[T^{r_0,c_0}\big] \\
&= q^{\theta} t_{ij} \big[T\big] .
\end{aligned}$$

Now it is left to compute θ more carefully.

$$\theta = \langle\!\langle j, J^1 \rangle\!\rangle - \langle\!\langle j, J^{01} \rangle\!\rangle + \langle\!\langle j, J^0 \rangle\!\rangle$$

We may assume $c_0 < c_1$. There are three cases: (a) $j < c_0$; (b) $c_0 < j < c_1$; and (c) $c_1 < j$. In all three cases, a simple calculation reduces θ to $\langle\!\langle j, J \rangle\!\rangle = \ell(j|J^j) - \ell(J^j|j)$ as needed. □

Corollary 48. *For all $a \in [n]$, interpret $\ell(a|a)$ as 0. Define a function $\mathfrak{k}$ by $\mathfrak{k}(A,B) = q^{\ell(A|B)-\ell(B|A)} = \prod_{a \in A, b \in B} q^{\ell(a|b)-\ell(b|a)}$. Suppose $R', C' \in \binom{[n]}{d}$ and $R, C \in \binom{[n]}{m}$ with $R' \subseteq R, C' \subseteq C$. The minors $[T_{R,C}]$ and $[T_{R',C'}]$ of T are related by the equation*

$$\big[T_{R,C}\big] \cdot \big[T_{R',C'}\big] = \mathfrak{k}(C', C) \big[T_{I,J}\big] \cdot \big[T\big] .$$

In particular, if $T_{R,C} = T_{[n],[n]} = T$ and $1 \leq j \leq n$, then

$$\big[T\big] \big[T^{ij}\big] = q^{n+1-2j} \big[T^{ij}\big] \big[T\big].$$

Proof. The first statement follows directly from the previous theorem and the definition of $\det_{\mathrm{II}}$ in terms of the t_{ij}. The second statement comes from some simple arithmetic which we reproduce below.

$$\begin{aligned}
\sum_{t \in [n] \setminus j} \langle\!\langle t, [n] \setminus t \rangle\!\rangle &= \Big\{ \sum_{t=1}^{n} \langle\!\langle t, [n] \setminus t \rangle\!\rangle \Big\} - \langle\!\langle j, [n] \setminus j \rangle\!\rangle \\
&= \Big\{ \sum_{t=1}^{n} (t-1) - (n-t) \Big\} - \Big\{ (j-1) - (n-j) \Big\} \\
&= \Big\{ n(n+1) + 2\frac{n(n+1)}{2} \Big\} + \Big\{ n+1-2j \Big\} \\
&= n+1-2j, \text{ as desired.}
\end{aligned}$$

□

5.8.2 Main result

We catalog the key discoveries made above for use in Section 7.2.

Proposition 49. *The pair* $(\mathcal{A}^{\mathrm{II}}(n), \det_{\mathrm{II}})$ *is an amenable pair* $(\mathcal{A}(n), \mathrm{Det})$ *with associated measuring functions given by*

- $\mathfrak{I}_{\mathfrak{r}}(a,b) = (-q)^{-\ell(a|b)}$ *and* $\mathfrak{I}_{\mathfrak{x}}(a,b) = (-1)^{-\ell(a|b)}$,

- $\mathfrak{K}_{\mathfrak{r}}(a,b) = 1$ *and* $\mathfrak{K}_{\mathfrak{x}}(a,b) = \mathfrak{k}(a,b) = q^{\ell(a|b)-\ell(b|a)}$,

In $\mathcal{A}^{\mathrm{II}}(n)$*, the column-minors* $\left\{[T_A] = \det_{\mathrm{II}} T_{[d],A} \mid A \in \binom{[n]}{d}\right\}$ *satisfy*

- $(\forall 1 \leq s \leq r < n)$ *If* $K, M \subseteq [n]$ *are subsets satisfying* $|K| = r+1, |M| = s-1$, *then*

$$0 = \sum_{k \in K \setminus M} (-1)^{-\ell(M|k)-\ell(k|K^k)} q^{-\ell(M|k)+\ell(k|M)} \cdot [T_{M \cup k}][T_{K \setminus k}] \tag{5.46}$$

- *If* $I, J, M \subseteq [n]$ $(|J| = s \leq r = |I|, |M| = u)$ *are pairwise disjoint, and if* J *can't distinguish* I *as columns, then for any* $i \in I$

$$[T_{J \cup M}][T_{I \cup M}] = \frac{q^{\ell(J|i)-\ell(i|J)}}{q^{\ell(J|I)-\ell(I|J)}} \frac{q^{\ell(M|J)-\ell(J|M)}}{q^{\ell(M|I)-\ell(I|M)}} \cdot [T_{I \cup M}][T_{J \cup M}] \tag{5.47}$$

Chapter 6

Quantum Flag Algebra of Taft-Towber

In this chapter, we study the implications of our quasideterminantal calculus on the important "quantum flags" of Taft and Towber. Throughout, F is a commutative field with distinguished element $q \neq 0$ and not equal to a root of unity.

6.1 Left & Right Quantum Plücker Coordinates

One important property of the quantum determinant which we have yet to mention is its behavior under transpose. From the row/column symmetry in (5.6)–(5.11), it is easy to see

Proposition 50. *If A is a q-generic matrix, then its transpose A^T is q-generic as well.*

It is somewhat harder, though still straightforward (cf. [48]), to see

Proposition 51. *If A is a q-generic matrix, then*

$$\det_q A = \det_q A^T.$$

With these propositions, we make an important reduction. If we are given a "q-generic point" $A(\Phi)$ in some left noncommutative flag $F\ell(\gamma)$, then we may equally well consider left (column) quantum Plücker coordinates of A or right (row) quantum Plücker coordinates of A^T, thought of as a q-generic point in some right noncommutative flag $F\ell(\gamma)$.

As alluded to earlier, there is some advantage to considering row coordinates over column coordinates for q-generic matrices (cf. (5.13) and the remark following). Here, we make explicit the connection between left and right coordinates of A and A^T.

Proposition 52. *Let A be a $d \times n$ q-generic matrix with $d < n$. The left quasi-Plücker coordinates of A are related to the right quasi-Plucker coordinates of A^T by the formula*

$$p_{ji}^K(A) = q^{\ell(i|j)-\ell(j|i)}(-q)^{2\ell(i|K)-2\ell(j|K)} \cdot r_{ij}^K(A^T)\,.$$

Proof. Below, we write $[A_{I,K}]$ for the quantum determinant $\det_q A_{I,K}$. Also, for $I \in \binom{[n]}{d}$, we choose the first d rows of A to build column coordinates $[A_{[d],I}]$ and the first d columns of A^T to build row coordinates $[(A^T)_{I,[d]}]$. We may write this last expression as $[(A_{[d],I})^T]$ and it will suit us to do so.

Fix an $n \times n$ q-generic matrix T. We summarize important facts which have come before. Letting $R, C \in \binom{[n]}{d}$ denote the rows and columns of the sub-matrix $X = T_{R,C}$, we have:

- (transpose) X^T is q-generic, and $[X] = [X^T]$.
- (factorization): $[X] = (-q)^{\ell(r|R^r)-\ell(c|C^c)} \times |X|_{rc} \times [X^{rc}]$, and the factors commute.
- (r_{ij}^K definition): For all $|K|+1 = d' \le d$, $r_{ij}^K(X) = |X_{i\cup K,[d']}|_{ic} \times (|X_{j\cup K,[d']}|_{jc})^{-1}$.
- (p_{ji}^K definition): For all $|K|+1 = d' \le d$, $p_{ji}^K(X) = (|X_{[d'],j\cup K}|_{rj})^{-1} \times |X_{[d'],i\cup K}|_{ri}$.

Now,

$$\begin{aligned}
r_{ij}^K(A^T) &= |(A_{[d],i\cup K})^T|_{ic} \times (|(A_{[d],j\cup K})^T|_{jc})^{-1} \\
&= (-q)^{\ell(c|[d]\setminus c)-\ell(i|K)} \cdot [(A_{[d],i\cup K})^T] \cdot [((A_{[d],i\cup K})^T)^{ic}]^{-1} \times \\
&\quad \Big((-q)^{\ell(c|[d]\setminus c)-\ell(j|K)} \cdot [(A_{[d],j\cup K})^T] \cdot [((A_{[d],j\cup K})^T)^{js}]^{-1}\Big)^{-1} \\
&= (-q)^{\ell(j|K)-\ell(i|K)} \cdot [(A_{[d],i\cup K})^T] \cdot [(A_{[d],j\cup K})^T]^{-1} \\
&= (-q)^{\ell(j|K)-\ell(i|K)} \cdot q^{\ell(j|i)-\ell(i|j)} [A_{[d],j\cup K}]^{-1} \cdot [A_{[d],i\cup K}]\,,
\end{aligned}$$

while

$$\begin{aligned}
p_{ji}^K(A) &= (|A_{[d],j\cup K}|_{rj})^{-1} \times |A_{[d],i\cup K}|_{ri} \\
&= \Big((-q)^{\ell(j|K)-\ell(r|[d]\setminus r)} \cdot [A_{[d],j\cup K}] \cdot [(A_{[d],j\cup K})^{rj}]^{-1}\Big)^{-1} \times \\
&\quad (-q)^{\ell(i|K)-\ell(r|[d]\setminus r)} \cdot [A_{[d],i\cup K}] \cdot [(A_{[d],i\cup K})^{ri}]^{-1} \\
&= (-q)^{-\ell(j|K)+\ell(i|K)} \cdot [A_{[d],j\cup K}] \cdot [A_{[d],i\cup K}]^{-1}\,.
\end{aligned}$$

$\square$

For the rest of the chapter, we concentrate on the quantum Plücker coordinates of a right flag. Also, we often suppress the column subscripts, taking $[I]$ to mean $[A_{I,\{1,2,\ldots,|I|\}}]$.

6.2 The Quantum Flag Algebra

Recall the definition of "pre–flag algebra" from Chapter 4.

Given a composition $\gamma \models n$, and a noncommutative algebra $\mathcal{A}(n)$ with amenable determinant, the right pre–flag algebra $\tilde{\mathcal{F}}(\gamma)$ associated to $\mathcal{A}(n)$ is the F-algebra with generators $\left\{\tilde{f}_I \mid I \in \binom{[n]}{d},\, d \in \|\gamma\|\right\}$ and relations given by equations

$$0 = \sum_{k\in K\setminus M} \frac{\mathfrak{I}_{\mathfrak{r}}(k,kM)}{\mathfrak{I}_{\mathfrak{r}}(k,K)\mathfrak{K}_{\mathfrak{r}}(K\setminus k,K)} \cdot \tilde{f}_{K\setminus k}\tilde{f}_{k\cup M} \tag{6.1}$$

whenever $K, M \in \mathcal{P}[n]$ with $|K|-1, |M|+1 \in \binom{n}{\|\gamma\|}$, and

$$\tilde{f}_{JM}\tilde{f}_{IM} = \frac{\rho_J \mathfrak{K}_{\mathfrak{r}}(J,I)}{\mathfrak{K}_{\mathfrak{r}}(\bar{K},K)} \cdot \frac{\mathfrak{K}_{\mathfrak{r}}(M,I)}{\mathfrak{K}_{\mathfrak{r}}(M,J)\mathfrak{K}_{\mathfrak{r}}([t],[t+s]\setminus[t+r])} \tilde{f}_{I\cup M}\tilde{f}_{J\cup M} \tag{6.2}$$

whenever J can't distinguish I, $J\cup M, I\cup M \in \binom{[n]}{\|\gamma\|}$ $(|J|=r, |I|=s, |M|=t)$, and I,J,M are pairwise disjoint.

In our present situation, we may add q-alternating relations, change the set of generators, and employ Proposition 28 to get a candidate definition of the homogeneous coordinate algebra for quantum flags. First, we characterize when J can't distinguish I.

Definition 35. Given two subsets $I, J \subseteq [n]$, we say J *surrounds* I, written $J \curvearrowright I$, if (i) $|J| \leq |I|$, and (ii) there exist disjoint subsets $\emptyset \subseteq J', J'' \subseteq J$ such that:[1]

a. $J \setminus I = J' \,\dot{\cup}\, J''$,

b. $J' \prec I \setminus J$ and $I \setminus J \prec J''$.

[1] In the literature, sets J and I sharing this relationship are called "weakly separated." I do not like this terminology because it does not indicate who separates whom. It should be pointed out that, working from the definition of $\det_q$ within $\mathcal{M}_q(n)$, Leclerc and [33] showed that quantum minors $[J]$ and $[I]$ q-commute if and only if they are weakly separated.

We may extend this notion to tuples instead of sets by letting J always surround I provided two indices of J or I are identical; otherwise, J surrounds I iff $\mathsf{set}(J)$ surrounds $\mathsf{set}(I)$.

Let us extend our notion of "can't distinguish" from sets to tuples in a manner analogous to the preceding definition. Then we have the following easy result.

Proposition 53. *Fix two tuples $J, I \in [n]^{\|\gamma\|}$. In the case of quantum determinants, we have J can't distinguish I (as rows) if and only if $J \curvearrowright I$.*

Proof. We need $\frac{-\mathfrak{I}_{\mathfrak{r}}(i,j)}{\mathfrak{I}_{\mathfrak{r}}(j,i)\mathfrak{K}_{\mathfrak{r}}(i,j)}$ to be constant across all $i \in I$ for each fixed $j \in J$. In the present setting, this expression becomes $-(-q)^{\ell(j|i)-\ell(i|j)}$. Now place J and I on the number line between 1 and n, and consider a fixed j. If there are elements i to the left and to the right of j, then $\ell(j|i) - \ell(i|j)$ is sometimes 1 and sometimes -1. If there are only elements i on the left (or on the right) of j, then $\ell(j|i) - \ell(i|j)$ is constantly 1 (or -1). $\square$

Remark. The same statement and proof hold for the two-parameter and multi-parameter deformations of $\mathrm{M}_n(\mathbb{C})$.

Definition 36 (Quantum Flag Algebra). Given a composition $\gamma \models n$, the quantum pre–flag algebra $\tilde{\mathcal{F}}(\gamma)$ associated to $\mathcal{M}_q(n)$ is the F-algebra with generators $\left\{ \tilde{f}_I \mid I \in [n]^d,\, d \in \|\gamma\| \right\}$ and relations given below.

- The *q-alternating* relations $(\mathcal{A}_I)$: For all $I \in [n]^{\|\gamma\|}$ with $I' = \mathsf{rect}(I)$,

$$\tilde{f}_I = \begin{cases} 0 & \text{if } I \text{ contains repeated indices} \\ (-q)^{-\ell(I)}\tilde{f}_{I'} & \text{otherwise} \end{cases} \tag{6.3}$$

- The *weak–Young symmetry* relations $(\mathcal{Y}_{I,J})_{(1)}$: For all $I, J \in \mathcal{P}[n]$ with $|I| - 1, |J| + 1 \in \binom{n}{\|\gamma\|}$,

$$0 = \sum_{k \in I} (-q)^{-\ell(I\setminus k|k)} \tilde{f}_{I\setminus k} \tilde{f}_{k|J}\,. \tag{6.4}$$

- The *q-commuting* relations $(\mathcal{C}_{J,I})$: For all $J, I \in \binom{[n]}{\|\gamma\|}$ with $J \curvearrowright I$,

$$\tilde{f}_J \tilde{f}_I = q^{|J''|-|J'|} \tilde{f}_I \tilde{f}_J\,. \tag{6.5}$$

Let us compare this algebra to the quantum flag algebra introduced by Taft and Towber in [45].

Definition 37. Given a composition $\gamma \models n$, the quantum flag algebra $\mathcal{F}_q(\gamma)$ is the F-algebra with generators $\left\{ f_I \mid I \in [n]^d,\, d \in \|\gamma\| \right\}$ and relations given below.

- The *q-alternating* relations $(\mathcal{A}_I)$: For all $I \in [n]^{\|\gamma\|}$ with $I' = \mathsf{rect}(I)$,

$$f_I = \begin{cases} 0 & \text{if } I \text{ contains repeated indices} \\ (-q)^{-\ell(I)} f_{I'} & \text{otherwise} \end{cases} \tag{6.6}$$

- The *Young symmetry* relations $(\mathcal{Y}_{I,J})_{(u)}$: For all $I, J \in \mathcal{P}[n]$ and all $u \in \mathbb{N}$ with $|I| - u, |J| + u \in \|\gamma\|$,

$$0 = \sum_{\Lambda \subseteq I, |\Lambda| = u} (-q)^{-\ell(I \backslash \Lambda | \Lambda)} f_{I \backslash \Lambda} f_{\Lambda | J} \tag{6.7}$$

- The *q-straightening* relations $(\mathcal{S}_{J,I})$: For all $J, I \in \binom{[n]}{\|\gamma\|}$ with $|J| \leq |I|$,

$$f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda| = |J|} (-q)^{\ell(\Lambda | I \backslash \Lambda)} f_{J | I \backslash \Lambda} f_{\Lambda} \tag{6.8}$$

Remark. Several comments are in order.

- The definition as it appears in [45] pertains only to full flags $F\ell((1^n))$. However, the definition and the theorem appearing below are readily extended to more general flags (cf. the papers of Hodge, Towber, and Taft [25, 49, 45, 46]).

- Technically, we should have taken I, J to be tuples instead of sets in (6.7) and (6.8). To repair, simply replace all instances of I with $\mathsf{tup}(I)$, etc.

- Notice that relation (6.8) is trivial when $|J| = |I|$, reading $f_J f_I = f_J f_I$. This leaves the coordinate algebra of the quantum Grassmannian with only two sets of relations (namely, (6.6) and (6.7) with $\|\gamma\| = \{d\}$).

- This algebra is the "correct one" as the following quantum version of the Basis Theorem indicates.

Theorem 54 (Taft-Towber, [45]). *The algebra $\mathcal{F}_q(\gamma)$ is isomorphic to the subalgebra of $\mathcal{M}_q(n)$ generated by the quantum minors $\Big\{\big[I\big] = \det_q T_{I,[d]} \mid I \in \binom{[n]}{d}, d \in \|\gamma\|\Big\}$ of the matrix of generators T.*

The quantum flag algebra has been well studied—mostly in its $\mathcal{M}_q(\gamma)$ incarnation—since its introduction (cf. [6, 12, 20, 29, 32]). In this chapter we focus on the discrepancy between (6.3)–(6.5) and (6.6)–(6.8).

6.3 Young Symmetry Relations

In the commutative setting for flag algebras it is known that all relations analogous to $(\mathcal{Y}_{I,J})_{(u)}$ with $u > 1$ (see (3.1)), are direct consequences of those with $u = 1$ (cf. [26] and [49]). The proofs published there rely heavily on the commutativity of the Plücker coordinates $\{p_I\}$ (and hence of the coordinate functions $\{f_I\}$). What follows is a proof of the same fact for quantum Plücker coordinates. In addition to giving a new proof for the classical case (set $q = 1$), it stands as an important result on its own.

Proposition 55. *Let I, J be ordered subsets of $[n]$ with respective sizes $s+u$ and $r-u$ $(1 \leq u \leq r \leq s)$. Then $(\mathcal{Y}_{I,J})_{(u)}$ can be written in terms of relations of type $(\mathcal{Y}_{L,M})_{(u-1)}$. Specifically, writing $Y_{I,J;(u)}$ for the right-hand side of relation $(\mathcal{Y}_{I,J})_{(u)}$, we have*

$$\sum_{k=1}^{s+u}\Big((-q)^{2(u-1)-\ell(I^{i_k}|i_k)}(-q)^{-\ell(i_k|J)}Y_{I^{i_k},i_k\cup J;(u-1)}\Big) = \left(\sum_{k=0}^{u-1}(-q)^{2k}\right)Y_{I,J;(u)}\,.$$

Proof. In terms of quantum minors, this reads

$$\sum_{k=1}^{s+u}(-q)^{2(u-1)-\ell(I^{(k)}|i_k)} \sum_{\substack{\Lambda^{(k)}\subset I^{(k)}\\ |\Lambda^{(k)}|=u-1}} (-q)^{-\ell(I^{(k)}\setminus\Lambda^{(k)}|\Lambda^{(k)})}\big[I^{(k)}\setminus\Lambda^{(k)}\big]\big[\Lambda^{(k)}|i_k|J\big]$$

$$= \left(\sum_{k=0}^{u-1}(-q)^{2k}\right)\sum_{\substack{\Lambda\subset I\\ |\Lambda|=u}}(-q)^{-\ell(I\setminus\Lambda|\Lambda)}\big[I\setminus\Lambda\big]\big[\Lambda|J\big].$$

Here, we have abused our standard set-operations notation as follows: if $A = \{a_1 < a_2 < \cdots < a_p\}$, then we write $A^{(k)}$ for A^{a_k} to increase legibility.

To demonstrate the equality, we simply take an arbitrary Λ and compare the coefficients on the left- and right-hand sides of the monomial $\big[I\setminus\Lambda\big]\big[\Lambda|J\big]$.

left-hand side:

$$\sum_{i_k \in \Lambda} (-q)^{2(u-1)-\ell(I^{(k)}|i_k)} (-q)^{-\ell(I^{(k)} \setminus \Lambda^{(k)}|\Lambda^{(k)})} \big[I \setminus \Lambda\big] \big[\Lambda^{(k)}|i_k|J\big]$$

$$= \left(\sum_{i_k \in \Lambda} (-q)^{2(u-1)-\ell(I^{(k)}|i_k)-\ell(I^{(k)} \setminus \Lambda^{(k)}|\Lambda^{(k)})-\ell(\Lambda^{(k)}|i_k)} \right) \big[I \setminus \Lambda\big] \big[\Lambda|J\big]$$

right-hand side:

$$\left(\sum_{k=0}^{u-1} (-q)^{2k-\ell(I\setminus\Lambda|\Lambda)} \right) \big[I \setminus \Lambda\big] \big[\Lambda|J\big].$$

Multiplying both sides by $(-q)^{+\ell(I\setminus\Lambda|\Lambda)}$ and using $\ell(I\setminus\Lambda|\Lambda) = \ell(I\setminus\Lambda|\Lambda^{(k)}) + \ell(I^{(k)}|i_k) - \ell(\Lambda^{(k)}|i_k)$, we are left with showing

$$\sum_{k=1}^{u} (-q)^{2(u-1)-2\ell(\Lambda^{(k)}|i_k)} = \sum_{k=1}^{u} (-q)^{2(k-1)}.$$

But $(u-1) - \ell(\Lambda^{(k)}|i_k)$ is exactly $k-1$. □

Remark. Note that this proof fails to work if q^2 is a u-th root of unity. In the case $q = 1$ it additionally fails if the characteristic of the field is u. Thus there is no improvement to the situation addressed in [49] in the commutative case.

Repeated application of this reduction proves the following important modification to the definition of the quantum flag algebra.

Corollary 56. *Equation (6.7) in the definition of $\mathcal{F}_q(\gamma)$ may be replaced with an abbreviated version—taking only $u = 1$.*

In particular, this settles the discrepancy between (6.4) and (6.7). As (6.8) is vacuous in the case of quantum Grassmannians, we have the following important theorem

Theorem 57. *For fixed $0 < d < n$, all identities holding among the quantum minors $\{[I] \mid I \in \binom{[n]}{d}\}$ of T are consequences of quasi-Plücker coordinate identities.*

This begs the question, what about relations $(\mathcal{C}_{J,I})$, which appear in the definition of $\tilde{\mathcal{G}}(d,n)$ but not in the definition of $\mathcal{G}_q(d,n)$? We will address these "missing relations" at the end of the next section.

6.4 q-Straightening and q-Commuting Relations

We have seen that the "Young symmetry" relations of Taft and Towber are consequences of relations known to hold among generic quasi-Plücker coordinates. As we will see below, this is not the case for the so-called "Commuting" relations of Taft and Towber (which we have labeled *q-straightening* relations here). However, a large subset of these relations may be so described. The question of whether and to what extent the remaining discrepancy may be fixed by finding new quasi-Plücker coordinate identities is an interesting one. Before presenting the main result of this section, it will be helpful to introduce some combinatorics.

6.4.1 POset paths

The elements of the power set $\mathcal{P}X$ have a partial ordering: for $A, B \in \mathcal{P}X$, we say $A < B$ if $A \subsetneq B$. In this section, we think of this POset as an edge-weighted, directed graph, and denote it by $\Gamma(X)$.

Definition 38. The graph $\Gamma(X) = \Gamma(X;\alpha) = (\{\mathcal{V},\mathcal{E}\};\alpha)$ has vertex set $\mathcal{V} = \mathcal{P}X$ and edge set $\mathcal{E} = \{(A,B) \mid A, B \in \mathcal{V},\ A \subsetneq B\}$. The function $\alpha : \mathcal{E} \to F$ assigns a weight α_A^B to each edge $(A,B) \in \mathcal{E}$.

Example. If $|X| = m$, then $\Gamma(X)$ has 2^m vertices and $\sum_{k=1}^{m} \binom{m}{k}(2^m - 1)$ edges. In Figure 6.1, we give an illustration of $\Gamma(\{1,5,6\})$, omitting some edges and edge weights for legibility.

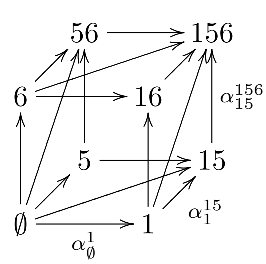

Figure 6.1: The graph $\Gamma(\{1,5,6\})$ (partially rendered).

For the remainder of the subsection, we will be interested in graphs arising from subsets $I, J \in [n]$ with $J \curvearrowright I$. To simplify notation, let us write $J = J'\dot{\cup}J'' = \{j_1 <$

$\ldots < j'_r\} \cup \{j_{r'+1} < \cdots < j_{r'+r''}\}$; also, put $|J| = r' + r'' = r$, $|I| = s$, and $s - r = t$. It will only be necessary to consider the case $J \cap I = \emptyset$, though the balance of this subsection may be repeated in greater generality with minimal effort. Write $\Gamma(J;I)$ for the graph built on the POset $\mathcal{P}J$ with edge-weight function given by

$$(\forall (A,B) \in \mathcal{E}) \quad \alpha_A^B := (-q)^{-\ell(J\backslash B|B\backslash A)-\ell(B\backslash A|A)+\left(2|J\backslash B|-|I|\right)|(B\backslash A)\cap J'|}. \tag{6.9}$$

Definition 39. In the graph $\Gamma(J;I)$, we consider *paths* $\mathfrak{P}_0$ and $\mathfrak{P}$ defined as follows:

$$\mathfrak{P}_0 = \{(A_1, A_2, \ldots, A_p) \mid A_i \subseteq J \text{ s.t. } \emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p \subsetneq J\},$$

and $\mathfrak{P} = \mathfrak{P}_0 \cup \hat{0} \cup \hat{1}$, where $\hat{0} = (\emptyset)$, and

$$\hat{1} = (\{j_{r'+1}\}, \{j_{r'+1}, j_{r'+2}\}, \ldots, J'', \{j_{r'}, \ldots, j_r\}, \ldots, \{j_2, \ldots, j_r\}, J).$$

The *weight* $\alpha(\pi)$ of a path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}_0$ is the product of edge weights of the augmented path $(\emptyset, \pi, J)$:

$$\alpha_{\emptyset}^{A_1} \cdot \alpha_{A_1}^{A_2} \cdots \alpha_{A_{p-1}}^{A_p} \cdot \alpha_{A_p}^{J}.$$

We extend the definition of α to all of $\mathfrak{P}$ as follows. Notice that if $B = A$ in (6.9), we get $\alpha_A^A = 1$. With this broader definition of the weight function α, we may define $\alpha(\pi) = \alpha(\emptyset, \pi, J)$ for $\pi = \hat{0}, \hat{1}$ as well. Writing $\hat{1} = (A_1, \ldots, A_{r=|J|})$, the path $(A_1, \ldots, A_{r-1}) \in \mathfrak{P}_0$ will also be important, we label this special path $\pi^{\hat{1}}$.

Definition 40. Given a subset $K \subseteq J$, define $\mathsf{mM}(K)$ as follows. If $K \cap J' \neq \emptyset$, put $\mathsf{mM}(K) = \min(K \cap J')$. Otherwise, put $\mathsf{mM}(K) = \max(K \cap J'')$.

For any path $\pi = (A_1, \ldots, A_p)$, put $A_0 = \emptyset$ and $A_{p+1} = J$. Notice that $\hat{1}$ has the property that $A_k \setminus A_{k-1} \neq \mathsf{mM}(A_{k+1} \setminus A_{k-1})$ for all $1 \leq k < r$, but $A_r = \mathsf{mM}(A_{r+1} \setminus A_{r-1})$.

Definition 41. Fix a length $1 \leq p \leq r-1$. A path $(A_1, \ldots, A_p) \in \mathfrak{P}_0$ shall be called *regular* (or *regular at position* i_0), if $(\exists i_0)(1 \leq i_0 \leq p)$ satisfying: (a) $|A_i| = i\, (\forall\, 1 \leq i \leq i_0)$; (b) $A_{i_0} \setminus A_{i_0-1} = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0-1})$ (again, taking $A_0 = \emptyset$ and $A_{p+1} = J$ if necessary). A sequence is called *irregular* if it is nowhere regular. Extend the notion of regularity to $\mathfrak{P}$ by calling $\hat{0}$ irregular and $\hat{1}$ regular.

Remark. The set $\mathfrak{P}$ is the disjoint union of its regular and irregular paths. We point out this tautology only to emphasize its importance in the coming proposition. Write $\mathfrak{P}'$ for the irregular paths, and $\mathfrak{P}''$ for the regular paths.

Proposition 58. *The subsets $\mathfrak{P}'$ and $\mathfrak{P}''$ of $\mathfrak{P}$ are equinumerous.*

Naturally, we will build a bijective map between the two sets. Given an irregular path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}_0$, we insert a new set B so that $\varphi(\pi)$ is regular at B:

1. Find the unique i_0 satisfying: $(|A_i| = i \quad \forall i \leq i_0) \wedge (|A_{i_0+1}| > i_0 + 1)$.

2. Compute $b = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0})$

3. Put $B = A_{i_0} \cup \{b\}$.

4. Define $\varphi(\pi) := (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A_p)$.

For the remaining irregular path $\hat{0}$, we put $\varphi(\hat{0}) = (\{j_1\})$, which agrees with the general definition of φ if we think of $\hat{0}$ as the empty path $()$ instead of the path consisting of the empty set.

Example. Table 6.1 illustrates the action of φ on $\mathfrak{P}$ when $J = \{1, 5, 6\}$.

π	$\varphi(\pi)$
$\hat{0}$	(1)
(5)	$(5, 15)$
(6)	$(6, 16)$
(15)	$(1, 15)$
(16)	$(1, 16)$
(56)	$(6, 56)$
$(5, 56)$	$\hat{1}$

Table 6.1: The pairing of $\mathfrak{P}'$ and $\mathfrak{P}''$ via φ.

Proof. We reach a proof in three steps.

Claim 1: $\varphi(\mathfrak{P}') \subseteq \mathfrak{P}''$.

Take a path $\pi \in \mathfrak{P}'$ (i.e. a path with no regular points). The effect of φ is to insert a regular point at position $i_0 + 1$ (the spot where B sits), so the claim is proven if we can show $\varphi(\pi) \in \mathfrak{P}$.

As $\varphi(\hat{0})$ clearly belongs to $\mathfrak{P}$, we may focus on those $\pi \in \mathfrak{P}_0$. Also, it is plain to see that $\pi^{\hat{1}}$ is irregular, and $\varphi(\pi^{\hat{1}}) = \hat{1}$. If φ is to be a bijection, we are left with the task of showing that $\varphi(\mathfrak{P}' \cap \mathfrak{P}_0 \setminus \pi^{\hat{1}}) \subseteq \mathfrak{P}_0$

When $|A_p| < r-1$, any B that is inserted will result in another path in $\mathfrak{P}_0$ (because $|B|$ must be less than r). When $|A_p| = r-1$, there is some concern that we will have to insert a B at the end of the path, resulting in J being the new terminal vertex—disallowed in $\mathfrak{P}_0$. This cannot happen:

Case $p < r-1$: At some point $1 \leq i_0 < p$, there is a jump in set-size greater than one when moving from A_{i_0} to A_{i_0+1}. Hence, the B to be inserted will not come at the end, but rather immediately after A_{i_0} to A_{i_0+1}

Case $p = r-1$: The only path $(A_1, A_2, \ldots, A_{r-1}) \in \mathfrak{P}_0$ which is nowhere regular is the path $\pi^{\hat{1}}$.

Claim 2: φ is 1-1.

Suppose $\varphi(A_1, \ldots, A_p) = \varphi(A'_1, \ldots, A'_{p'})$, and suppose we insert B and B' respectively. By the nature of φ, we have $p = p'$ and $i_0 \neq i'_0$. Take $i_0 < i'_0$. Also notice that $(A'_1, \ldots, A'_{p'}) = (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A'_{i'_0}, \ldots A'_{p'})$ In particular, B is a regular point of $(A'_1, \ldots, A'_{p'})$, and consequently, $(A'_1, \ldots, A'_{p'}) \notin \mathfrak{P}'$.

Claim 3: φ is onto.

Consider a path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}''$. If $p = 1$, then it is plain to see that the only irregular path is $\pi = (\{j_1\})$, which is the image of $(\emptyset)$ under φ. So we consider $\pi \in \mathfrak{P}''$ with $p > 1$. Note that $|A_1| = 1$, for otherwise π cannot have any regular points. Now, locate the first $1 \leq i_0 \leq p$ with (a) $|A_{i_0}| = i_0$; and (b) $A_{i_0} \setminus A_{i_0-1} = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0-1}$. The path $\pi' = (A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k)$ is in $\mathfrak{P}'$ and moreover, $\varphi(\pi') = \pi$. $\square$

Certainly one could cook up other bijections between the regular and irregular paths in $\mathfrak{P}$. The map we have used has an additional nice property.

Proposition 59. *The bijection φ from the proof of Proposition 58 is path-weight preserving.*

The result rests on the following

Lemma. Let $\emptyset \subseteq A \subseteq B \subseteq C \subseteq J$. Writing $\hat{B} = B \setminus A$ and $\hat{C} = C \setminus B$, we have

$$\alpha_A^B \alpha_B^C = \left[(-q)^{2\ell(B' \cap J'|C') - 2\ell(C'|B' \cap J'')}\right] \alpha_A^C \,. \tag{6.10}$$

Proof. From the definition of α_B^C, we have

$$\begin{aligned}
\alpha_A^B &= (-q)^{-\ell(J\setminus B|\hat{B}) - \ell(\hat{B}|A) + \left(2|J\setminus B| - |I|\right)|\hat{B}\cap J'|} \\
\alpha_B^C &= (-q)^{-\ell(J\setminus C|\hat{C}) - \ell(\hat{C}|B) + \left(2|J\setminus C| - |I|\right)|\hat{C}\cap J'|} \\
\alpha_A^C &= (-q)^{-\ell(J\setminus C|\hat{B}\cup\hat{C}) - \ell(\hat{B}\cup\hat{C}|A) + \left(2|J\setminus C| - |I|\right)|(\hat{B}\cup\hat{C})\cap J'|}
\end{aligned}$$

Let us compare the exponents of α_A^C and $\alpha_A^B \alpha_B^C$:

$$\begin{aligned}
\exp(\alpha_A^C) &= -\ell(J \setminus C|\hat{B}) - \ell(J \setminus C|\hat{C}) - \ell(\hat{C}|A) - \ell(\hat{B}|A) + \\
&\quad \left(2|J \setminus A| - 2|\hat{C}| - 2|\hat{B}| - |I|\right)\left(|\hat{B} \cap J'| + |\hat{C} \cap J'|\right) ,
\end{aligned} \tag{6.11}$$

while

$$\begin{aligned}
\exp(\alpha_A^B \alpha_B^C) &= -\ell(J \setminus B|\hat{B}) - \ell(J \setminus C|\hat{C}) - \ell(\hat{B}|A) - \ell(\hat{C}|B) + \\
&\quad \left(2|J \setminus B| - |I|\right)|\hat{B} \cap J'| + \left(2|J \setminus C| - |I|\right)|\hat{C} \cap J'| \\
&= -\Big\{\ell(J \setminus C|\hat{B}) + \ell(\hat{C}|\hat{B})\Big\} - \ell(J \setminus C|\hat{C}) - \ell(\hat{B}|A) - \\
&\quad \Big\{\ell(\hat{C}|A) + \ell(\hat{C}|\hat{B})\Big\} + \Big\{2|J \setminus A| - 2|\hat{B}| - |I|\Big\}|\hat{B} \cap J'| + \\
&\quad \Big\{2|J \setminus A| - 2|\hat{B}| - 2|\hat{C}| - |I|\Big\}|\hat{C} \cap J'| \\
&= 2|\hat{C}||\hat{B} \cap J'| - 2\ell(\hat{C}|\hat{B}) + \Big\{\exp(\alpha_A^C)\Big\} .
\end{aligned} \tag{6.12}$$

Notice that $2|\hat{C}||\hat{B} \cap J'| = 2\ell(\hat{C}|\hat{B} \cap J') + 2\ell(\hat{B} \cap J'|\hat{C})$, and that $-2\ell(\hat{C}|\hat{B}) = -2\ell(\hat{C}|\hat{B} \cap J') - 2\ell(\hat{C}|\hat{B} \cap J'')$. The discrepancy between (6.12) and (6.11) becomes $2\ell(\hat{B} \cap J'|\hat{C}) - 2\ell(\hat{C}|\hat{B} \cap J'')$, as desired. $\square$

Now the proposition follows by comparing $\alpha(A_{i_0}, A_{i_0+1})$ and $\alpha(A_{i_0}, B, A_{i_0+1})$.

Proof of Proposition. Suppose that $\pi = (\ldots, A, C, \ldots)$, and that $\varphi(\pi)$ inserts B immediately after A. Then $B = A \cup \mathsf{mM}(C \setminus A)$. Writing $b = \mathsf{mM}(C \setminus A)$, (6.10) implies

$$\alpha(\varphi(\pi)) = \left[(-q)^{2\ell(b\cap J'|\hat{C}) - 2\ell(\hat{C}|b\cap J'')}\right] \cdot \alpha(\pi)\,.$$

Now, if $b \cap J' \neq \emptyset$, then b is the smallest element in $C \setminus A$, and in particular, $\ell(b|\hat{C}) = 0$. In this same case, $b \cap J'' = \emptyset$, so $\ell(\hat{C}|b \cap J'') = 0$ too. An analogous argument works for the case $b \cap J' = \emptyset$. □

One more interesting fact about $\Gamma(J; I)$ and $\mathfrak{P}$ is worth mentioning. When calculating $\alpha(\pi^{\hat{1}})$ using (6.10), the twos introduced in the exponents there all disappear.

Proposition 60. *Given, J, J', J'', and $\pi^{\hat{1}}$ as above, we have*

$$\alpha(\pi^{\hat{1}}) = (-q)^{|J'|(|J'|-1)-|J''|(|J''|-1)} \times \alpha_{\emptyset}^{J} . \tag{6.13}$$

Proof. Applying 6.10 repeatedly to the expression $\alpha(\pi^{\hat{1}})$ we see that

$$\begin{aligned}
\alpha(\pi^{\hat{1}}) &= \left[(-q)^{2\ell(j_{r'+1} \cap J'|j_{r'+2}) - 2\ell(j_{r'+2}|j_{r'+1} \cap J'')}\right] \times \\
&\quad \alpha_{\emptyset}^{j_{r'+1}j_{r'+2}} \alpha_{j_{r'+1}j_{r'+2}}^{j_{r'+1}j_{r'+2}j_{r'+3}} \cdots \alpha_{j_2 \cdots j_r}^{J} \\
&= (-q)^{-2(1)} \left[(-q)^{2\ell(j_{r'+2} \cap J'|j_{r'+3}) - 2\ell(j_{r'+3}|j_{r'+2} \cap J'')}\right] \times \\
&\quad \alpha_{\emptyset}^{j_{r'+1}j_{r'+2}j_{r'+3}} \cdots \alpha_{j_2 \cdots j_r}^{J} \\
&= (-q)^{-2(1)-2(2)} \left[(-q)^{2\ell(j_{r'+3} \cap J'|j_{r'+4}) - 2\ell(j_{r'+4}|j_{r'+3} \cap J'')}\right] \times \\
&\quad \alpha_{\emptyset}^{j_{r'+1}j_{r'+2}j_{r'+3}j_{r'+4}} \cdots \alpha_{j_2 \cdots j_r}^{J} \\
&\vdots \\
&= (-q)^{-2(1)-\cdots-2(|J''|-1)} \left[(-q)^{2\ell(j_r \cap J'|j_{r'}) - 2\ell(j_{r'}|j_r \cap J'')}\right] \times \\
&\quad \alpha_{\emptyset}^{j_{r'} \cdots j_r} \cdots \alpha_{j_2 \cdots j_r}^{J} \\
&= (-q)^{-2\frac{(|J''|-1)|J''|}{2}} (-q)^{0-0} \left[(-q)^{2\ell(j_{r'} \cap J'|j_{r'-1}) - 2\ell(j_{r'-1}|j_{r'} \cap J'')}\right] \times \\
&\quad \alpha_{\emptyset}^{j_{r'-1} \cdots j_r} \cdots \alpha_{j_2 \cdots j_r}^{J} \\
&= (-q)^{2(1)} (-q)^{-|J''|(|J''|-1)} \left[(-q)^{2\ell(j_{r'-1} \cap J'|j_{r'-2}) - 2\ell(j_{r'-2}|j_{r'-1} \cap J'')}\right] \times \\
&\quad \alpha_{\emptyset}^{j_{r'-2} \cdots j_r} \cdots \alpha_{j_2 \cdots j_r}^{J} \\
&\vdots \\
&= (-q)^{2(1)+\cdots+2(|J'|-1)} (-q)^{-|J''|(|J''|-1)} \times \alpha_{\emptyset}^{J} \\
&= (-q)^{|J'|(|J'|-1)-|J''|(|J''|-1)} \times \alpha_{\emptyset}^{J} ,
\end{aligned}$$

the desired expression. □

6.4.2 $(\mathcal{C}_{J,I})$ versus $(\mathcal{S}_{J,I})$

We are now ready to study the final discrepancy between our pre–flag algebra $\tilde{\mathcal{F}}$ and the quantum flag algebra $\mathcal{F}_q$ of Taft and Towber.

Theorem 61. *Suppose $I, J \in \binom{[n]}{\|\gamma\|}$ and moreover $J \curvearrowright I$. Writing $|J| = r$, $|I| = s$, and $s - r = t$, we have*

$$\tilde{f}_J \tilde{f}_I = \sum_{\Lambda \subseteq I, |\Lambda| = r} (-q)^{\ell(\Lambda|I\setminus\Lambda)} \tilde{f}_{J|I\setminus\Lambda} \tilde{f}_\Lambda \,. \tag{6.14}$$

In other words, a weak version of the q-straightening relations hold for strictly quasideterminantal reasons. As the proof will make clear, the complimentary statement is stronger: the quantum flag algebra of Taft and Towber, with relations $(\mathcal{A}_I)$, $(\mathcal{Y}_{I,J})$, and $(\mathcal{S}_{J,I})$, implicitly satisfies the relations $(\mathcal{C}_{J,I})$.

In the sequel, it will be convenient to abbreviate the right-hand side of (6.4) (and its $u > 1$ versions, known to be true after the results of Section 6.3) by $Y_{I,J;(u)}$. Also, we will abbreviate the difference $(lhs - rhs)$ in (6.5) as $C_{J,I}$, and the difference $(lhs - rhs)$ in (6.14) as $S_{J,I}$.

Using a Muir's Law argument as in the proof of Proposition 24, any statement we say about the case $J \cap I = \emptyset$ may be immediately extended to the more general case. As in the proof there, the extension to the general case will only introduce coefficients like $\mathfrak{K}_{\mathfrak{r}}(M, I), \mathfrak{K}_{\mathfrak{r}}(L, K)$, etc. But these all equal 1 by Proposition 28. So we focus on the case $J \cap I = \emptyset$.[2]

The proof will proceed as a linear algebra argument, writing $S_{J,I}$ as a linear combination of relations of type $C_{J,I}$ and $Y_{L,K;(u)}$. Before we dive in, we define a new quantity $CS_{J,I}(\theta)$. In the first step below, we replace $\ell(\Lambda|I \setminus \Lambda)$ with $|I \setminus \Lambda||\Lambda| - \ell(I \setminus \Lambda|\Lambda)$ and $\ell(J|I \setminus \Lambda)$ with $|J||I \setminus \Lambda| - \ell(I \setminus \Lambda|J)$. In the last step below, we factor to make the quantity inside the parentheses look like a $Y_{L,K;(u)}$ expression.

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[2]The reader may also consult [30], where a quantum determinant version of Muir's Law is stated. There, it is proven using the quasideterminant version of Muir's Law, cf. Theorem 7 of this thesis.

$$\begin{aligned}
C_{J,I} - S_{J,I} &= -q^{|J''|-|J'|}\tilde{f}_I\tilde{f}_J + \left(\sum_{i_1\le\lambda_1<\cdots<\lambda_r\le i_s} (-q)^{\ell(\Lambda|I\setminus\Lambda)}\tilde{f}_{J|I\setminus\Lambda}\tilde{f}_\Lambda\right) \\
&= \sum_{\Lambda\subseteq I}(-q)^{|J'|t}(-q)^{-\ell(I\setminus\Lambda|\Lambda)}\tilde{f}_{J\cup(I\setminus\Lambda)}\tilde{f}_\Lambda - q^{|J''|-|J'|}\tilde{f}_I\tilde{f}_J \\
&= \sum_{\Lambda\subseteq I}(-q)^{|J'|t+|J''||J|}(-q)^{-\ell((J\cup I)\setminus\Lambda|\Lambda)}\tilde{f}_{(J\cup I)\setminus\Lambda}\tilde{f}_\Lambda - q^{|J''|-|J'|}\tilde{f}_I\tilde{f}_J \\
CS_{J,I}(\theta) &= (-q)^{|J'|t+|J''||J|}\left(\sum_{\Lambda\subseteq I}(-q)^{-\ell((J\cup I)\setminus\Lambda|\Lambda)}\tilde{f}_{(J\cup I)\setminus\Lambda}\tilde{f}_\Lambda - \theta\tilde{f}_I\tilde{f}_J\right).
\end{aligned}$$

We prove the theorem in steps:

Proposition 62. *Let I and J be two sets satisfying the conditions of Theorem 61. With $SC_{J,I}(\theta)$ and $Y_{L,K;(u)}$ as defined above, there are constants $\{\eta_K \mid \emptyset \subseteq K \subsetneq J\}$ such that*

$$SC_{J,I}(\theta) = \sum_{\emptyset\subseteq K\subsetneq J}\eta_K Y_{(I\cup J)\setminus K,K;(r-|K|)}$$

for some $\theta \in \mathbb{Z}[q,q^{-1}]$.

Proposition 63. *In the previous proposition, $\theta = (-q)^{-|J'|t-|J''||J|}q^{|J''|-|J'|}$.*

As $C_{J,I}$ and $Y_{(I\cup J)\setminus K,K}$ are zero for quasideterminantal reasons, i.e. zero in our pre–flag algebra, these two propositions and the calculation preceding them constitute a proof of Theorem 61.

Example. An example before the proof:

$$[1][234] = (-q)^{\ell(2|34)}[134][2] + (-q)^{\ell(3|24)}[124][3] + (-q)^{\ell(4|23)}[123][4].$$

$$\begin{aligned}
C_{234,1} - S_{234,1} &= -q^{-1}[234][1] + [134][2] - q^1[124][3] + q^2[123][4] \\
&= q^2\left([123][4] + q^{-1}[124][3] + q^{-2}[134][2] - q^{-3}[234][1]\right) \\
&= q^2 Y_{1234,\emptyset;(1)}.
\end{aligned}$$

Definition 42. Let $\mathfrak{X} = \{(A,B) \mid A\cup B = I\cup J, A\cap B = \emptyset, \text{ and } |B| = r\}$. Define V to be the vector space over F with basis $\{e_{A,B} \mid (A,B)\in\mathfrak{X}\}$.

There is an obvious F-linear map $\mu : V \to \tilde{\mathcal{F}}$, sending $e_{A,B}$ to $\tilde{f}_A\tilde{f}_B$. We will pull back the expressions $SC_{J,I}(\theta)$ and $Y_{I\cup J\setminus K,K;(r-|K|)}$ to suitable preimages in V and work there. We write

$$v^\theta := (-q)^{|J'|t+|J''||J|}\left(\sum_{\Lambda\subseteq I}(-q)^{-\ell((I\cup J)\setminus\Lambda|\Lambda)}e_{(I\cup J)\setminus\Lambda,\Lambda} - \theta e_{I,J}\right),$$

and

$$v^K := \sum_{\Lambda\subsetneq(I\cup J),|\Lambda|=r-|K|}(-q)^{-\ell((I\cup J\setminus K)\setminus\Lambda|\Lambda)}(-q)^{-\ell(\Lambda|K)}e_{(I\cup J)\setminus(K\cup\Lambda),K\cup\Lambda}$$

for each $\emptyset \subseteq K \subsetneq J$. Notice that $\mu(v^\theta) = SC_{J,I}(\theta)$ and $\mu(v^K) = Y_{(I\cup J)\setminus K,K}$.

Proposition 62 will be proven if we can show that $\tilde{v}^\theta = v^\theta(-q)^{-|J''||J|-|J'|t}$ is a linear combination of the v^K for some θ. To this end, we introduce a grading on our vector space.

Definition 43. For each $K \in \mathcal{P}J$, let $V_{(K)} = \operatorname{span}_F\{e_{A,B} \mid B\cap J = K\}$. Clearly, V is graded by the POset $\mathcal{P}J$, i.e., $V = \bigoplus_{K\in\mathcal{P}J}V_{(K)}$. For each $K \in \mathcal{P}J$, define the distinguished element e^K by

$$e^K = \sum_{\Lambda\subseteq I,|\Lambda|=r-|K|}(-q)^{-\ell((I\cup J)\setminus(K\cup\Lambda)|\Lambda)}(-q)^{-\ell(\Lambda|K)}e_{(I\cup J)\setminus(\Lambda\cup K),\Lambda\cup K}.$$

For any $v \in V$, write $(v)_{(K)}$ for the component of v in $V_{(K)}$, that is, $v = \sum_K (v)_{(K)}$.

Notice that $e^J = e_{I,J}$, and that

$$e^\emptyset = \sum_{\lambda\in I,|\Lambda|=|J|}(-q)^{-\ell((I\cup J)\setminus\Lambda|\Lambda)}e_{(I\cup J)\setminus\Lambda,\Lambda}$$

In other words, $\tilde{v}^\theta = e^\emptyset - \theta e^J$. A more remarkable fact is that the $v^{K'}$ may also be expressed in terms of the e^K.

Lemma. For each $K' \in \mathcal{P}J \setminus J$, there are constants $\alpha^K_{K'} \in F$ satisfying

$$v^{K'} = \sum_{K\in\mathcal{P}J}\alpha^K_{K'}e^K.$$

Remark. The notation is intended to be suggestive of the edge-weight function on $\Gamma(J;I)$. It so happens that $\alpha^K_{K'} = 0$ if $K' \not< K$ in the POset $\mathcal{P}J$, and $\alpha^K_{K'} \neq 0$ otherwise. This important feature will become clear in the proof below.

Proof of Lemma. Fixing a K', if $K \supsetneq K'$, we write $\hat{K} = K \setminus K'$. Similarly, we will write $\hat{\Lambda} = \Lambda \setminus J$. Studying $v^{K'}$, we see that

$$\begin{aligned}
v^{K'} &= \sum_{\substack{\Lambda \subseteq (I \cup J)\setminus K' \\ |\Lambda| = r - |K'|}} (-q)^{-\ell((I \cup J \setminus K')\setminus\Lambda|\Lambda)}(-q)^{-\ell(\Lambda|K')} e_{(I\cup J)\setminus(\Lambda\cup K'),\Lambda\cup K'} \\
&= \sum_{K \in \mathcal{P}J} (v^{K'})_{(K)} \\
&= \sum_{K\in\mathcal{P}J} \sum_{\substack{\lambda \subseteq (I\cup J)\setminus K' \\ \Lambda\cap J = \hat{K}}} (-q)^{-\ell((I\cup J)\setminus(\hat{\Lambda}\cup K)|\hat{\Lambda}\cup\hat{K})} \times \\
&\quad (-q)^{-\ell(\hat{\Lambda}\cup\hat{K}|K')} e_{(I\cup J)\setminus(\hat{\Lambda}\cup K),\hat{\Lambda}\cup K} \\
&= \sum_{K\in\mathcal{P}J} (-q)^{-\ell((I\setminus\hat{\Lambda})\cup(J\setminus K)|\hat{K})}(-q)^{-\ell(\hat{K}|K')} \times \\
&\quad \left(\sum_{\substack{\hat{\lambda}\subseteq I \\ |\hat{\Lambda}| = r - |K|}} (-q)^{-\ell((I\cup J)\setminus(\hat{\Lambda}\cup K)|\hat{\Lambda})}(-q)^{-\ell(\hat{\Lambda}|K')} e_{(I\cup J)\setminus(\hat{\Lambda}\cup K),\hat{\Lambda}\cup K} \right).
\end{aligned}$$

Why can we perform this last step? Because $J \curvearrowright I$, the expression $\ell(I \setminus \hat{\Lambda}|\hat{K})$ does not actually depend on $\hat{\Lambda}$, only on $|\hat{\Lambda}|$. Indeed, it equals $|I \setminus \hat{\Lambda}| \cdot |\hat{K} \cap J'|$. Multiplying and dividing by $(-q)^{-\ell(\hat{\Lambda}|\hat{K})}$, we rewrite this last expression as

$$\begin{aligned}
v^{K'} &= \sum_{K} (-q)^{-\ell((I\setminus\hat{\Lambda})\cup(J\setminus K)|\hat{K})}(-q)^{-\ell(\hat{K}|K')+\ell(\hat{\Lambda}|\hat{K})} \times \\
&\quad \left(\sum_{\hat{\lambda}\subseteq I, |\hat{\Lambda}| = r-|K|} (-q)^{-\ell((I\cup J)\setminus(\hat{\Lambda}\cup K)|\hat{\Lambda})}(-q)^{-\ell(\hat{\Lambda}|K)} e_{(I\cup J)\setminus(\hat{\Lambda}\cup K),\hat{\Lambda}\cup K} \right) \\
&= \sum_{K' \leq K} (-q)^{(2|J\setminus K| - |I|)|\hat{K}\cap J'| - \ell(J\setminus K|\hat{K}) - \ell(\hat{K}|K')} \left(e^K\right) \\
&= \sum_{K'\leq K} \alpha^K_{K'} e^K .
\end{aligned}$$

□

As the reader may suspect, this is precisely the same value given to $\alpha^K_{K'}$ in the previous subsection. In particular, notice that $(v^K)_{(K)} = 1 \cdot e^K$. This yields the important

Corollary 64. *For any $v^{K'}, v^K$ with $K' < K$ in the POset $\mathcal{P}J$, and for the same constants $\alpha^K_{K'}$ as defined above, we have*

$$(v^{K'} - \alpha^K_{K'} v^K)_{(K)} = 0.$$

Proof of Proposition 62. We will use this key fact to perform a certain Gaussian elimination on the "matrix" of v^K's. Table 6.2 on the POset $\mathcal{P}(\{1,5,6\})$ should make our intentions clear.

	$e^{\emptyset}$	e^1	e^5	e^6	e^{15}	e^{16}	e^{56}	e^{156}
v^{15}					1			α_{15}^{156}
v^{16}						1		α_{16}^{156}
v^{56}							1	α_{56}^{156}
v^1		1			α_1^{15}	α_1^{16}		α_1^{156}
v^5			1		α_5^{15}		α_5^{56}	α_5^{156}
v^6				1		α_6^{16}	α_6^{56}	α_6^{156}
$v^{\emptyset}$	1	$\alpha_{\emptyset}^1$	$\alpha_{\emptyset}^5$	$\alpha_{\emptyset}^6$	$\alpha_{\emptyset}^{15}$	$\alpha_{\emptyset}^{16}$	$\alpha_{\emptyset}^{56}$	$\alpha_{\emptyset}^{156}$

Table 6.2: Writing the vectors $v^{K'}$ in terms of the e^K.

Performing Gaussian elimination between the rows in the first two layers, we see that the new rows in the second layer—who began their life with $|J|+1$ nonzero entries—now have exactly two nonzero entries:

$$\begin{aligned} (v^{J\setminus j_1\setminus j_2})' &= v^{J\setminus j_1\setminus j_2} - \alpha_{J\setminus j_1\setminus j_2}^{J\setminus j_1} v^{J\setminus j_1} - \alpha_{J\setminus j_1\setminus j_2}^{J\setminus j_2} v^{J\setminus j_2} \\ &= e^{J\setminus j_1\setminus j_2} + \left(\alpha_{J\setminus j_1\setminus j_2}^{J} - \alpha_{J\setminus j_1\setminus j_2}^{J\setminus j_1}\alpha_{J\setminus j_1}^{J} - \alpha_{J\setminus j_1\setminus j_2}^{J\setminus j_2}\alpha_{J\setminus j_2}^{J}\right) e^J \end{aligned}$$

Marching down the layers of this matrix one-by-one, we see that the new final row is given by $(v^{\emptyset})' = e^{\emptyset} + \theta e^J = \tilde{v}^{\theta}$ for some θ. $\square$

Proof of Proposition 63. Careful bookkeeping shows that

$$\begin{aligned} \theta &= \alpha_{\emptyset}^J - \left(\sum_{\emptyset\subsetneq K\subsetneq J} \alpha_{\emptyset}^K \alpha_K^J\right) + \left(\sum_{\emptyset\subsetneq K_1\subsetneq K_2\subsetneq J} \alpha_{\emptyset}^{K_1}\alpha_{K_1}^{K_2}\alpha_{K_2}^J\right) - \cdots \\ &\quad \cdots + (-1)^{|J|-1}\left(\sum_{\emptyset\subsetneq K_1\subsetneq\cdots\subsetneq K_{|J|-1}\subsetneq J} \alpha_{\emptyset}^{K_1}\alpha_{K_1}^{K_2}\cdots\alpha_{K_{r-1}}^J\right). \end{aligned} \tag{6.15}$$

In other words, θ is a signed sum of path weights $\alpha(\pi)$, π running over all paths in $\mathfrak{P}$ save for $\hat{1}$. As the sign attached to π is the same as the length of π, and as the bijection φ from subsection 6.4.1 increases length by one but preserves path weight, we immediately conclude

$$\begin{aligned}
\theta &= (-1)^{|J|-1}\alpha(\pi^{\hat{1}}) \\
&= (-1)^{|J|-1}(-q)^{|J'|(|J'|-1)-|J''|(|J''|-1)} \times \alpha_{\emptyset}^{J} \\
&= (-1)^{|J|-1}(-q)^{|J''|-|J'|}(-q)^{|J'||J'|-|J''||J''|-|I||J'|} \\
&= q^{|J''|-|J'|}(-q)^{|J'||J'|-|J''||J''|-|J''||J'|-|I||J'|-(|J'|+|J''|+t)|J'|+|J''||J'|} \\
&= q^{|J''|-|J'|}(-q)^{-|J'|t-|J''||J|},
\end{aligned}$$

as needed. ☐

With Proposition 63 proven, the main theorem is finally demonstrated.

6.4.3 Missing relations

Let us analyze the proof above when $\gamma = (d, n-d)$. Recall that for Grassmannians, the right-hand side of $(\mathcal{S}_{J,I})$ is simply $f_J f_I$, same as the left-hand side. In this case, $SC_{J,I}$ above becomes $g_J g_I - q^{|J''|-|J'|} g_I g_J$, i.e. the q-commuting relations already known to hold in the pre–flag algebra. However, what the proof says is that $C_{J,I} = SC_{J,I}$ is a linear combination of the Young symmetry relations $Y_{I\cup J\setminus K,K}$. In particular, the q-commuting property of quantum Plücker coordinates, made explicit in the definition of $\tilde{\mathcal{G}}(d,n)$, implicitly holds within the algebras $\mathcal{G}_q(d,n)$ of Taft and Towber.

Example. Two further examples after the fact:

1. $[156][234] = q^{2-1}[234][156]$:

$$\begin{aligned}
C_{156,234} &= q^6 Y_{123456,\emptyset} \\
&\quad + q^5 Y_{23456,1} + q^5 Y_{12346,5} - q^6 Y_{12345,6} \\
&\quad - q^3 Y_{2346,15} + q^4 Y_{2345,16} + q^4 Y_{1234,56}.
\end{aligned}$$

2. $[134][156] = q^{0-2}[156][134]$:

$$\begin{aligned}
C_{134,156} &= q^2 Y_{13456,1} \\
&\quad - q^0 Y_{1456,13} + q^1 Y_{1356,14}.
\end{aligned}$$

The same argument shows that, in $\mathcal{F}_q$, $(\mathcal{C}_{J,I})$ is a consequence of $(\mathcal{Y}_{I,J})$ and $(\mathcal{S}_{J,I} \mid J \curvearrowright I)$. In particular, the quantum flag algebras $\mathcal{F}_q(\gamma)$ are quotients of the pre–flag algebras $\tilde{\mathcal{F}}(\gamma)$. It would seem there are not relations missing from $\mathcal{F}_q$ after all, rather there are relations missing from $\tilde{\mathcal{F}}$.

Chapter 7

Noncommutative Flags Algebras

7.1 The "pre" Prefix

In Chapter 6, we have seen that our pre–flag algebra $\tilde{\mathcal{F}}(\gamma)$ needn't be the true homogeneous coordinate algebra for the flags in any particular noncommutative setting. Still, it is clearly a good starting point in the task of building that algebra in any (amenable) noncommutative setting of interest; especially in light of Theorem 57. Here, we give another compelling result: as defined, the pre–flag algebra is good enough to completely capture the complex flags' coordinate functions.

Theorem 65. *The pre–flag algebra $\tilde{\mathcal{F}}(\gamma)$ for the classic flag variety $F\ell(\gamma)$ over $\mathbb{C}$ is isomorphic to $\mathcal{F}(\gamma)$, the ring of homogeneous coordinate functions introduced in Chapter 3.*

Proof. The algebra $\mathcal{F}(\gamma)$ is the *commutative* algebra with generators $\left\{f_I \mid I \in \binom{[n]}{\|\gamma\|}\right\}$ and relations given by

$$(\forall I, J, u : |I| - u, |J| + u \in \|\gamma\|) \quad 0 = \sum_{\substack{\Lambda \in I \setminus J \\ |\Lambda| = u}} (-1)^{\ell(I\setminus\Lambda|\Lambda)}(-1)^{\ell(\Lambda|J)} f_{I\setminus\Lambda} f_{\Lambda\cup J}\,.$$

For those terms $f_{I\setminus\lambda} f_{\lambda\cup J}$ appearing above satisfying $\{\lambda\} \cap J \neq \emptyset$, we understand $f_{\lambda\cup J}$ as zero.

Turning to the pre–flag algebra, first note that in the present setting $\mathfrak{K}_{\mathfrak{r}} = \mathfrak{K}_{\mathfrak{x}} = 1$ while $\mathfrak{I}_{\mathfrak{r}}(a, A^a) = \mathfrak{I}_{\mathfrak{x}}(a, A^a) = (-1)^{\ell(a|A^a)}$. In particular, the left and right pre–flag algebras are isomorphic, as we expect over $\mathbb{C}$. We focus on the right flags. $\tilde{\mathcal{F}}(\gamma)$ is the *noncommutative* algebra with generators $\left\{\tilde{f}_I \mid I \in \binom{[n]}{\|\gamma\|}\right\}$ and relations given by

$$(\forall I, J : |I| - 1, |J| + 1 \in \|\gamma\|) \quad 0 = \sum_{\lambda \in I\setminus J} (-1)^{\ell(I\setminus\lambda|\lambda)}(-1)^{\ell(\lambda|J)} \tilde{f}_{I\setminus\lambda} \tilde{f}_{\lambda\cup J}\,,$$

and by

$$(\text{when } J \text{ can't distinguish } I) \quad \tilde{f}_{JM}\tilde{f}_{IM} = \tilde{f}_{IM}\tilde{f}_{JM}\,.$$

The first set of relations is identical after the comments preceding Proposition 55. We only have the problem that $\mathcal{F}(\gamma)$ is a commutative algebra while $\tilde{\mathcal{F}}(\gamma)$ is a noncommutative algebra. However, as $\rho_j(i) = -\frac{\mathfrak{I}_{\mathfrak{r}}(i,j)}{\mathfrak{I}_{\mathfrak{r}}(j,i)\mathfrak{K}_{\mathfrak{r}}(i,j)} = -(-1)^{\pm 1}$ for all $i, j \in [n]$, this last set of relations really reads

$$(\forall I, J : |I|, |J| \in \|\gamma\|) \quad \tilde{f}_J\tilde{f}_I = \tilde{f}_I\tilde{f}_J,$$

i.e. all generators are central. □

In the next section, we strengthen our case for the study of the pre–flag algebra by considering a construction of Frenkel and Jardim. The final section introduces a related algebra more closely aligned with the noncommutative coordinate algebra introduced in Section 3.4.

7.2 Two "pre" Examples

Here we summarize how quasideterminants and the pre–Grassmannian algebra can be used to describe a construction of Frenkel and Jardim [13]. The construction arose from a new attempt to build *quantum instantons.*[1]

7.2.1 A quantum Grassmannian

Definition 44. The *quantum compactified complexified Minkowski space* $\mathfrak{M}_{p,q}$ is the graded $\mathbb{C}$-algebra generated by $z_{11'}, z_{12'}, z_{21'}, z_{22'}, D, D'$ satisfying the relations (7.1) to

[1]The reader will forgive our not defining this term. It is not important to the result presented here. Worse, including the requisite background from [40, 2, 13] and the citations therein would take us too far off course. For a little more background (not much!), the reader may consult Chapter 8.

(7.5) below.

$$\begin{aligned} z_{11'}z_{12'} &= z_{12'}z_{11'} \\ z_{11'}z_{21'} &= z_{21'}z_{11'} \\ z_{12'}z_{22'} &= z_{22'}z_{12'} \\ z_{21'}z_{22'} &= z_{22'}z_{21'} \\ z_{12'}z_{21'} &= z_{21'}z_{12'} \end{aligned} \tag{7.1}$$

$$q^{-1}(z_{11'}z_{22'} - z_{12'}z_{21'}) = q(z_{22'}z_{11'} - z_{12'}z_{21'}) \tag{7.2}$$

$$\begin{aligned} Dz_{11'} &= pq^{-1}z_{11'}D & D'z_{11'} &= p^{-1}q^{-1}z_{11'}D' \\ Dz_{12'} &= pq^{-1}z_{12'}D & D'z_{12'} &= p^{-1}qz_{12'}D' \\ Dz_{21'} &= pqz_{21'}D & D'z_{21'} &= p^{-1}q^{-1}z_{21'}D' \\ Dz_{22'} &= pqz_{22'}D & D'z_{22'} &= p^{-1}qz_{22'}D' \end{aligned} \tag{7.3}$$

$$p^{-1}DD' = pD'D \tag{7.4}$$

$$q^{-1}(z_{11'}z_{22'} - z_{12'}z_{21'}) = p^{-1}DD' \tag{7.5}$$

Relations (7.1)-(7.4) are commutation relations, while (7.5) plays the role of the quadric that defines $Gr_{\mathbb{C}}(2,4)$ as a subvariety of $\mathbb{P}^5$. In other words, the algebra $\mathfrak{M}_{p,q}$ can be regarded as a quantum Grassmannian.

Remark. In [13], it is stated that the relations (7.1)-(7.5) may be expressed in R-matrix form. This is not easy to see, and indeed it wouldn't look like the RTT-algebra construction above (e.g. because 6, the number of generators here, is not n^2 for any integer n). The details will not be important to us, and the interested reader is urged to consult [13]. Briefly:

- Beginning with the standard one-parameter 2×2 quantum matrix $T = (t_{ij})$, introduce a formal noncommuting parameter δ with the formula:

$$\mathrm{diag}(\delta,\delta) \cdot T \cdot \mathrm{diag}(q^{\frac{1}{4}}, q^{-\frac{1}{4}})^2 = \mathrm{diag}(q^{\frac{1}{4}}, q^{-\frac{1}{4}})^2 \cdot T \cdot \mathrm{diag}(\delta,\delta).$$

- Label the diagonal matrices above Δ and Q; put $X = Q^{-1}\Delta T Q$ and $Y = Q\Delta^{-1}TQ^{-1}$.

- The relations on X and Y are given in terms of an R-matrix because the relations on T are. For example (viewing X and R as acting via left-multiplication), the R-matrix

$$\begin{pmatrix} p^{-1} & & & \\ & q & p^{-1}-q & \\ & p^{-1}-q^{-1} & q^{-1} & \\ & & & p^{-1} \end{pmatrix},$$

  and the identity $R_{23}X_{12}X_{13} = X_{13}X_{12}R_{23}$ reproduce the relations for X.

- View the $x_{ij'}$, $y_{ij'}$ as coordinates in two "affine patches" $(1 : x_{ij'} : D\delta)$ and $(\frac{\delta}{D'} : y_{ij'} : 1)$ in a quantum $\mathbb{P}^5$. Compactify this picture by introducing variables $z_{ij'}$ and demanding

$$x_{ij'} = \frac{z_{ij'}}{D} \qquad \text{and} \qquad y_{ij'} = \frac{z_{ij'}}{D'}$$

- Conclude that Z will have R-matrix type relations because X and Y do.

In a moment, we will see a more straightforward way to give $\mathfrak{M}_{p,q}$ an R-matrix structure. First, replace the indices $(1, 2, 1', 2')$ with $(1, 2, 3, 4)$, and write $D = z_{12}, D' = z_{34}$. Taking $p = q$, we may define $\mathfrak{M}_q$ as follows.

Definition 45. The quantum Grassmannian $\mathfrak{M}_q$ is the graded $\mathbb{C}$-algebra generated by $z_{ij} \quad (1 \le i < j \le 4)$ with relations:

$$\begin{array}{rclcrcl} z_{13}z_{12} &=& z_{12}z_{12} & \qquad & z_{34}z_{13} &=& q^{-2}z_{13}z_{34} \\ z_{14}z_{12} &=& z_{12}z_{14} & & z_{34}z_{14} &=& z_{14}z_{34} \\ z_{23}z_{12} &=& q^{-2}z_{12}z_{23} & & z_{34}z_{23} &=& q^{-2}z_{23}z_{34} \\ z_{24}z_{12} &=& q^{-2}z_{12}z_{24} & & z_{34}z_{24} &=& z_{24}z_{34} \end{array} \tag{7.6}$$

$$z_{34}z_{12} = q^{-2}z_{12}z_{34} \tag{7.7}$$

$$\begin{aligned} z_{14}z_{13} &= z_{13}z_{14} \\ z_{23}z_{13} &= z_{13}z_{23} \\ z_{24}z_{14} &= z_{14}z_{24} \\ z_{23}z_{14} &= z_{14}z_{23} \\ z_{24}z_{23} &= z_{23}z_{24} \end{aligned} \tag{7.8}$$

$$z_{24}z_{13} = q^{-2}z_{13}z_{24} + (1-q^{-2})z_{14}z_{23} \tag{7.9}$$

$$z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0 \tag{7.10}$$

Here relations (7.6)–(7.9) are presented in a form conducive to using Bergman's Diamond Lemma, while relation (7.10) is precisely the classical Plücker relation (Young symmetry relation) for $Gr_{\mathbb{C}}(2,4)$.

It is worth mentioning that this is a *new* quantum Grassmannian in that it is not isomorphic to the quantum Grassmannian of Taft and Towber. We give a proof of this and several related facts in Chapter 8.

7.2.2 An R-matrix realization

We want to view all six symbols z_{ij} as being 2×2 minors of a 2×4 matrix $T=(t_{ij})$. To that end, let $\mathcal{A}^{\mathrm{I}}(n)$ be the $\mathbb{C}(q)$-algebra given in Section 5.6 with generators $\{t_{ij}\}_{1\le i,j\le n}$ and relations given by the R-matrix

$$R = q^{-2}\sum_{i=1}^{n}\mathbb{E}_{ii}\otimes\mathbb{E}_{ii} + \sum_{i<j}\left(\mathbb{E}_{jj}\otimes\mathbb{E}_{ii} + q^{-2}\mathbb{E}_{ii}\otimes\mathbb{E}_{jj} + (q^{-2}-1)\mathbb{E}_{ij}\otimes\mathbb{E}_{ji}\right).$$

Recall $\operatorname{Det} T_{I,J} = \sum_{\pi\in\mathfrak{S}_d}(-1)^{\ell(\pi)}t_{i_1j_{\pi1}}\cdots t_{i_dj_{\pi d}}$ for $I,J\in\binom{[n]}{d}$. In this setting, cf. Proposition 36, we deduce

Theorem 66. *The left pre–Grassmannian algebra $\tilde{\mathcal{G}}(2,4)$ associated to $\mathcal{A}^{\mathrm{I}}(4)$ is the $\mathbb{C}$-algebra with generators $\left\{\tilde{f}_{ij}\mid\{i,j\}\in\binom{[4]}{2}\right\}$ and relations given by (7.6)–(7.10) (the z's being replaced by $\tilde{f}$'s). That is, $\tilde{\mathcal{G}}(2,4)\simeq\mathfrak{M}_q(2,4)$.*

Proof. We leave it to the reader to verify the majority of the relations. Below we spell out two q-commuting relations and two Young symmetry relations.

(q-Commuting Relations): In the setting $\mathcal{A}^{\mathrm{I}}(n)$, for all $I, J \in \binom{[n]}{d}$, J can't distinguish I (as columns) iff $J \curvearrowright I$. In this case, writing $M = I \cap J, \hat{J} = J \setminus M$, and $\hat{I} = I \setminus M$, (5.31) becomes

$$[T_{[d],J}][T_{[d],I}] = (q^2)^{|\hat{J}''|(1-|\hat{I}|)+\binom{|\hat{J}|}{2}+\ell(M|\hat{J})-\ell(M|\hat{I})}[T_{[d],I}][T_{[d],J}] \,.$$

In particular, when $n = 4, d = 2, I = \{1, 2\}, J = \{2, 3\}$, we have

$$\tilde{f}_{23}\tilde{f}_{12} = q^{-2}\tilde{f}_{12}\tilde{f}_{23} \,,$$

and when $I = \{3, 4\}, J = \{1, 2\}$, we have

$$\tilde{f}_{34}\tilde{f}_{12} = q^{-2}\tilde{f}_{12}\tilde{f}_{34} \,.$$

This accounts for (7.6.c) and (7.7).

(Young Symmetry Relations): Recall from Proposition 36 that in the setting $\mathcal{A}^{\mathrm{I}}(n)$, the (column) Young symmetry relations take the form

$$0 = \sum_{k \in K \setminus M} (-1)^{\ell(M|k)+\ell(k|K^k)}(q^2)^{-\ell(M|k)}[T_{[d],M\cup k}][T_{[d],K\setminus k}] \,.$$

In particular, when $n = 4, d = 2, K = \{2, 3, 4\}, M = \{1\}$, we have

$$\tilde{f}_{12}\tilde{f}_{34} - \tilde{f}_{13}\tilde{f}_{24} + \tilde{f}_{14}\tilde{f}_{23} = 0.$$

This accounts for (7.10). Before continuing, let us rewrite (7.9) with (7.10) and (7.7) to get:

$$-z_{14}z_{23} + z_{24}z_{13} - z_{34}z_{12} = 0.$$

Compare this to the Young symmetry relation with $n = 4, d = 2, K = \{1, 2, 3\}, M = \{4\}$:

$$-q^{-2}\tilde{f}_{14}\tilde{f}_{23} + q^{-2}\tilde{f}_{24}\tilde{f}_{13} - q^{-2}\tilde{f}_{34}\tilde{f}_{12} = 0.$$

$\square$

7.2.3 A non-R-matrix realization

We have demonstrated the Grassmannian of Frenkel and Jardim as the Grassmannian associated to an RTT-algebra. Interestingly, it is also the Grassmannian for an algebra that cannot be described by an R-matrix, the algebra $\mathcal{A}^{\mathrm{II}}(n)$ of Section 5.8. The proof is the same as the one outlined above, resting on *coincidentally identical* q-commuting and Young symmetry relations in the $Gr(2,4)$ case. In the next section, the two settings will diverge somewhat.

Proposition 67. *Equations (5.30) & (5.31) coincide with (5.46) & (5.47) when* $n = 4$ *and* $\gamma = (2,2)$. □

7.3 Affine coordinate rings

We begin this section with an elementary result on noncommutative localization.

Definition 46. Fix a ring A and a subset $X \subseteq A$. A ring homomorphism $\alpha : A \to B$ is called X-inverting if $\alpha(x)$ has a two-sided inverse for all $x \in X$.

Proposition 68. *For any* A *and* X *as above, there is a unique ring* A_X *and a ring map* $\varepsilon : A \to A_X$ *satisfying the following universal property:*

$$\begin{tikzcd} A_X \arrow[r, "\exists!\lambda"] & B \\ A \arrow[u, "\varepsilon"] \arrow[ur, "\forall\alpha : X\text{-inverting}"'] & \end{tikzcd}$$

That is, if $\alpha : A \to B$ *is an* X*-inverting ring homomorphism, then there is a unique map* λ *so that* $\alpha = \lambda\varepsilon$ *(as ring maps).*

The proof may be found in any text discussing noncommutative localization, cf. [8, 31]. One simply starts with A, adjoins formal noncommuting variables x' for each $x \in X$ and then adds the relations $xx' = x'x = 1$ to this new ring.

For us the method of construction of A_X is just as important as its existence. From it we deduce:

- If X generates A, then X, together with X^{-1}, generates A_X.
- If there is an X-inverting map α which is injective on X, that is, $(\forall x, x' \in X)\, \alpha(x - x') = 0 \Rightarrow x = x'$, then ε must be injective.

This is the situation for our pre–flag algebras $\tilde{\mathcal{F}}(\gamma)$ below.

7.3.1 GL_d invariance

For the construction of our quantized flags, we explicitly demanded that we be able to invert $\operatorname{Det} T_{I,J}$ for (at least a very large number) $I, J \in \mathcal{P}[n],\ |I| = |J|$. We discovered in $\mathcal{T}(n)$ some relations among the expressions $\{\operatorname{Det} T_{I,[d]} : |I| = d\}$, and based on these, we defined pre–flag algebras associated to $\mathcal{T}(n)$ (as F-algebras on generators $X = \left\{\tilde{f}_I\right\}$ with the same relations). In short, for each γ, there is an algebra homomorphism $\alpha : \tilde{\mathcal{F}}(\gamma) \to \mathcal{T}(n)$ that is both X-inverting and X-injective.

We conclude that for noncommutative settings giving rise to amenable determinants, there is an algebra $\bar{\mathcal{F}}$ associated to $\tilde{\mathcal{F}}$ (namely, $\tilde{\mathcal{F}}_X$) with $\binom{n}{\|\gamma\|}$ distinct, invertible generators $\left\{\bar{f}_I \mid I \in \binom{[n]}{\|\gamma\|}\right\}$. While ε may not be injective, we may still treat $\bar{\mathcal{F}}$ as a nice localization of $\tilde{\mathcal{F}}$ since the image $\varepsilon(\tilde{\mathcal{F}})$ generates $\bar{\mathcal{F}}$. We focus on $\bar{\mathcal{F}}$ for the remainder of the section.

Proposition 69. *The ratios $\bar{f}_I/\bar{f}_J$ with $|I| = |J| = d \in \|\gamma\|$ (dividing on the left or right in accordance with viewing the generators as left or right coordinate functions) may be viewed as functions on the "q-generic" points of $F\ell(\gamma)$.*

Remark. We have identified $\bar{f}_I$ with the coordinate function $[T_I]$ on $F\ell(\gamma)$ (taken to mean $[T_{A,[d]}]$ if we are considering right flags and to mean $[T_{[d],A}]$ if we are considering left flags). Note that in the commutative case, and for $g \in \mathrm{P}_\gamma^+$, $f(T_I \cdot g) = f(T_I) \det g'$ where g' is the $|I| \times |I|$ upper-left block of g. Evidently the dependence on g drops out in the ratio of two such homogeneous coordinate functions, and as a result we have a legitimate function on $F\ell(\gamma)$ (or at least on the affine patch defined by the non-vanishing of the coordinate $[T_J]$. For noncommutative determinants, this g-intertwining property need not hold.

Proof. The idea is to replace $\big[I\big]\big[J\big]^{-1}$ by a product of ratios that look more like quasideterminants.

Writing $I = \{i_1 < i_2 < \cdots < i_d\}$ and J similarly, construct from (I, J) the longer sequence of subsets $(I = A_0, A_1, A_2, \ldots, A_{d-1}, A_d = J)$ by taking $A_{t+1} = A_t \setminus i_{d-t} \cup j_{d-t}$.

Example. The idea is to move from I to J one index at a time. Here are two examples that should be illuminating.

$$(123, 456) \rightsquigarrow (123, 126, 156, 456) \text{ and } (126, 346) \rightsquigarrow (126, 126, 146, 346).$$

We have assumed in $\mathcal{T}(n)$ the existence of $[T_A]^{-1}$ for all $A \in \binom{[n]}{\|\gamma\|}$, in particular for all A_i appearing in the sequence above. Let us stick to right flags for the remainder of the proof, and abbreviate, e.g., $[T_{A,[d]}]$ as $\big[A\big]$. Now we may view $\bar{f}_I/\bar{f}_J$ as the function returning $\big[I\big]\big[J\big]^{-1}$, or

$$(\big[A_0\big]\big[A_1\big]^{-1})(\big[A_1\big]\big[A_2\big]^{-1})\big[A_2\big] \;\cdots\; \big[A_{d-1}\big]^{-1}(\big[A_{d-1}\big]\big[A_d\big]^{-1}).$$

By first properties of adequate determinants, this last expression may be viewed, up to some coefficients $\mathfrak{I}_*, \mathfrak{K}_* \in F \setminus \{0\}$, as ratios of quasideterminants of the form

$$|T_{A_t,[d]}|_{i_{d-t}s}|T_{A_{t+1},[d]}|^{-1}_{j_{d-t}s}\,,$$

each of which are GL_n invariant by Proposition 13. □

7.3.2 Dehomogenization & Function Fields

Fix a composition $\gamma \models n$ with $\ell(\gamma) = r$. We follow the commutative case—also Kelley-Lenagan-Rigal [29]—and introduce a means for defining affine patches on quantized flags.

Definition 47. Say a chain of subsets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_r = [n]$ *has characteristic* γ if $|A_i| = \|\gamma\|_i = \sum_{j\leq i} \gamma_j$ for all $1 \leq i < r$. A sequence of sets $(A_1, A_2, \ldots, A_{r-1})$ shall be called an *affine patch* if the sequence $A_1 \subsetneq \cdots \subsetneq A_{r-1} \subsetneq [n]$ describes a chain of characteristic γ.

The previous result suggests that there are more appropriate rings associated to $F\ell(\gamma)$ than $\tilde{\mathcal{F}}(\gamma)$. Given an affine patch $\pi = (A_1, \ldots, A_{r-1})$, we define the *dehomogenization of* $F\ell(\gamma)$ *at* π to be the subring $\mathrm{Dhom}(\gamma, \pi)$ of the localization $\bar{\mathcal{F}}(\gamma)$ generated by the ratios $\{\tilde{f}_I/\tilde{f}_{A_d} : |I| = |A_d| = d\}$. This should be viewed as a piece of the function field $\mathcal{K}(\gamma)$ associated to $F\ell(\gamma)$—to whatever extent this object even exists.[2]

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[2] If $\mathcal{T}(\backslash)$ is a skew-field, a natural place to look for $\mathcal{K}(\gamma)$ is as a sub–skew field of $\mathcal{T}(n)$. However, $\mathcal{T}(n)$ needn't be embeddable in a skew field, even when $\mathcal{T}(n)$ is a domain. If this is the case, we may be out of luck when trying to construct $\mathcal{K}(\gamma)$. For more on this difficult problem of embedding domains into skew fields, the reader is directed to the discussions in [31] and [8].

Remark. An element x in a ring R is called *normal* if $xR = Rx$. A word or two about the normal elements of $\tilde{\mathcal{F}}$ is in order. If J can't distinguish A for all $J \in \binom{[n]}{\|\gamma\|}$, then f_A is a normal element. For in this case, Theorem 21 implies that f_A "q-commutes" with X, and hence is q-central in $\tilde{\mathcal{F}}$. In all of the examples we have addressed, the property $J \curvearrowright A$ is sufficient to guarantee that f_A q-commutes with f_J. In particular, $\{f_{[\|\gamma\|_1]}, f_{[\|\gamma\|_2]}, \ldots, f_{\|\gamma\|_{r-1}}\}$ is a collection of normal elements in $\tilde{\mathcal{F}}$.

Kelley, Lenagan, and Rigal [29] study the case of quantum Grassmannians and dehomogenizations at affine patches π whose associated generators f_{A_i} are normal elements. As observed in the previous paragraph, such patches exist for Grassmannians in all of the examples considered in Chapter 5. Many of the results in [29] should go through in these other quantized settings.

Normal or no, we may piece together the numerous dehomogenizations of $F\ell(\gamma)$ and view the result as a substantial piece of the field of functions on $F\ell(\gamma)$—again, to whatever extent the latter even exists.

Definition 48. The *ring of functions* on $F\ell(\gamma)$ is the subalgebra $\bar{\mathcal{K}}(\gamma)$ of $\bar{\mathcal{F}}(\gamma)$ generated by $\mathrm{Dhom}(\gamma, \pi)$, π running over all affine patches of γ.

The next result shows that this definition fits into the quasideterminant picture very well.

Proposition 70. *There is an F-algebra homomorphism φ from the ring $\mathcal{Q}(\gamma)$ of quasi-Plücker coordinates to the ring $\bar{\mathcal{K}}(\gamma)$ of functions on $F\ell(\gamma)$ in any amenable setting.*

Proof. We build the map from $\mathcal{Q}(\gamma)$ to $\bar{\mathcal{F}}(\gamma)$, then show the image lies in $\bar{\mathcal{K}}(\gamma)$. The proof amounts to a review of Chapters 3 and 4. Recall that by construction, $\bar{\mathcal{F}}(\gamma)$ satisfies every relation in the generators $\bar{f}$ that $\tilde{\mathcal{F}}(\gamma)$ satisfies in the generators $\tilde{f}$.

As a reminder, $\mathcal{Q}(\gamma)$ (for right-flags) is the F-algebra generated by symbols $\{r_{ij}^K \mid i, j \in [n],\ K \in \mathcal{P}[n],\ j \notin K,\ |K| + 1 \in \binom{[n]}{\|\gamma\|}\}$ subject to the relations:

- The *idempotent* relations ($\mathcal{I}_{i,j,M}$):

$$r_{ij}^M = \begin{cases} 0 & \text{if } i \in M \\ 1 & \text{if } i = j,\ i \notin M \end{cases} \tag{7.11}$$

- The *cancellation* relations ($\mathcal{C}_{i,j,k,M}$):

$$r_{ij}^{M} r_{jk}^{M} = r_{ik}^{M} \quad (j, k \notin M) \tag{7.12}$$

- The *skew-symmetry* relations ($\mathcal{S}_{i,j,k,M}$):

$$r_{ij}^{k\cup M} r_{jk}^{i\cup M} r_{ki}^{j\cup M} = -1 \quad (i, j, k \notin M) \tag{7.13}$$

- The *quasi-Plücker* relations ($\mathcal{P}_{i,L,M}$):

$$\sum_{j\in L} r_{ij}^{L\backslash j} r_{ji}^{M} = 1 \quad (|M| \leq |L| - 1,\ i \notin M) \tag{7.14}$$

Below, we will only check the skew-symmetry relation. Equation (4.5) informs us what the map should be:

$$\varphi(r_{ij}^{M}) = \frac{\mathfrak{I}_{\mathfrak{r}}(i, M)}{\mathfrak{I}_{\mathfrak{r}}(j, M)} \bar{f}_{i\cup M} \bar{f}_{j\cup M}^{-1} .$$

We must check that

$$\frac{\mathfrak{I}_{\mathfrak{r}}(i, kM)}{\mathfrak{I}_{\mathfrak{r}}(j, kM)} \frac{\mathfrak{I}_{\mathfrak{r}}(j, iM)}{\mathfrak{I}_{\mathfrak{r}}(k, iM)} \frac{\mathfrak{I}_{\mathfrak{r}}(k, jM)}{\mathfrak{I}_{\mathfrak{r}}(i, jM)} \bar{f}_{ikM} \bar{f}_{jkM}^{-1} \bar{f}_{ijM} \bar{f}_{ikM}^{-1} \bar{f}_{jkM} \bar{f}_{ijM}^{-1} = -1.$$

Call the coefficient appearing above $(\circlearrowleft_{i\,j}^{k})$. Then by the weak q-commuting property, this reduces to showing

$$\begin{aligned} -1 \quad &= \quad (-1) \frac{\mathfrak{I}_{\mathfrak{r}}(i, j) \mathfrak{K}_{\mathfrak{r}}(j, ikM)}{\mathfrak{I}_{\mathfrak{r}}(j, i) \mathfrak{K}_{\mathfrak{r}}(i, jkM)} \times \\ &\qquad (\circlearrowleft_{i\,j}^{k}) \bar{f}_{jkM}^{-1} \bar{f}_{ikM} \bar{f}_{ijM} \bar{f}_{ikM}^{-1} \bar{f}_{jkM} \bar{f}_{ijM}^{-1} . \end{aligned}$$

or

$$\begin{aligned} -1 \quad &= \quad -\frac{\mathfrak{I}_{\mathfrak{r}}(i, j) \mathfrak{K}_{\mathfrak{r}}(j, ikM)}{\mathfrak{I}_{\mathfrak{r}}(j, i) \mathfrak{K}_{\mathfrak{r}}(i, jkM)} \frac{\mathfrak{I}_{\mathfrak{r}}(j, k) \mathfrak{K}_{\mathfrak{r}}(k, ijM)}{\mathfrak{I}_{\mathfrak{r}}(k, j) \mathfrak{K}_{\mathfrak{r}}(j, ikM)} \frac{\mathfrak{I}_{\mathfrak{r}}(k, i) \mathfrak{K}_{\mathfrak{r}}(i, jkM)}{\mathfrak{I}_{\mathfrak{r}}(i, k) \mathfrak{K}_{\mathfrak{r}}(k, ijM)} \times \\ &\qquad (\circlearrowleft_{i\,j}^{k}) \bar{f}_{jkM}^{-1} \bar{f}_{ikM} \bar{f}_{ikM}^{-1} \bar{f}_{ijM} \bar{f}_{ijM}^{-1} \bar{f}_{jkM} . \end{aligned}$$

Now it is just a matter of checking that the introduced constants cancel $(\circlearrowleft_{i\,j}^{k})$, which occurs because $\mathfrak{I}_{\mathfrak{r}}, \mathfrak{K}_{\mathfrak{r}}$ are measuring functions. $\square$

7.4 The ring of quasi-Plücker coordinates

Proposition 70, coupled with the discussion preceding Proposition 69, indicates that a closer study of $\mathcal{Q}(\gamma)$ is merited. In this section, we begin this study by exhibiting a basis for $\mathcal{Q}(\gamma)$. We work over $\mathbb{Q}$.

Definition 49. An *inadmissible* word of length two is a word of the form

$$r_{ij}^{M_1} r_{jk}^{M_2} \quad \text{with } |M_1| \geq |M_2| \text{ and } j > \max\{M_1 \setminus (M_2 \cup k)\}\,.$$

A word in the symbols r_{ij}^M is *inadmissible* if it contains an inadmissible subword of length two. A word in the symbols r_{ij}^M is *admissible* if it contains no inadmissible subwords.

Theorem 71. *A basis for $\mathcal{Q}(\gamma)$ as a vector space over $\mathbb{Q}$ is given by all admissible words.*

The proof uses Bergman's Diamond Lemma. Before diving in, we prove an identity in $\mathcal{Q}(\gamma)$ that we will include in our reduction system in place of $(\mathcal{P}_{i,L,M})$.

Lemma. Suppose the integers $i, j \in [n]$ and sets $M, L \in \mathcal{P}[n]$ satisfy (i) $j \notin M$, (ii) $|M| \leq |L| - 1$, and (iii) $|M| + 1, |L| \in \binom{n}{\|\gamma\|}$. Then the following identity holds in $\mathcal{Q}(\gamma)$, call it $(\mathcal{P}_{i,L,M,j})$.

$$\sum_{k \in L} r_{ik}^{L \setminus k} r_{kj}^{M} = r_{ij}^{M}\,. \tag{7.15}$$

Proof. This follows by rewriting $(\mathcal{P}_{i,L,M})$ using the other relations, especially $(\mathcal{S}_{a,b,c,L\setminus abc})$.

First, suppose $i \in L$; then (7.15) becomes $r_{ii}^{L\setminus i} r_{ij}^{M} = r_{ij}^{M}$, which is simply relation $(\mathcal{I}_{i,i,L\setminus i})$.

In the remaining case, note that $L \setminus M$ is nonempty. Choose a particular element, say l_0 and, excluding $r_{il_0}^{L\setminus l_0} r_{l_0 j}^{M}$, rewrite each summand as

$$-r_{il_0}^{k\cup(L\setminus kl_0)} r_{l_0 k}^{i\cup(L\setminus kl_0)} r_{kj}^{M}$$

using relation $(\mathcal{S}_{a,b,c,L\setminus abc})$. Next, factor out $r_{il_0}^{L\setminus l_0}$ on the left and $r_{l_0 j}^{M}$ on the right—which we may do because $j, l_0 \notin M$—and get

$$\begin{aligned}
\sum_{k\in L} r_{ik}^{L\setminus k} r_{kj}^{M} &= r_{il_0}^{L\setminus l_0}\left(-\sum_{k\in L\setminus l_0} r_{l_0 k}^{i\cup(L\setminus kl_0)} r_{kl_0}^{M} + 1\right) r_{l_0 j}^{M} \\
&= r_{il_0}^{L\setminus l_0}\left(r_{l_0 i}^{L\setminus l_0} r_{il_0}^{M} - \sum_{k\in i\cup(L\setminus l_0)} r_{l_0 k}^{(i\cup L\setminus l_0)\setminus k} r_{kl_0}^{M} + 1\right) r_{l_0 j}^{M} \\
&= r_{il_0}^{L\setminus l_0} r_{l_0 i}^{L\setminus l_0} r_{il_0}^{M} r_{l_0 j}^{M} \\
&= r_{ij}^{M}\ .
\end{aligned}$$

$\square$

Starting the Diamond Lemma calculations with the set of reductions presented in (7.11)–(7.14) leads to an ambiguity that doesn't resolve. This is in fact how identity $(\mathcal{P}_{i,L,M,j})$ first revealed itself to me.

Proof of Theorem. Let X denote the set of all generators for $\mathcal{Q}(\gamma)$ and let $\langle X\rangle$ denote the set of all words w of finite length $\ell(w)$ in the elements of X. We need a reduction system $\mathcal{R} \subseteq \langle X\rangle \times \mathbb{Q}\langle X\rangle$ and a semigroup partial ordering $\leqslant$ on $\langle X\rangle$ that is compatible with $\mathcal{R}$.

Notation. For a word $w = r_{i_1 j_1}^{M_1} \cdots r_{i_t j_t}^{M_t}$ in $\langle X\rangle$, let $\ell(w)$ denote its *length* t. Also, take its *lower indices* to be the sequence $L(w) = (i_1, j_1, i_2, j_2, \ldots, i_t, j_t)$ and its *upper indices* to the the sequence $U(w) = (M_1, M_2, \ldots, M_t)$.

$(\leqslant)$: Given two words $w, w' \in X$, say $w \leqslant w'$ if $\ell(w) \leq \ell(w')$ and, if their lengths are equal and equal to t, $\sum_{1\leq i\leq t} L(w)_i \leq \sum_{1\leq i\leq t} L(w')_i$. The relation $\leqslant$ is a semigroup partial order on $\langle X\rangle$.

($\mathcal{R}$): We rewrite the relations in $\mathcal{Q}(\gamma)$ as follows:

$$r_{ij}^{M} = 0 \quad (i \neq j,\, i \in M) \tag{7.16}$$

$$r_{ii}^{M} = 1 \quad (i \notin M) \tag{7.17}$$

$$r_{ij}^{M} r_{jk}^{M} = r_{ik}^{M} \quad (j,k \notin M) \tag{7.18}$$

$$r_{ij}^{k\cup M} r_{jk}^{i\cup M} = -r_{ik}^{j\cup M} \quad (i,j,k \notin M) \tag{7.19}$$

$$r_{ij}^{L\setminus j} r_{jk}^{M} = r_{ik}^{M} - \sum_{l\in L} r_{il}^{L\setminus l} r_{lk}^{M} \quad \begin{pmatrix} |M| \leq |L|-1 \\ j = \max\{L \setminus (M \cup k)\} \end{pmatrix} \tag{7.20}$$

We take (7.16)–(7.20) to be $\mathcal{R}$. With the exception of the final reduction, it is plain to see that $\leqslant$ is compatible with $\mathcal{R}$. On the right hand side of the last reduction, there seem to be some words $r_{il}^{L\setminus l} r_{lk}^{M}$ with $\sum_p L\left(r_{il}^{L\setminus l} r_{lk}^{M}\right)_p > \sum_p L\left(r_{ij}^{L\setminus j} r_{jk}^{M}\right)_p$. However, the restriction $j = \max\{L \setminus (M \cup k)\}$ implies these words are not actually of length two:

$$r_{il}^{L\setminus l} r_{lk}^{M} = \begin{cases} 0 & \text{if } l \in M \\ r_{ik}^{L\setminus k} & \text{if } l = k \end{cases}$$

Here is an example of one of these degenerate cases that will play a prominent role in the calculations below:

$$r_{st}^{k\cup M} r_{tk}^{j\cup M} = r_{sk}^{j\cup M} - r_{sk}^{t\cup M} \quad (j,k,s,t \notin M).$$

To spare the reader, we will only verify that three overlap ambiguities resolve,

$$\text{(A)}\ r_{sj}^{L\setminus j} r_{jt}^{k\cup M} r_{tk}^{j\cup M} \qquad \text{(B)}\ r_{sk}^{N\setminus k} r_{kj}^{L\setminus j} r_{jt}^{M} \qquad \text{(C)}\ r_{js}^{L\setminus j} r_{sj'}^{L\setminus j'} r_{j't}^{M},$$

applying reductions to the first two symbols (12) or the last two symbols (23):

$\text{(A)}_{(23)}$:

$$\begin{aligned} r_{sj}^{L\setminus j} r_{jt}^{k\cup M} r_{tk}^{j\cup M} &= r_{sj}^{L\setminus j}\left(- r_{jk}^{t\cup M}\right) \\ &= -r_{sk}^{t\cup M} + \sum_{j'\neq j} r_{sj'}^{L\setminus j'} r_{j'k}^{t\cup M} \end{aligned}$$

$(\mathrm{A})_{(12)}$:

$$\begin{aligned}
r_{sj}^{L\setminus j} r_{jt}^{k\cup M} r_{tk}^{j\cup M} &= \Big(r_{st}^{k\cup M} - \sum_{j'\neq j} r_{sj'}^{L\setminus j'} r_{j't}^{k\cup M}\Big) r_{tk}^{j\cup M} \\
&= \Big(r_{sk}^{j\cup M} - r_{sk}^{t\cup M}\Big) - \sum_{j'\neq j} r_{sj'}^{L\setminus j'} r_{j't}^{k\cup M} r_{tk}^{j\cup M} \\
&= r_{sk}^{j\cup M} - r_{sk}^{t\cup M} - \sum_{j'\neq j} r_{sj'}^{L\setminus j'} \Big(r_{j'k}^{j\cup M} - r_{j'k}^{t\cup M}\Big) \\
&= (\mathrm{A})_{(23)} + r_{sk}^{j\cup M} - \sum_{j'\neq j} r_{sj'}^{L\setminus j'} r_{j'k}^{j\cup M} \\
&= (\mathrm{A})_{(23)} + 0, \text{ because } r_{sj}^{L\setminus j} r_{jk}^{j\cup M} \text{ is zero.}
\end{aligned}$$

$(\mathrm{B})_{(12)}$:

$$\begin{aligned}
r_{sk}^{N\setminus k} r_{kj}^{L\setminus j} r_{jt}^{M} &= \Big(r_{sj}^{L\setminus j} - \sum_{k'\neq k} r_{sk'}^{N\setminus k'} r_{k'j}^{L\setminus j}\Big) r_{jt}^{M} \\
&= r_{sj}^{L\setminus j} r_{jt}^{M} - \sum_{k'\neq k} r_{sk'}^{N\setminus k'} \Big(r_{k't}^{M} - \sum_{j'\neq j} r_{k'j'}^{L\setminus j'} r_{j't}^{M}\Big) \\
&= \Big(r_{st}^{M} - \sum_{j'\neq j} r_{sj'}^{L\setminus j'} r_{j't}^{M}\Big) - \sum_{k'\neq k} r_{sk'}^{N\setminus k'} \Big(r_{k't}^{M} - \sum_{j'\neq j} r_{k'j'}^{L\setminus j'} r_{j't}^{M}\Big)
\end{aligned}$$

$(\mathrm{B})_{(23)}$:

$$\begin{aligned}
r_{sk}^{N\setminus k} r_{kj}^{L\setminus j} r_{jt}^{M} &= r_{sk}^{N\setminus k} \Big(r_{kt}^{M} - \sum_{j'\neq j} r_{kj'}^{L\setminus j'} r_{j't}^{M}\Big) \\
&= r_{sk}^{N\setminus k} r_{kt}^{M} - \sum_{j'\neq j} \Big(r_{sj'}^{L\setminus j'} - \sum_{k'\neq k} r_{sk'}^{N\setminus k'} r_{k'j'}^{L\setminus j'}\Big) r_{j't}^{M}
\end{aligned}$$

The difference becomes

$$(\mathrm{B})_{(12)} - (\mathrm{B})_{(23)} = r_{st}^{M} - \sum_{k'\neq k} r_{sk'}^{N\setminus k'} r_{k't}^{M} - r_{sk}^{N\setminus k} r_{kt}^{M}\,. \tag{7.21}$$

Now, either k or one of the $k' \neq k$ satisfies $k'' = \max\{N \setminus (M \cup t)\}$. Applying reduction (7.20) to the appropriate term above reduces (7.21) to zero.

$(\mathrm{C})_{(12)}$:

$$r_{js}^{L\setminus j} r_{sj'}^{L\setminus j'} r_{j't}^{M} = \Big(-r_{jj'}^{s\cup L^{jj'}}\Big) r_{j't}^{M}$$

$(\mathrm{C})_{(23)}$:

$$\begin{aligned}
r_{js}^{L\setminus j} r_{sj'}^{L\setminus j'} r_{j't}^{M} &= r_{js}^{L\setminus j}\Big(r_{st}^{M} - \sum_{j''\neq j'} r_{sj''}^{j\cup L^{jj''}} r_{j''t}^{M}\Big)\\
&= r_{js}^{L^j} r_{st}^{M} - \sum_{j''\neq j'}\Big(r_{js}^{j''\cup L^{jj''}} r_{sj''}^{j\cup L^{jj''}}\Big) r_{j''t}^{M}\\
&= r_{js}^{L^j} r_{st}^{M} - \sum_{j''\neq j'}\Big(- r_{jj''}^{s\cup L^{jj''}}\Big) r_{j''t}^{M}
\end{aligned}$$

The reasoning for $(\mathrm{B})_{(12)} - (\mathrm{B})_{(23)}$ finishes the job for $(\mathrm{C})_{(12)} - (\mathrm{C})_{(23)}$. $\square$

We conclude this section with a key step toward the goal of computing the number of admissible words in $\mathcal{Q}(\gamma)$ of any given length.

Proposition 72. *The number of inadmissible words of length two is*

$$\sum_{1\leq j\leq n}\ \sum_{\substack{m_2<m_1\\ m_2+1,m_1+1\in\|\gamma\|}}\ \sum_{0\leq m_2'\leq \min\{n-j,m_2\}} (n-1-m_1)\binom{n-j}{m_2'}\binom{j-1}{m_2-m_2'}\times$$
$$\left\{(j-1-m_2+m_2')\binom{j-1+m_2'}{m_1} + (n-j-m_2')\binom{j+m_2'}{m_1}\right\}$$

Proof. We count all of the choices for i, j, k, M_1, M_2 we can make in order for $w = r_{ij}^{M_1} r_{jk}^{M_2}$ to be an admissible word. To start the count, we fix j, $|M_2|$, and $|M_1|$—this explains the first two sums appearing above.

Next, we divide M_2 into two pieces: $M_2' \subseteq \{j+1,\ldots,n\}$, $M_2'' \subseteq \{1,2,\ldots,j-1\}$. This gives

$$\binom{n-j}{|M_2'|}\binom{j-1}{|M_2|-|M_2'|}$$

choices for M_2. Depending on whether $k<j$ or $k>j$, we have a different number of choices for k:

$$\begin{aligned}
&\text{Case } k<j: \qquad (j-1-|M_2|+|M_2'|)\\
&\text{Case } k<j: \qquad (n-j-|M_2|+|M_2'|)\,.
\end{aligned}$$

For w to be inadmissible, we need all elements of M_1 greater than j to also appear in $M_2\cup\{k\}$. Depending on whether $k<j$ or $k>j$, we have a different number of choices for M_1:

$$\begin{array}{ll} \text{Case } k < j : & \dbinom{j-1+|M_2'|}{|M_1|} \\ \text{Case } k < j : & \dbinom{j-1+|M_2'|+1}{|M_1|}. \end{array}$$

In all cases for k and M_1, we have $(n-1-|M_1|)$ choices for i. Putting these considerations together with $m_1 = |M_1|$, $m_2 = |M_2|$, and $m_2' = |M_2'|$, we get the advertised formula. $\square$

Chapter 8

The Frenkel-Jardim Flag

As we mentioned in the previous chapter, the construction of Frenkel and Jardim arose from a new attempt to build quantum instantons. Simply put, an *instanton* is a solution to the so-called anti-self-dual Yang-Mills (ASDYM) equation. In [2], Atiyah, Drinfeld, Hitchin, and Manin (ADHM) constructed solutions over Minkowski space-time, $\mathbb{R}^3 \oplus \mathbb{R}$. Their solutions were parametrized in terms of some linear data. Moreover, they proved that, up to gauge transformation, all solutions were of this type.

Frenkel and Jardim [13] follow the program of R. Penrose [40] by looking for solutions in compactified complexified Minkowski space; or rather, a quantized version of this space. With the construction outlined in Section 7.2 of (the algebra of functions on) compactified complexified Minkowski space, they are able to build solutions to the quantum ASDYM which are directly parametrized by the very same classic ADHM linear data.

In their paper, Frenkel and Jardim also introduce indeterminants $z_i, z_{i'}$ and piece together a quantum flag algebra $\mathfrak{F}_q$ around the quantum Grassmannian $\mathfrak{M}_{p,q}$. As in the commutative case, they are able to view the extension as adding *twistors* to the picture. For further background and motivation, cf. loc. cit.

The content of this chapter is a pair of negative results: (i) the quantum Grassmannian $\mathcal{G}_q$ of Taft and Towber is not isomorphic to $\mathfrak{M}_q$; (ii) the pre–Grassmannian algebras which were shown to be isomorphic to $\mathfrak{M}_q$ cannot be naturally extended to capture the flag $\mathfrak{F}_q$. One could argue that the first result is a positive one—it verifies that the Frenkel-Jardim construction truly is a new quantum Grassmannian—but the second is certainly discouraging. Were an isomorphism of the type $(\mathfrak{M}_q \hookrightarrow \mathfrak{F}_q) \simeq (\tilde{\mathcal{G}} \hookrightarrow \tilde{\mathcal{F}})$ to exist, it would put the Frenkel-Jardim construction squarely under the umbrella of straightforward, easy to define quantized flag algebras.

8.1 Two quantum Grassmannians

Here we show that the Taft-Towber quantum Grassmannian $\mathcal{G}_q(2,4)$ is not isomorphic to the Frenkel-Jardim quantum Grassmannian $\mathfrak{M}_q$. The table below strongly suggests this is the case, but we'll give a rigorous proof just the same.

Let $\mathfrak{M}_q$ have generators $z_{ij} : 1 \leq i < j \leq 4$ and $\mathcal{G}_q = \mathcal{G}_q(2,4)$ have generators $f_{ij} : 1 \leq i < j \leq 4$. Table 8.1 displays their relations.

$z_{14}z_{13} = z_{13}z_{14}$ $z_{23}z_{13} = z_{13}z_{23}$ $z_{24}z_{14} = z_{14}z_{24}$ $z_{24}z_{23} = z_{23}z_{24}$	$f_{14}f_{13} = qf_{13}f_{14}$ $f_{23}f_{13} = qf_{13}f_{23}$ $f_{24}f_{14} = qf_{14}f_{24}$ $f_{24}f_{23} = qf_{23}f_{24}$
$z_{13}z_{12} = z_{12}z_{13}$ $z_{14}z_{12} = z_{12}z_{14}$ $z_{23}z_{12} = q^{-2}z_{12}z_{23}$ $z_{24}z_{12} = q^{-2}z_{12}z_{24}$	$f_{13}f_{12} = qf_{12}f_{13}$ $f_{14}f_{12} = qf_{12}f_{14}$ $f_{23}f_{12} = qf_{12}f_{23}$ $f_{24}f_{12} = qf_{12}f_{24}$
$z_{34}z_{13} = q^{-2}z_{13}z_{34}$ $z_{34}z_{14} = z_{14}z_{34}$ $z_{34}z_{23} = q^{-2}z_{23}z_{34}$ $z_{34}z_{24} = z_{24}z_{34}$	$f_{34}f_{13} = qf_{13}f_{34}$ $f_{34}f_{14} = qf_{14}f_{34}$ $f_{34}f_{23} = qf_{23}f_{34}$ $f_{34}f_{24} = qf_{24}f_{34}$
$z_{34}z_{12} = q^{-2}z_{12}z_{34}$ $z_{23}z_{14} = z_{14}z_{23}$ $z_{24}z_{13} = z_{13}z_{24} + (q^{-2} - 1)z_{12}z_{34}$ $z_{14}z_{23} = z_{13}z_{24} - z_{12}z_{34}$	$f_{34}f_{12} = q^2 f_{12}f_{34}$ $f_{23}f_{14} = f_{14}f_{23}$ $f_{24}f_{13} = q^2(f_{13}f_{24} + (q^{-1} - q)f_{12}f_{34})$ $f_{14}f_{23} = qf_{13}f_{24} - q^2 f_{12}f_{34}$

Table 8.1: Relations on the Grassmannian coordinates.

Proposition 73. *The algebras $\mathfrak{M}_q$ and $\mathcal{G}_q$ are not isomorphic.*

Proof. One striking feature of the relations in the left-hand column of Table 8.1 is that z_{14} is central in $\mathfrak{M}_q$. Exploiting this feature gives the result.

Suppose $\varphi : \mathfrak{M}_q \to \mathcal{G}_q$ is an isomorphism. Writing $\varphi(z_{14}) = a_0 + \sum_{i<j} a_{ij} f_{ij} + \sum_{i<j,k<l} a_{ij,kl} f_{ij} f_{kl} + \cdots$, we see what conditions we must impose on a's in order that $\varphi(z_{14})$ is central. We may assume the only nonzero constants $a_{i_1 j_1, i_2 j_2, \ldots, i_r j_r}$ appearing are attached to those monomials which are part of a $\mathbb{C}$-basis for $\mathcal{G}_q$. Let us write $a_{i_1 j_1, \ldots, i_r j_r} f_{i_1 j_1} \cdots f_{i_r j_r}$ as $a_K f^K$ for some $K = (K_1, K_2, \ldots, K_r) \in \binom{[4]}{2} \times \cdots \times \binom{[4]}{2}$ (r copies). Also, let us denote this set of tuples by $\mathcal{B}(r)$.

One striking feature of the relations in the right-hand column of Table 8.1 is that, for any $K \in \mathcal{B}(r)$, for any $r \geq 1$,

$$f^K f_{12} = q^{r - \#\{K_i = \{1,2\}\} + \#\{K_i = \{3,4\}\}} f_{12} f^K.$$

Let us demand that $\left[\varphi(z_{14}), f_{12}\right] = 0$:

$$\begin{aligned} \varphi(z_{14}) f_{12} &= \left(a_0 + \sum_{r \geq 1} \sum_{K \in \mathcal{B}(r)} a_K f^K \right) f_{12} \\ &= f_{12} \left(a_0 + \sum_{r \geq 1} \sum_{K \in \mathcal{B}(r)} q^{r - \#\{K_i = \{1,2\}\} + \#\{K_i = \{3,4\}\}} a_K f^K \right). \end{aligned}$$

Thus, the only nonzero a_K which may appear must have all $K_i = (1, 2)$. For if this is not the case, we may subtract this last expression from the desired result and get

$$f_{12} \left(\sum_{r \geq 1} \sum_{K \in \mathcal{B}(r)} a_K (1 - q^{m_K}) f^K \right) = 0,$$

for those particular a_K not satisfying $K_i = (1, 2)\, \forall i$—here the m_K are strictly positive integers. But the monomials appearing in the second factor were assumed to be part of a basis, so that sum isn't zero. Also, $\mathcal{G}_q$ is a domain (cf. [29]) so the product isn't zero.

We conclude that the only nonzero a_K are those with each $K_i = (1, 2)$. Now try to commute f_{13} past $\varphi(z_{14})$ and discover that the only choice for these a's is also zero. Finally, we see that the center of $\mathcal{G}_q$ is $\mathbb{C}$, too small for an isomorphism to exist. □

8.2 The quantum flag of Frenkel-Jardim

In their paper, Frenkel and Jardim define a quantum flag with two parameters p, q with $p = q^{\pm 1}$.

Definition 50. The quantum flag $\mathfrak{F}_{p,q}$ associated to $\mathfrak{M}_{p,q}$ is the $\mathbb{C}$-algebra with generators $\{z_i, z_{jk} \mid 1 \le i \le 4, 1 \le j < k \le 4\}$ and relations given by (7.6)–(7.10) together with

$$\left[z_i, z_j\right] = 0, \tag{8.1}$$

$$\begin{aligned} \left[z_1, z_{13}\right] &= \left[z_1, z_{14}\right] = 0 \\ \left[z_2, z_{23}\right] &= \left[z_2, z_{24}\right] = 0 \\ \left[z_3, z_{13}\right] &= \left[z_3, z_{23}\right] = 0 \\ \left[z_4, z_{14}\right] &= \left[z_4, z_{24}\right] = 0 \end{aligned} \tag{8.2}$$

$$\begin{aligned} z_1 z_{12} &= pq z_{12} z_1 & z_1 z_{34} &= z_{34} z_1 \\ z_2 z_{12} &= pq^{-1} z_{12} z_2 & z_2 z_{34} &= z_{34} z_2 \\ z_3 z_{12} &= z_{12} z_3 & z_3 z_{34} &= p^{-1} q z_{34} z_3 \\ z_4 z_{12} &= z_{12} z_4 & z_4 z_{34} &= p^{-1} q^{-1} z_{34} z_4 \end{aligned} \tag{8.3}$$

$$\begin{aligned} z_2 z_{13} &= pq^{-1} z_{13} z_2 + (1 - pq^{-1}) z_{23} z_1 \\ z_2 z_{14} &= pq^{-1} z_{14} z_2 + (1 - pq^{-1}) z_{24} z_1 \\ z_3 z_{14} &= p^{-1} q z_{14} z_3 + (1 - p^{-1} q) z_{13} z_4 \\ z_3 z_{24} &= p^{-1} q z_{24} z_3 + (1 - p^{-1} q) z_{23} z_4 \end{aligned} \tag{8.4}$$

$$\begin{aligned} z_1 z_{23} &= pq z_{23} z_1 + (1 - pq) z_{13} z_2 \\ z_1 z_{24} &= pq z_{24} z_1 + (1 - pq) z_{14} z_2 \\ z_4 z_{13} &= p^{-1} q^{-1} z_{13} z_4 + (1 - p^{-1} q^{-1}) z_{14} z_3 \\ z_4 z_{23} &= p^{-1} q^{-1} z_{23} z_4 + (1 - p^{-1} q^{-1}) z_{24} z_3 \end{aligned} \tag{8.5}$$

$$\begin{aligned} z_{12} z_3 &= q(z_{13} z_2 - z_{23} z_1) \\ z_{12} z_4 &= q(z_{14} z_2 - z_{24} z_1) \\ z_{34} z_1 &= q^{-1}(-z_{13} z_4 + z_{14} z_3) \\ z_{34} z_2 &= q^{-1}(-z_{23} z_4 + z_{24} z_3) \end{aligned} \tag{8.6}$$

In keeping with our discussion in Section 7.2, we will assume $p = q$ in the sequel and denote the algebra as $\mathfrak{F}_q$. Were the results of the next section of a more positive nature, it might be worth exploring the case $p = q^{-1}$ as well. Naïvely, it should be the result of taking row minors instead of column minors; or taking as the definition of determinant, an object which counts row permutations instead of column permutations. However, as we will show presently, the Frenkel-Jardim flag construction does not extend the Grassmannian construction in a manner consistent with the extension from pre-Grassmannians to pre-flags implicit in Sections 5.6 and 5.8.

8.3 Four quantum flags

Let $\tilde{\mathcal{F}}^{\mathrm{I}}$, $\tilde{\mathcal{F}}^{\mathrm{II}}$ denote the the pre–flag algebras on $\gamma = (1^4)$ introduced in Sections 5.6 and 5.8 respectively.

Proposition 74. *There is no algebra map* $\varphi : \tilde{\mathcal{F}}^{\mathrm{II}} \to \mathfrak{F}_q$ *which prolongs the isomorphism* $\tilde{\mathcal{G}}^{\mathrm{II}} \simeq \mathfrak{M}_q$ *exhibited in Section 7.2.*

Proof. In Section 7.2 we had $\varphi(\tilde{f}_{ij}) = z_{ij}$. It is left to find images for the new variables $\tilde{f}_i$. In Table 8.2, we display the new relations that must be respected.

One striking feature of the relations in the right-hand column of Table 8.2 is that $\tilde{f}_{12}\tilde{f}_i = q^{\pm 1}\tilde{f}_i\tilde{f}_{12}$ for all (ij). Let us follow the proof of Proposition 73 and write $\varphi(f_i) = a_0^{(i)} + \sum_j a_j^{(i)} z_j + \sum_{kl} a_{kl}^{(i)} z_{kl} + \sum_{r\geq 2}\sum_{K\in\mathcal{B}(r)} a_K^{(i)} z^K$. Diverge from the notation there by setting $\mathcal{B}(1) := \binom{[4]}{\{1,2\}}$ instead of $\binom{[4]}{2}$; keep $\mathcal{B}(r)$ equal to $\mathcal{B}(1)^r$. Again, we may assume the only $a_K^{(i)}$ appearing are those attached to monomials z^K comprising a linearly independent set in $\mathfrak{F}_q$.

One striking feature of the relations in the left-hand column (of Tables 8.1 and 8.2) is that z_{12} commutes with everything up to a power of q^2.

Let us compare $\varphi(f_{12}f_i - q^{\pm 1} f_i f_{12})$ to zero, bearing in mind that we have fixed $\varphi(f_{12}) = z_{12}$. The calculation reduces to

$$0 = \left((1-q^{\pm 1})a_0^{(i)} + \sum_{r\geq 1}\sum_{K\in\mathcal{B}(r)} (q^{2m_K} - q^{\pm 1}) a_K^{(i)} z^K\right) z_{12}\,,, \tag{8.7}$$

for some integers m_K depending on $K = (K_1, K_2, \ldots, K_r)$. Now, the monomials appearing in the first factor were assumed to be part of a basis, so that sum isn't zero—excluding the case q is a root of unity. If $\mathfrak{F}_q$ is a domain, we reach a contradiction and

$[z_i, z_j] = 0$	$[\tilde{f}_i, \tilde{f}_j] = 0$
$z_{12}z_1 = q^{-2}z_1z_{12}$ $z_{13}z_1 = z_1z_{13}$ $z_{14}z_1 = z_1z_{14}$ $z_{34}z_1 = z_1z_{34}$	$\tilde{f}_{12}\tilde{f}_1 = q^{-1}\tilde{f}_1\tilde{f}_{12}$ $\tilde{f}_{13}\tilde{f}_1 = q^{-1}\tilde{f}_1\tilde{f}_{13}$ $\tilde{f}_{14}\tilde{f}_1 = q^{-1}\tilde{f}_1\tilde{f}_{14}$ $\tilde{f}_{34}\tilde{f}_1 = q^{-1}\tilde{f}_1\tilde{f}_{34}$
$z_{12}z_2 = z_2z_{12}$ $z_{23}z_2 = z_2z_{23}$ $z_{24}z_2 = z_2z_{24}$ $z_{34}z_2 = z_2z_{34}$	$\tilde{f}_{12}\tilde{f}_2 = q\tilde{f}_2\tilde{f}_{12}$ $\tilde{f}_{23}\tilde{f}_2 = q^{-1}\tilde{f}_2\tilde{f}_{23}$ $\tilde{f}_{24}\tilde{f}_2 = q^{-1}\tilde{f}_2\tilde{f}_{24}$ $\tilde{f}_{34}\tilde{f}_2 = q^{-1}\tilde{f}_2\tilde{f}_{34}$
$z_{12}z_3 = z_3z_{12}$ $z_{13}z_3 = z_3z_{13}$ $z_{23}z_3 = z_3z_{23}$ $z_{34}z_3 = z_3z_{34}$	$\tilde{f}_{12}\tilde{f}_3 = q\tilde{f}_3\tilde{f}_{12}$ $\tilde{f}_{13}\tilde{f}_3 = q^{-1}\tilde{f}_3\tilde{f}_{13}$ $\tilde{f}_{23}\tilde{f}_3 = q^{-1}\tilde{f}_3\tilde{f}_{23}$ $\tilde{f}_{34}\tilde{f}_3 = q^{-1}\tilde{f}_3\tilde{f}_{34}$
$z_{12}z_4 = z_4z_{12}$ $z_{14}z_4 = z_4z_{14}$ $z_{24}z_4 = z_4z_{24}$ $z_{34}z_4 = q^2z_4z_{34}$	$\tilde{f}_{12}\tilde{f}_4 = q\tilde{f}_4\tilde{f}_{12}$ $\tilde{f}_{14}\tilde{f}_4 = q\tilde{f}_4\tilde{f}_{14}$ $\tilde{f}_{24}\tilde{f}_4 = q\tilde{f}_4\tilde{f}_{24}$ $\tilde{f}_{34}\tilde{f}_4 = q\tilde{f}_4\tilde{f}_{34}$
$z_{23}z_1 = z_1z_{23} + (q - q^{-1})z_3z_{12}$ $z_{24}z_1 = z_1z_{24} + (q - q^{-1})z_4z_{12}$ $z_{13}z_4 = z_4z_{13} - (q - q^{-1})z_1z_{34}$ $z_{23}z_4 = z_4z_{23} - (q - q^{-1})z_2z_{34}$	$\tilde{f}_{23}\tilde{f}_1 = q^{-1}\tilde{f}_1\tilde{f}_{23}$ $\tilde{f}_{24}\tilde{f}_1 = q^{-1}\tilde{f}_1\tilde{f}_{24}$ $\tilde{f}_{13}\tilde{f}_4 = q\tilde{f}_4\tilde{f}_{13}$ $\tilde{f}_{23}\tilde{f}_4 = q\tilde{f}_4\tilde{f}_{23}$
$z_{13}z_2 = z_1z_{23} + qz_3z_{12}$ $z_{14}z_2 = z_1z_{24} + qz_4z_{12}$ $z_{14}z_3 = z_4z_{13} + q^{-1}z_1z_{34}$ $z_{24}z_3 = z_4z_{23} + q^{-1}z_2z_{34}$	$\tilde{f}_2\tilde{f}_{13} = \tilde{f}_1\tilde{f}_{23} + \tilde{f}_3\tilde{f}_{12}$ $\tilde{f}_2\tilde{f}_{14} = \tilde{f}_1\tilde{f}_{24} + \tilde{f}_4\tilde{f}_{12}$ $\tilde{f}_3\tilde{f}_{14} = \tilde{f}_1\tilde{f}_{34} + \tilde{f}_4\tilde{f}_{13}$ $\tilde{f}_3\tilde{f}_{24} = \tilde{f}_2\tilde{f}_{34} + \tilde{f}_4\tilde{f}_{23}$

Table 8.2: Relations on the flag coordinates.

are finished. We can get away with less. First let us rewrite (8.7) as follows

$$\begin{aligned} 0 \quad = \quad & \left((1-q^{\pm 1})a_0^{(i)} z_{12}\right) + \\ & \left(\sum_{j\in[4]} (q^{2m_j} - q^{\pm 1}) a_j^{(i)} z_j z_{12} + \sum_{kl \in \binom{[4]}{2}} (1 - q^{-2m_{kl}\pm 1}) a_{kl}^{(i)} z_{12} z_{kl}\right) + \\ & \sum_{r\geq 2} \left(\sum_{K\in\mathcal{B}(r)} (q^{2m_K} - q^{\pm 1}) a_K^{(i)} z^K z_{12}\right) \end{aligned}$$

Because the relations in $\mathfrak{F}_q$ are homogeneous, we know each of these graded pieces must be zero independently. We focus on the first two, and argue that $\{z_{12}, z_1 z_{12}, \ldots, z_4 z_{12}, z_{12}z_{12}, z_{12}z_{13}, \ldots, z_{12}z_{34}\}$ is a linearly independent set in $\mathfrak{F}_q$.

Claim 1: The set $\mathfrak{X}$ of monomials $z_{i_1} z_{i_2} \cdots z_{i_r} z_{j_1 k_1} z_{j_2 k_2} \cdots z_{j_s k_s}$ satisfying

1. $i_t \leq i_{t+1}\,(1 \leq t < r)$
2. $j_t < k_t$ and $j_t \leq j_{t+1}$ and $k_t \leq k_{t+1}\,(1 \leq t < s)$

span $\mathfrak{F}_q$ as a $\mathbb{C}$ vector space.

The relations in the left-hand column of Tables 8.1 and 8.2 indicate that any word not belonging to $\mathfrak{X}$ may be written as a linear combination of words in $\mathfrak{X}$. Perhaps the members of $\mathfrak{X}$ are not linearly independent, but anyhow they certainly span $\mathfrak{F}_q$.

Claim 2: The monomials $\mathfrak{Z} = \{z_{12}, z_1 z_{12}, \ldots, z_4 z_{12}, z_{12}z_{12}, z_{12}z_{13}, \ldots, z_{12}z_{34}\}$ are part of a basis for $\mathfrak{F}_q$.

The relations in the left-hand column of Tables 8.1 and 8.2 are presented in a form conducive for applying Bergman's Diamond Lemma [4]. The candidate basis is precisely those words included in $\mathfrak{X}$. As we mentioned above, there may be relations among some of these words. However, because the relations in $\mathfrak{F}_q$ are homogeneous of degree two, any new relations which we must introduce while implementing the Diamond Lemma—coming from overlap ambiguities that don't resolve—will be homogeneous of degree strictly greater than two. So no matter what percentage of $\mathfrak{X}$ survives as the true basis of $\mathfrak{F}_q$, we are guaranteed that $\mathfrak{Z}$ will be a part of it.

We conclude that $\varphi(f_i) \subseteq \bigoplus_{j\geq 2} (\mathfrak{F}_q)_{(j)}$ for all i (letting $(\mathfrak{F}_q)_{(j)}$ denote the degree

j graded piece of $\mathfrak{F}_q$). Finally, as $\varphi(f_{ij}) = z_{ij}$ and $\varphi(1) = 1$, we are left with a 4-dimensional piece of $(\mathfrak{F}_q)_{(1)}$ which is unaccounted for…not a promising quality for a purported onto map. $\square$

Proceeding in a manner analogous to that above, one should conclude that $\mathcal{F}_q, \tilde{\mathcal{F}}^{\mathrm{I}}, \tilde{\mathcal{F}}^{\mathrm{II}}$, and $\mathfrak{F}_q$ are four pairwise non-isomorphic quantizations of the homogeneous coordinate algebra for $F\ell(4)$.

References

[1] Michael Artin, William Schelter, and John Tate, *Quantum deformations of* GL_n, Comm. Pure Appl. Math. **44** (1991), no. 8-9, 879–895.

[2] M. F. Atiyah, N. J. Hitchin, V. G. Drinfel′d, and Yu. I. Manin, *Construction of instantons*, Phys. Lett. A **65** (1978), no. 3, 185–187.

[3] Arkady Berenstein and Vladimir Retakh, *Noncommutative double Bruhat cells and their factorizations*, in preparation (27 pages, 2004), arXiv: math.QA/0407010.

[4] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218.

[5] Jonathan Brundan and Alexander Kleshchev, *Parabolic presentations of the Yangian* $Y(\mathfrak{gl}_n)$, Comm. Math. Physics (to appear), preprint (64 pages, 2002), arXiv: math.QA/0407011.

[6] Philippe Caldero, *A multiplicative property of quantum flag minors*, Represent. Theory **7** (2003), 164–176 (electronic).

[7] Vyjayanthi Chari and Andrew Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995, Corrected reprint of the 1994 original.

[8] Paul Moritz Cohn, *Skew field constructions*, London Mathematical Society Lecture Note Series, no. 27, Cambridge University Press, Cambridge, 1977.

[9] Alain Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994.

[10] V. G. Drinfel′d, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064.

[11] Pavel Etingof, Israel Gelfand, and Vladimir Retakh, *Factorization of differential operators, quasideterminants, and nonabelian Toda field equations*, Math. Res. Lett. **4** (1997), no. 2-3, 413–425.

[12] R. Fioresi, *Quantum deformation of the Grassmannian manifold*, J. Algebra **214** (1999), no. 2, 418–447.

[13] Igor B. Frenkel and Marcos Jardim, *Quantum instantons with classical moduli spaces*, Comm. Math. Phys. **237** (2003), no. 3, 471–505.

[14] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry.

[15] I. M. Gel′fand and V. S. Retakh, *Determinants of matrices over noncommutative rings*, Funktsional. Anal. i Prilozhen. **25** (1991), no. 2, 13–25, 96.

[16] ______, *Theory of noncommutative determinants, and characteristic functions of graphs*, Funktsional. Anal. i Prilozhen. **26** (1992), no. 4, 1–20, 96.

[17] I. M. Gelfand and V. S. Retakh, *Quasideterminants, I*, Selecta Math. (N.S.) **3** (1997), no. 4, 517–546.

[18] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson, *Quasideterminants*, Adv. in Math. **193** (2005), no. 1, 56–141.

[19] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, and Jean-Yves Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), no. 2, 218–348.

[20] K. R. Goodearl and T. H. Lenagan, *Quantum determinantal ideals*, Duke Math. J. **103** (2000), no. 1, 165–190.

[21] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original.

[22] D. I. Gurevich, *Algebraic aspects of the quantum Yang-Baxter equation*, Algebra i Analiz **2** (1990), no. 4, 119–148, Trans. in *Leningrad Math. J.* 2 (1991), no. 4, 801–828.

[23] Michiel A. Hazewinkel, *Multiparameter quantum groups and multiparameter R-matrices*, Acta Appl. Math. **41** (1995), no. 1-3, 57–98, Geometric and algebraic structures in differential equations.

[24] A. Heyting, *Die theorie der linearen gleichungen in einer zahlenspezies mit nichtkommutativer multiplikation*, Math. Ann. **98** (1928), 465–490.

[25] W. V. D. Hodge, *Some enumerative results in the theory of forms*, Proc. Cambridge Philos. Soc. **39** (1943), 22–30.

[26] W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry. Vol. I*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1994, Book I: Algebraic preliminaries, Book II: Projective space, Reprint of the 1947 original.

[27] Kenji Iohara, *Bosonic representations of Yangian double $\mathcal{DY}_\hbar(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$*, J. Phys. A **29** (1996), no. 15, 4593–4621.

[28] Christian Kassel, *Quantum groups*, Springer-Verlag, New York, 1995.

[29] A. C. Kelly, T. H. Lenagan, and L. Rigal, *Ring theoretic properties of quantum Grassmannians*, J. Algebra Appl. **3** (2004), no. 1, 9–30.

[30] Daniel Krob and Bernard Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. **169** (1995), no. 1, 1–23.

[31] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.

[32] Bernard Leclerc, Maxim Nazarov, and Jean-Yves Thibon, *Induced representations of affine Hecke algebras and canonical bases of quantum groups*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 115–153.

[33] Bernard Leclerc and Andrei Zelevinsky, *Quasicommuting families of quantum Plücker coordinates*, Kirillov's seminar on representation theory, Amer. Math. Soc. Transl. Ser. 2, vol. 181, Amer. Math. Soc., Providence, RI, 1998, pp. 85–108.

[34] A. I. Malcev, *On the immersion of an algebraic ring in a skew field*, Math. Ann. **113** (1937), 686–691.

[35] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.

[36] A. I. Molev, *Yangians and their applications*, Handbook of algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 907–959.

[37] Alexander Molev and Vladimir Retakh, *Quasideterminants and Casimir elements for the general linear Lie superalgebra*, Int. Math. Res. Not. (2004), no. 13, 611–619.

[38] Peter Olver, *On multivariate interpolation*, preprint (35 pages, 2004), http://www.math.umn.edu/~olver.

[39] Oystein Ore, *Linear equations in noncommutative rings*, Ann. of Math. **32** (1931), 463–477.

[40] Roger Penrose, *The twistor programme*, Rep. Mathematical Phys. **12** (1977), no. 1, 65–76.

[41] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz **1** (1989), no. 1, 178–206, English transl.: *Leningrad Math. J.* **1** (1990), no. 1, 193–225.

[42] Christophe Reutenauer, *Inversion height in free fields*, Selecta Math. (N.S.) **2** (1996), no. 1, 93–109.

[43] Zoran Škoda, *Localizations for construction of quantum coset spaces*, Noncommutative geometry and quantum groups (Warsaw, 2001), Banach Center Publ., vol. 61, Polish Acad. Sci., Warsaw, 2003, pp. 265–298.

[44] J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 2, 171–216 (electronic).

[45] Earl Taft and Jacob Towber, *Quantum deformation of flag schemes and Grassmann schemes, I. A q-deformation of the shape-algebra for* $\mathrm{GL}(n)$, J. Algebra **142** (1991), no. 1, 1–36.

[46] Earl J. Taft, *Quantum deformation of the flag variety*, Quantum Groups (Leningrad, 1990), Lecture Notes in Mathematics, vol. 1510, Springer, Berlin, 1992, pp. 138–141.

[47] Mitsuhiro Takeuchi, *A two-parameter quantization of* $\mathrm{GL}(n)$ *(summary)*, Proc. Japan Acad. Ser. A Math. Sci. **66** (1990), no. 5, 112–114.

[48] ———, *A short course on quantum matrices*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, Cambridge, 2002, Notes taken by Bernd Strüber, pp. 383–435.

[49] Jacob Towber, *Young symmetry, the flag manifold, and representations of* $\mathrm{GL}(n)$, J. Algebra **61** (1979), no. 2, 414–462.

Vita

Aaron Lauve

2005 Ph.D. in Mathematics, *Rutgers University*

1993–1999 B.Sc. in Mathematics, cum laude, *University of Oklahoma*

1993–1999 B.Sc. in Physics, *University of Oklahoma*

1993 H.S. Diploma, *Missisippi School for Mathematics and Science*

2004–2005 Bevier Dissertation fellow, Department of Mathematics, Rutgers University

2001–2004 Teaching assistant, Department of Mathematics, Rutgers University

1999–2001 VIGRE fellow, Department of Mathematics, Rutgers University