A Quasideterminantal Approach to Quantized Flag Varieties

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■ Key Idea	■ Proceed with Caution	■ Watch Out!

Problem Statement

Provide a means to construct noncommutative flag varieties in a variety of noncommutative settings via the quasideterminant.

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- "It would be very important to define noncommutative flag spaces for quantum groups." [Manin, '88]
- Mind the Gap!
- The case of Grassmannians has satisfactory results.

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Provide a means to construct noncommutative flag varieties in a variety of noncommutative settings via the quasideterminant.

- Traditionally, noncommutative geometry is studied by proxy:

$$\{\text{topological spaces } X\} \leftrightarrow \{\text{rings of functions } R(X) \text{ on } X\}$$

- e.g. call a noncommutative algebra the “ring of functions” for some (phantom, noncommutative) variety.

Problem Statement

Provide a means to construct noncommutative flag varieties in a variety of noncommutative settings via the quasideterminant.

- In settings of "quantum group" type. . .
- and only those settings possessing an "amenable determinant."

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- “A main organizing tool in noncommutative algebra.” [Gelfand-G-Retakh-Wilson, ‘02]
- In the commutative case, it looks like

$$\pm \frac{\det A}{\det A^{ij}}.$$

- Has a Cramer’s Rule.
- Is zero when matrix isn’t of full rank.

⋮

Notation

- Denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

- Fix an $n \times n$ matrix A.
  - If $i, j \in [n]$ then A^{ij} denotes the deletion of row i and column j.
  - If $I, J \subseteq [n]$ then $A_{I,J}$ indicates we keep only rows I and columns J.
  - If $I \subseteq [n]$ with $|I| = d$, we abbreviate $A_{I,[d]}$ by A_I.

- Fix two *sets* $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_s\}$ and $k \in [n] \setminus I$.
  - We write kI for $\{k\} \cup I$.
  - We write $I|J$ for the *sequence* $(i_1, \ldots, i_r, j_1, \ldots, j_s)$.
  - We write $\ell(I|J)$ for the length of the derangement $I|J$ (the min. number of adjacent swaps needed to put $I|J$ in increasing order).

Flag Varieties

- Fix an integer $n > 1$ and a sequence $\gamma = (\gamma_1, \ldots, \gamma_r)$ of positive integers summing to n. Fix a vector space $V = \mathbb{C}^n$ with basis $\mathfrak{B}$.

Definition (Flags). *A* flag Φ *of shape* γ *is a left coset representative of* $F\ell(\gamma) := \mathrm{GL}_n(\mathbb{C})/\mathrm{P}_\gamma^+$ *where*

$$\mathrm{P}_\gamma^+ = \begin{bmatrix} \blacksquare & & & * \\ & \blacksquare & & \\ & & \ddots & \\ 0 & & & \blacksquare \end{bmatrix}$$

- Focus on $\gamma = (1, 1, \ldots, 1)$ for simplicity. Write $F\ell(n)$ in this case.

- Another special case is $\gamma = (d, n-d)$. It describes the Grassmannian $Gr(d, n)$, the set of d-dimensional subspaces of V.

- $F\ell(n)$ is made into a (projective) variety by the **Plücker embedding:**

$$\eta : A \mapsto \{\det A_I \mid I \subseteq [n],\ |I| = d,\ 1 \leq d < n\},$$

a map into $\mathbb{P}_\gamma := \mathbb{PC}^{\binom{n}{1}} \times \mathbb{PC}^{\binom{n}{2}} \times \cdots \times \mathbb{PC}^{\binom{n}{n-1}}$.

Plücker Coordinates

- A point $\pi = (p_I) \in \mathbb{P}_\gamma$ belongs to $\eta(F\ell(n))$ iff π satisfies:

Definition (The Young Symmetry Relations $(\mathcal{Y}_{L,M})_{(u)}$). *Given $L, M \subseteq [n]$ with $|L| = s+u, |M| = t-u$ and $s \geq t$*

$$0 = \sum_{\substack{\Lambda \subset L \\ |\Lambda| = u}} (-1)^{\ell(L\backslash\Lambda|\Lambda)+\ell(\Lambda|M)} p_{L\backslash\Lambda} p_{\Lambda\cup M} \, .$$

- or add alternating relations for the symbols p_I and rewrite as

$$0 = \sum_{\substack{\Lambda \subset L \\ |\Lambda| = u}} (-1)^{\ell(L\backslash\Lambda|\Lambda)} p_{L\backslash\Lambda} p_{\Lambda|M} \, .$$

- In this case, call the coordinates of π **Plücker coordinates.**

Theorem (Hodge-Pedoe, '47). *A homogeneous polynomial F in the homogeneous coordinate ring $\mathbb{C}[f_I]$ for $\mathbb{P}_\gamma$ is zero on η if and only if it is in the ideal generated by the (right-hand sides of the) relations $(\mathcal{Y}_{L,M})_{(u)}$ (replacing p's with f's).*

Flag Algebra

Definition (Flag Algebra). *The flag algebra* $\mathcal{F}(n)$*, the homogeneous coordinate ring for* $F\ell(n)$*, is the* $\mathbb{C}$*-algebra with generators* $\left\{f_I \mid I \in [n]^d, 1 \leq d < n\right\}$ *and relations*

Alternating $(\mathcal{A}_I)$*: For all* $I \in [n]^d$

$$f_I = \begin{cases} 0 & \text{if the } d \text{ elements of } I \text{ are not distinct.} \\ (-1)^{\ell(\sigma)} f_{\sigma I} & \text{if } \sigma \in \mathfrak{S}_d \text{ “straightens” the } d\text{-tuple } I. \end{cases}$$

Young symmetry $(\mathcal{Y}_{L,M})_{(u)}$*:* $(\forall L, M \subseteq [n],\ u > 0)$ *s.t.* $|M| + u \leq |L| - u$

$$0 = \sum_{\Lambda \subset L, |\Lambda| = u} (-1)^{-\ell(L \setminus \Lambda | \Lambda)} f_{L \setminus \Lambda} f_{\Lambda | M}\,.$$

Commuting $(C_{J,I})$ $\quad (\forall I, J \subsetneq [n])$

$$f_J f_I = f_J f_I\,.$$

A q-Deformation ("Algebra B")

- Fix a field $\mathbb{K}_q$ with a distinguished invertible element q.

Definition (Taft-Towber, '91). *The quantum flag algebra $\mathcal{F}_q(n)$ is the $\mathbb{K}_q$-algebra with generators $\left\{ f_I \mid I \in [n]^d, 1 \le d < n \right\}$ and relations*

Alternating $(\mathcal{A}_I)$*: For all $I \in [n]^d$*

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Young symmetry $(\mathcal{Y}_{L,M})_{(u)}$*: $(\forall L, M \subseteq [n],\ u > 0)$ s.t. $|M| + u \le |L| - u$*

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q-Straightening $(S_{J,I})$ $\quad (\forall I, J \subsetneq [n])$ *s.t.* $|J| \le |I|$

$$f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda| = |J|} (-q)^{\ell(\Lambda | I \setminus \Lambda)} f_{J | I \setminus \Lambda} f_{\Lambda} \,.$$

Key Features

Theorem (T-T, '91). *The quantum flag algebra $\mathcal{F}_q(n)$ satisfies*

- $\mathcal{F}_q(n)$ *reduces to* $\mathcal{F}(n)$ *when* $q \to 1$.
- $\mathcal{F}_q(n)$ *and* $\mathcal{F}(n)$ *are graded domains sharing the same basis and rate of growth.*
- $\mathcal{F}_q(n)$ *is a comodule algebra for the quantum groups* $\mathrm{GL}_q(n)$ *and* $\mathrm{SL}_q(n)$.

$\vdots$

View $\mathcal{F}_q(n)$ as an answer for Manin (for these particular quantum groups). After this theorem, one may safely say, the quantum flag algebra of Taft and Towber is *the correct deformation* for this noncommutative setting.

Different Approach: Noncommutative Flags

- Try to deform the flags themselves, not the algebra of functions on them.
- Hopefully arrive at the same algebra $\mathcal{F}_q(n)$.

Preliminary Steps are Identical

- Fix a skew-field D and a free D-module $V = D^n$ (must choose: left or right?)
- A suitable notion of a (left/right) flag Φ exists.
- A matrix representation $A(\Phi)$ exists.
- $A(\Phi)$ is unique up to (left/right) multiplication by triangular matrices over D.

Questions

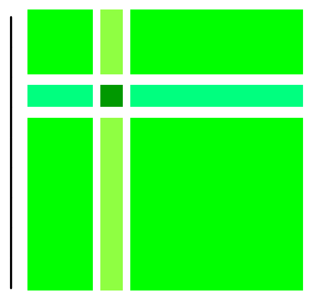
1. Can we find a description of these flags $F\ell(n)$ in terms of coordinates?
2. Can we find a set of relations among the coordinates that characterize $F\ell(n)$?

Quasideterminants

- Fix a matrix $A = (a_{kl}) \in M_n(R)$ for some (noncommutative) ring R. Write A^{ij} for the submatrix built from A by deleting row i and column j.

Definition (Gelfand-Retakh, '91). *The (ij)-quasideterminant $|A|_{ij}$ is defined whenever A^{ij} is invertible, and in that case,*

$$|A|_{ij} \quad =$$



$_{ij}$

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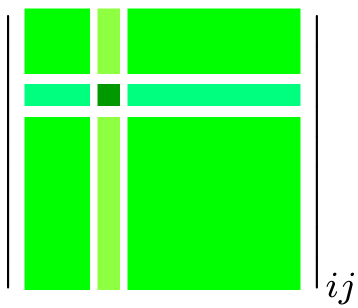

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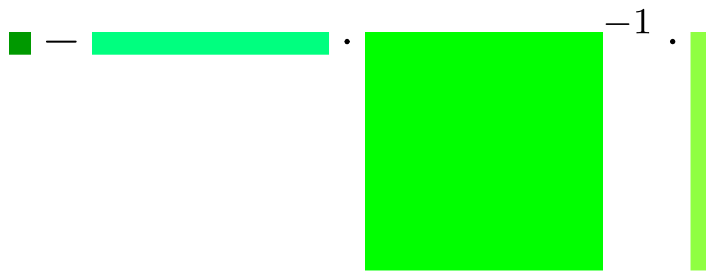

- 2×2 Example: $|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}$.

Quasi-Plücker Coordinates

Definition. *Given an $n \times n$ matrix A and an integer $0 < d < n$, the (right)* ***quasi-Plücker coordinates*** *of size d are given by*

$$\left\{ r_{ij}^{K}(A) := |A_{iK}|_{is} |A_{jK}|_{js}^{-1} \;\middle|\; i, j \in [n],\ K \subseteq [n] \setminus j,\ |K| = d - 1 \right\}$$

Theorem (G-R, '97). *The quasi-Plücker coordinates $r_{ij}^{K}(A)$ satisfy*

- $r_{ij}^{K}(A)$ *is independent of s (appearing in definition above)*
- $r_{ij}^{K}(A \cdot g) = r_{ij}^{K}(A)$ *for all $g \in U_n^+$*
- *If $F(A)$ is some rational function in the a_{ij} which is U_n^+-invariant, then F is a rational function in the $r_{ij}^{K}(A)$.*
- ***Quasi-Plücker Relations*** $(\mathcal{P}_{i,L,M})$*: If $L, M \subseteq [n]$, $i \in [n] \setminus M$, $|M| = |L| - 1$, then:*

$$1 = \sum_{j \in L} r_{ij}^{L \setminus j}(A) \cdot r_{ji}^{M}(A)\,.$$

q-Generic Flags

- Fix D and the flags $F\ell(n)$ over D.

Definition. *A flag Φ is called **q-generic** if there is some matrix representation $A(\Phi)$ whose entries a_{ij} satisfy the defining relations of the quantum matrix algebra $\mathrm{M}_q(n)$ built on a square matrix T. Let X_q denote the set of q-generic flags of $F\ell(n)$.*

Definition. *There is a notion of quantum determinant $\det_q(\text{ - })$ for T and its submatrices $T_{J,K}$. We call the collection $\{\det_q A_I \mid I \subseteq [n]\}$ the (row) **quantum Plücker coordinates** of A (of Φ).*

- Another Key Feature of $\mathcal{F}_q(n)$:

Theorem (T-T, '91). *The quantum flag algebra $\mathcal{F}_q(n)$ is isomorphic to the subalgebra of $\mathrm{M}_q(n)$ generated by the quantum Plücker coordinate functions $\{\det_q T_I \mid I \subseteq [n]\}$ for X_q.*

Quasi $\rightsquigarrow$ Quantum ("Algebra A")

Theorem (G-R, '91 and Krob-Leclerc, '95). *Given any* $i \in I \subseteq [n]$ *and* $j \in J \subseteq [n]$, *there is a **Determinant Factorization:** putting* $B = A_{I,J}$, *we have*

$$\det_q B = (-q)^{\ell(i|I)-\ell(j|J)} \, |B|_{ij} \det_q B^{ij} \,,$$

and the factors commute.

- In particular: $|A_{i\cup K}|_{is}|A_{j\cup K}|_{js}^{-1} = q^{\pm 1}(\det_q A_{i\cup K})(\det_q A_{j\cup K})^{-1}$.
- Try to reconstruct "algebra B" from facts about quasi-Plücker coordinate functions r_{ij}^{K}.

Definition (Algebra A, First Try). *Let* $\tilde{\mathcal{F}}_q(n)$ *be the* $\mathbb{K}_q$*-algebra given by generators* $\tilde{f}_{iK}\tilde{f}_{jK}^{-1}$ *and quasi-Plücker relations* $(\mathcal{P}_{i,L,M})$ *with* $|M| = |L| - 1$*:*

$$1 = \sum_{j\in L} \tilde{f}_{iL\setminus j}\tilde{f}_L^{-1}\tilde{f}_{jM}\tilde{f}_{iM}^{-1} \,.$$

The Vast Gulf

Algebra A:

- Generators are coupled.
- No flag Young symmetry relations.
- No hint of q-straightening relations.

Algebra B:

- Too many Young symmetry relations.

Not evidently a problem yet, but...

- No q-commuting relations..

Step ①

$$0 = \sum_{\Lambda \subseteq L} (-q)^{-\ell(L \setminus \Lambda | \Lambda)} f_{L \setminus \Lambda} f_{\Lambda | M}$$

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$$\downarrow$$

$$0 = \sum_{j \in L} (-q)^{-\ell(L\setminus j|j)} f_{L\setminus j} f_{j|M}$$

- Found a way to express $(\mathcal{Y}_{L,M})_{(u)}$ in terms of particular $(\mathcal{Y}_{I,J})_{(1)}$'s.
- **Novelty:** A proof in the commutative case that does not require $[f_I, f_J] = 0$.

Step ① in Detail

Theorem. *If q is not a root of unity in $\mathbb{K}_q$, then the Young symmetry relation $(\mathcal{Y}_{L,M})_{(u)}$ of $\mathcal{F}_q(n)$ is a consequence of the Young symmetry relations $\big\{(\mathcal{Y}_{L\setminus j,j|M})_{(u-1)} \mid j \in L\big\}$*

Sketch of Proof:

- Write the right-hand sides of the expressions as $Y_{I,J;(v)}$.
- Show
$$Y_{L,M;(u)} = \sum_{j\in L} \frac{(-q)^{2(u-1)-\ell(L^j|j)}}{1+q^2+\cdots+q^{2(u-1)}} Y_{L^j,j|M;(u-1)} \,.$$
- Fix a particular Λ and simply compare the coefficients of $f_{L\setminus\Lambda} f_{\Lambda|M}$ appearing above.

Step ① in Detail

Clearing the denominator on the right-hand side we have

on the left

$$\sum_{k=0}^{u-1}(-q)^{2k-\ell(L\setminus\Lambda|\Lambda)},$$

and on the right

$$\sum_{j\in\Lambda}(-q)^{2(u-1)-\ell(L\setminus j|j)-\ell(L\setminus\Lambda|\Lambda\setminus j)-\ell(\Lambda\setminus j|j)}.$$

But $\ell(L \setminus \Lambda|\Lambda) = \ell(L \setminus \Lambda|\Lambda \setminus j) + \ell(L \setminus j|j) - \ell(\Lambda \setminus j|j)$. We are left needing

$$\sum_{k=1}^{u}(-q)^{2(u-1)-2\ell(\Lambda\setminus\lambda_k|\lambda_k)} = \sum_{k=1}^{u}(-q)^{2(k-1)},$$

which is true because

$$(u-1) = \ell(\Lambda \setminus \lambda_k|\lambda_k) + \ell(\lambda_k|\Lambda \setminus \lambda_k) = \ell(\Lambda \setminus \lambda_k|\lambda_k) + (k-1).$$

□

Step ②

$$\begin{cases} generators: & \tilde{f}_{iK}\tilde{f}_{jK}^{-1} \\ relations: & 1 = \sum_{j \in L} \tilde{f}_{iL\setminus j}\tilde{f}_{L}^{-1}\tilde{f}_{jM}\tilde{f}_{iM}^{-1} \quad (\text{for } |M| = |L| - 1) \end{cases}$$

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$$\downarrow$$

$$\begin{cases} \textit{gens}: & \tilde{f}_{I} \\ \textit{rels}: & 0 = \sum_{j\in L} (-q)^{-\ell(L\setminus j|j)}\tilde{f}_{L\setminus j}\tilde{f}_{j\cup M} \quad (\forall\, 0 \le |M| < |L|-1) \end{cases}$$

- Found a *weak q-commuting* law, allowing me to decouple the generators.
- **Novelty:** Found a Laplace expansion proof of $(\mathcal{P}_{i,L,M})$ that allowed M to have any cardinality smaller than $|L|$.
- **Novelty:** Answers *Question 1:* the quasi-Plücker coordinates are suitable for flags, not just Grassmannians.

Steps ① & ②

Theorem (No Gap for Grassmannians). *In case $\gamma = (d, n)$, the pre–flag algebra $\tilde{\mathcal{F}}_q(\gamma)$ is isomorphic to the Taft-Towber flag algebra $\mathcal{F}_q(\gamma)$.*

Step ③

$$\begin{cases} 0 = \sum_{j \in L} (-q)^{-\ell(L\setminus j|j) - \ell(j|M)} f_{L\setminus j} f_{j \cup M} \\ f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda| = |J|} (-q)^{\ell(\Lambda | I\setminus\Lambda) - \ell(J|I\setminus\Lambda)} f_{J\cup I\setminus\Lambda} f_\Lambda \end{cases}$$

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$$\begin{cases} 0 = \sum_{j \in L} (-q)^{-\ell(L\setminus j|j) - \ell(j|M)} f_{L\setminus j} f_{j \cup M} \\ f_J f_I = q^{|J''| - |J'|} f_I f_J \quad \text{whenever } J \curvearrowright I, \text{ otherwise:} \\ f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda| = |J|} (-q)^{\ell(\Lambda|I\setminus\Lambda) - \ell(J|I\setminus\Lambda)} f_{J \cup I \setminus \Lambda} f_\Lambda \end{cases}$$

- Found a way to rewrite q-straightening relations as q-commuting relations.
- **Novelty:** Found the "missing relations" within $\mathcal{F}_q(n)$: q-commuting relations were known to hold within $\mathrm{M}_q(n)$, but were not included in relations defining $\mathcal{F}_q(n)$.

Step ④

$$\tilde{f}_{jK}\tilde{f}_{iK} = q^{\pm 1}\tilde{f}_{iK}\tilde{f}_{jK}$$

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$$\tilde{f}_{jK}\tilde{f}_{iK} = q^{\pm 1}\tilde{f}_{iK}\tilde{f}_{jK}$$

$$\downarrow$$

$$(\mathcal{C}_{J,I}) : \qquad \tilde{f}_J\tilde{f}_I = q^{|J''|-|J'|}\tilde{f}_I\tilde{f}_J \quad \text{whenever } J \curvearrowright I$$

- From quasi-Plücker relations, managed to bootstrap my way up to *strong* q*-commuting law.*
- **Novelty:** Saw past "algebra B" to what was really going on (amenable determinants).

The (Persistent) Canal

$$(\forall J \not\curvearrowright I) \quad f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda|=|J|} (-q)^{\ell(\Lambda|I\setminus\Lambda)} f_{J|I\setminus\Lambda} f_\Lambda \,.$$

- Comes from fact that quantum determinant has row and column Laplace expansions.
- Looked briefly for such a proof for quasi-Plücker coordinates.
- Preliminary computer calculations suggest there are no more quasi-Plücker coordinate identities to be discovered.
  - *Proving This:* gives a positive answer for *Question 2*.
  - *Disproving This:* moves toward a positive answer for *Question 2* and also (likely) closes the "canal."
- Awaiting closure, we call $\tilde{\mathcal{F}}_q(\gamma)$ a "pre"–flag algebra and turn our attention to other noncommutative settings.

Problem Statement (Refined)

- **Given:** an algebra $\mathcal{M}(n)$ on n^2 generators $T = (t_{ij})$; a T-injective map into a skew field $\mathcal{M}(n) \to D$; "q-generic" relations on the generators T; and a determinant function $\mathrm{Det}(\text{ - })$ for T and its submatrices.
- **Construct:** the homogeneous ring of coordinate functions, the "flag algebra," for the q-generic points of $F\ell(D^n, \gamma)$.

- **Solution:** if Det is an amenable determinant, then the pre–flag algebra for $\mathcal{M}(n)$ is given by generators $\tilde{f}_I$ and relations of the form $(\mathcal{C}_{J,I})$ and $(\mathcal{Y}_{L,M})_{(1)}$ whose precise form comes from the expression of Det in terms of the quasideterminant.

Conjecture. For any amenable setting, the "pre" prefix may be dropped in the construction of quantized Grassmannians. That is, the basis and graded-piece growth are identical to those in the classical algebra for $Gr(d, n)$.

Amenable Determinants

- Fix a $\mathbb{K}$-algebra $\mathcal{M}(n)$ on n^2 generators $T = (t_{ij})$ and "q-generic" relations.

Definition. *Let* Det *be a map from square submatrices of* T *to* $\mathcal{M}(n)$*. Write* $\mathrm{Det}\, T_{R,C} = [T_{R,C}]$ *for short. Call* Det *an* ***amenable determinant*** *if there are measuring functions* $\mathfrak{K}_{\mathfrak{r}}, \mathfrak{K}_{\mathfrak{x}}, \mathfrak{I}_{\mathfrak{r}}, \mathfrak{I}_{\mathfrak{x}} : \mathcal{P}[n] \times \mathcal{P}[n] \to \mathbb{K} \setminus \{0\}$ *associated to* Det *satisfying:*

1. $(\forall r, c \in [n]) \quad [T_{r,c}] = t_{rc}$.
2. $(\forall r, r' \in R) \quad \sum_{c \in C} t_{rc} \frac{\mathfrak{I}_{\mathfrak{x}}(c,C)}{\mathfrak{I}_{\mathfrak{r}}(r',R)} [(T_{R,C})^{r'c}] = [T_{R,C}] \cdot \delta_{rr'}$.
3. $(\forall R' \subseteq R)(\forall C' \subseteq C) \quad [T_{R,C}][T_{R',C'}] = \frac{\mathfrak{K}_{\mathfrak{x}}(C',C)}{\mathfrak{K}_{\mathfrak{r}}(R',R)} [T_{R',C'}][T_{R,C}]$.

Theorem. *If* $\mathcal{M}(n) \to D$ *is a homomorphism to a ring* D *over which (enough) square submatrices of* T *may be inverted, then* $\mathcal{M}(n)$ *has a pre–flag algebra.*

Amenable Examples

- The usual commutative determinant [(?)Cramer, 1750]
- The quantum determinant [Kulish-Sklyanin, ‘82; Manin ‘89]
- The two- parameter quantum determinant [Takeuchi, ‘90]
- The multi-parameter quantum determinant [Artin-Schelter-Tate, ‘91]
- The Yangian determinant [Izergin-Korepin, ‘81; K-S, ‘82]
- The super determinant (Berezinian) [Berezin, ‘83]
- A construction in “quantized Minkowski space” [Frenkel-Jardim, ‘03]

$\vdots$

Future Directions

Continue Manin's Program:

- Build other quantized determinantal varieties using the quasideterminant.
- Build flag varieties for quantized groups not of type A.

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Study the Generic Noncommutative Flag:

- Attempt to exhaust all quasi-Plücker coordinate identities (*Question 2*).
- Study the resulting "noncommutative flag algebra" $\mathbb{K}\langle r_{ij}^{K} \mid \ldots \rangle$.

Any Questions?