A-Wedge and Weak A-Wedge FK-Spaces

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Abstract. The purpose of this paper is to study topological sequence spaces in which the A-transform of coordinate vectors (weakly) converge to zero, where A is a nonnegative regular factorable matrix. We call these spaces (weak) A-wedge FK-spaces.

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1. Introduction

Wedge FK-spaces are defined to be topological sequence spaces in which the coordinate vectors converge to zero. Several characterizations of these spaces are given by Bennett in [1]. Ince [3] studied the topological sequence spaces in which the arithmetic means of coordinate vectors converge to zero. Some characterizations of these spaces may be found in [3]. In this paper, we study A-wedge FK-spaces which are topological sequence spaces in which the A-transform of coordinate vectors converge to zero and give some characterizations of these spaces.

In Section 2, we give notation and terminology while in Section 3 we give some preliminary results. Section 4 deals with A-wedge FK-spaces and Section 5 is devoted to weak A-wedge FK-spaces.

2. Notations and Preliminaries

Let w denote the space of all real- or complex-valued sequences $x = (x_n)_{n=1}^{\infty}$. A K-space is a locally convex vector subspace of w with continuous coordinates. An FK-space is a K-space which is also a Fréchet space (complete linear metric space). A BK-space is a normed FK-space. The basic properties of such spaces can be found in [6], [7], and [9].

By m, c, and c_0 we denote the spaces of all bounded sequences, convergent, and null sequences, respectively. These are BK-spaces under the norm

$\|x\|_\infty = \sup_n |x_n|$. By $l^p, (1 \le p < \infty)$, we shall denote the BK-space of all absolutely p-summable sequences. As usual, l^1 is denoted simply by l.

Throughout the paper δ^j, $j = 1, 2, \ldots$, denotes the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with the one in the j-th position. Let $\phi := l.hull\left\{\delta^k : k \in N\right\}$. The topological dual of X is denoted by X'. A K-space X is said to have the property AD if ϕ is a dense subset of X and it is said to have the property AK, if $X \supset \phi$ and for each $x \in X$, we have $x^{(n)} \to x$ in X, where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k = (x_1, x_2, \ldots, x_n, 0, \ldots)$.

Let X be an FK-space containing ϕ. Following [6],

$$X^f = \left\{x \in w : x = \left(f(\delta^1), f(\delta^2), \ldots, f(\delta^k), \ldots\right), f \in X'\right\},$$

$$X^\beta = \left\{x \in w : \sum_{k=1}^{\infty} x_k y_k \text{ exists for every } y \in X\right\}.$$

Following Bennett [1] we say that a K-space (X, τ) containing ϕ is a wedge space if $\delta^j \to 0$ in (X, τ) and it is a weak wedge space if $\delta^j \to 0$ weakly in X.

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation $y = Ax$ of sequences $x = (x_k)$ by the formula $y_i = (Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j, (i = 1, 2, 3, \ldots)$.

For an FK-space X generated by seminorms $\{q_1, q_2, \ldots\}$, we consider the summability domain

$$X_A = \{x \in w : Ax \in X\}.$$

Then X_A is an FK-space under the seminorms (see e.g., [7] and [8])

(i) $p_i = |x_i|, (i = 1, 2, \ldots)$,

(ii) $h_i(x) = \sup_m \left|\sum_{j=1}^{m} a_{ij} x_j\right|, (i = 1, 2, \ldots)$,

(iii) $(q_i o A)(x) = q_i(Ax), (i = 1, 2, \ldots)$.

Throughout the paper $\{a_n\}$ and $\{b_n\}$ will be positive real-valued sequences such that

(i) $\{a_n\}$ is strictly decreasing with $\lim_n a_n = 0$,

(ii) $\{b_n\}$ is nondecreasing with $\lim_n a_n b_n = 0$ and

(iii) $\lim_n a_n \sum_{k=1}^{n} b_k = 1$.

In this case, a factorable matrix $A = (a_{nk})$ defined by $a_{nk} = a_n b_k$ if $1 \le k \le n$, and zero otherwise, is a regular matrix; that is, $c \subset c_A$ and $\lim_{i\to\infty} x_i = \lim_{i\to\infty} (Ax)_i$ for all $x \in c$.

For example, the sequences $\{b_n\}$ and $\{a_n\}$, with $b_n = \binom{n+\alpha}{n}$ and $a_n = \left(\sum_{k=1}^{n} b_k\right)^{-1}$ satisfy these conditions. More generally, all Riesz (weighted mean) matrices (see, e.g., Section 3.2 of [2]) $A = (a_{nk})$ with $a_{nk} = \frac{p_k}{P_n}$ for $k \le$

n (and zero otherwise), where $\{p_n\}$ is positive nondecreasing, $P_n = \sum_{k=1}^{n} p_k$, and $\frac{p_n}{P_n} \to 0$ (as $n \to \infty$), satisfies these conditions.

We define A-wedge and weak A-wedge using the matrix A given above.

Definition 2.1. Let (X, τ) be an K-space containing ϕ. Then (X, τ) is called an A-wedge space if the sequence of matrix transformation under the matrix A of coordinate vectors converges to zero in (X, τ); that is, the sequence

$$a_n b^{(n)} = a_n \sum_{k=1}^{n} b_k \delta^k = (a_n b_1, a_n b_2, a_n b_3, \ldots, a_n b_n, 0, 0, \ldots),$$

converges to zero in (X, τ).

Observe that these are the rows of the matrix A. Because of the regularity of the factorable matrix A, every wedge space is an A-wedge space but the converse doesn't hold. Note that, for $a_n = \frac{1}{n}$ and $b_n = 1$, the FK-spaces $c_0, c, m, l^p, (p > 1)$ as well as the space of bounded sequences of bounded variation bv are A-wedge spaces but not wedge spaces.

Definition 2.2. A K-space X containing ϕ is called a weak A-wedge space if the sequence

$$a_n b^{(n)} = a_n \sum_{k=1}^{n} b_k \delta^k = (a_n b_1, a_n b_2, a_n b_3, \ldots, a_n b_n, 0, 0, \ldots)$$

converges weakly to zero in (X, τ).

Every weak wedge space is a weak A-wedge space but the converse doesn't hold. For example, $bv_0 = bv \cap c_0$ is A-wedge and weak A-wedge space but it is neither wedge nor weak wedge.

3. Preliminary Results

In this section, we give preliminary results that we need in later sections.

Theorem 3.1. *(i) A closed subspace, containing ϕ, of an A-wedge (resp., weak A-wedge) FK-space is an A-wedge (resp., weak A-wedge) FK-space.*

(ii) An FK-space which contains an A-wedge (resp., weak A-wedge) FK-space must be an A-wedge (resp., weak A-wedge) FK-space.

(iii) A countable intersection of A-wedge (resp., weak A-wedge) FK-spaces is an A-wedge (resp., weak A-wedge) FK-space.

Proof. The elementary properties of FK-spaces yield that the proof (see e.g., Chapter 4 of [6]). □

Definition 3.2. Let $z \in w$. The space $V_0^b(z)$ consists of all sequences $x \in c_0$ for which

$$\sum_{n=1}^{\infty} |z_n| \left|\Delta(\frac{x_n}{b_n})\right| < \infty$$

Lemma 3.3. *Under the norm*

$$\|x\|_{V_0^b(z)} = \sum_{n=1}^{\infty} |z_n| \left|\Delta(\frac{x_n}{b_n})\right| + \sup_n |x_n| \tag{3.1}$$

$V_0^b(z)$ is an FK-AK space.

Proof. Define the matrix V by $v_{nk} = \frac{z_n}{b_k}$ for $k = n$, $v_{nk} = \frac{-z_n}{b_k}$ for $k = n+1$, and $v_{nk} = 0$ elsewhere. Since $V_0^b(z) = l_V \cap c_0$, then $V_0^b(z)$ is an FK-space under the norm (3.1) by Theorems 4.3.1 of [6]. For all $x \in V_0^b(z)$, we have

$$\left\|x - x^{(m)}\right\|_{V_0^b(z)} = \sum_{n=m+1}^{\infty} |z_n| \left|\Delta(\frac{x_n}{b_n})\right| + \sup_{n\geq m} |x_n|.$$

Since $x \in V_0^b(z)$, then $\|x - x^{(m)}\|_{V_0^b(z)} \to 0$ as $m \to \infty$. Hence $V_0^b(z)$ is an AK-space. □

Theorem 3.4. *The space $V_0^b(z)$ is an A-wedge space if and only if $z_n = o(\frac{1}{a_n})$.*

Proof. We have for each n,

$$\left\|a_n b^{(n)}\right\|_{V_0^b(z)} = a_n|z_n| + a_n \sup_{k\leq n} |b_k| = a_n |z_n| + a_n b_n.$$

Since $\lim_n a_n b_n = 0$, the sequence $a_n b^{(n)}$ converges to zero in $V_0^b(z)$ if and only if $\lim_n a_n z_n = 0$. □

Definition 3.5. For $z_n = \frac{1}{a_n}$, let $h_a^b = V_0^b(z)$.

Lemma 3.6. *The spaces h_a^b is an FK-AK space under the norm*

$$\|x\|_{h_a^b} = \sum_{n=1}^{\infty} \frac{1}{a_n} \left|\Delta(\frac{x_n}{b_n})\right|. \tag{3.2}$$

Proof. By Lemma 3.3, we know that the spaces $h_a^b = V_0^b(\frac{1}{a_n})$ is an FK-AK space under the $V_0^b(\frac{1}{a_n})$ norm

$$\|x\|_{V_0^b(\frac{1}{a_n})} = \sum_{n=1}^{\infty} \frac{1}{a_n} \left|\Delta(\frac{x_n}{b_n})\right| + \sup_n |x_n|.$$

We show that the h_a^b and $V_0^b(\frac{1}{a_n})$ norms are equivalent by proving

$$\|x\|_{h_a^b} \leq \|x\|_{V_0^b(\frac{1}{a_n})} \leq (1+M) \|x\|_{h_a^b}, \tag{3.3}$$

where $M = \sup_n a_n b_n$. The first inequality is obvious. Let $x \in h_a^b$ and $n = 1, 2, 3, \ldots$. Since $\lim_m \frac{x_m}{b_m} = 0$, we have $\frac{x_n}{b_n} = \sum_{j=n}^{\infty} \Delta(\frac{x_j}{b_j})$. Thus

$$\frac{|x_n|}{b_n} = \left|\frac{x_n}{b_n}\right| \leq \sum_{j=n}^{\infty} \left|\Delta(\frac{x_j}{b_j})\right| = \sum_{j=n}^{\infty} a_j \frac{1}{a_j} \left|\Delta(\frac{x_j}{b_j})\right|.$$

Since $\{a_n\}$ is decreasing,

$$\frac{|x_n|}{b_n} \leq \sum_{j=n}^{\infty} a_n \frac{1}{a_j} \left|\Delta(\frac{x_j}{b_j})\right| = a_n \sum_{j=n}^{\infty} \frac{1}{a_j} \left|\Delta(\frac{x_j}{b_j})\right|.$$

Thus $\frac{|x_n|}{a_n b_n} \leq \sum_{j=n}^{\infty} \frac{1}{a_j} \left|\Delta(\frac{x_j}{b_j})\right| \leq \sum_{j=1}^{\infty} \frac{1}{a_j} \left|\Delta(\frac{x_j}{b_j})\right| = \|x\|_{h_a^b}$, which implies $\sup_m |x_m| \leq M \sup_m \frac{|x_m|}{a_m b_m} \leq M \|x\|_{h_a^b}$. This is sufficient to prove the second inequality of (3.3). □

Theorem 3.7. *The space h_a^b is not an A-wedge space.*

Proof. Since $h_a^b = V_0^b(z)$ for $z = \frac{1}{a_n}$, the result follows from Theorem 3.4. □

The following spaces relate to duality of h_a^b.

Lemma 3.8. *The spaces $(\sigma_0)_a^b$ and $(\sigma_\infty)_a^b$ given by*

$$(\sigma_0)_a^b = \left\{x \in w : \lim_n a_n \sum_{k=1}^{n} b_k x_k = 0\right\},$$

and

$$(\sigma_\infty)_a^b = \left\{x \in w : \sup_n a_n \left|\sum_{k=1}^{n} b_k x_k\right| < \infty\right\}$$

are FK- spaces under the norm

$$\|x\|_{(\sigma_\infty)_b^a} = \sup_n a_n \left|\sum_{k=1}^{n} b_k x_k\right|. \tag{3.4}$$

In addition $(\sigma_0)_a^b$ is an AK-space under the norm (3.4).

Proof. Define the sequence Ax by $(Ax)_n = a_n \sum_{k=1}^{n} b_k x_k$. Then $(\sigma_0)_a^b = (c_0)_A$ and $(\sigma_\infty)_a^b = m_A$. Since A is a triangular matrix, then $(\sigma_0)_a^b$ and $(\sigma_\infty)_a^b$ are FK-spaces under the norm (3.4) by Theorem 4.3.12 in [6]. For $x \in (\sigma_0)_a^b$ and $\epsilon > 0$, there exists N such that $\sup_n a_n \left|\sum_{k=1}^{n} b_k x_k\right| < \epsilon$ whenever

$n > N$. To show that $(\sigma_0)^b_a$ is an AK-space, we show $\left\|x - x^{(m)}\right\|_{(\sigma_\infty)^b_a} = \sup_{n\geq m} a_n \left|\sum_{k=m+1}^{n} b_k x_k\right| < 2\epsilon$ whenever $m > N$. Let $n \geq m > N$. Then

$$a_n \left|\sum_{k=m+1}^{n} b_k x_k\right| \leq a_n \left|\sum_{k=1}^{n} b_k x_k\right| + a_n \left|\sum_{k=1}^{m} b_k x_k\right| < \epsilon + a_m \left|\sum_{k=1}^{m} b_k x_k\right| < 2\epsilon.$$

□

Lemma 3.9. $(h^b_a)^f = (h^b_a)^\beta = (h^b_a)' = (\sigma_\infty)^b_a$, *and*
$((\sigma_0)^b_a)^f = ((\sigma_0)^b_a)^\beta = ((\sigma_0)^b_a)' = h^b_a$.

Proof. Since h^b_a is an AK-space, then $(h^b_a)^f = (h^b_a)^\beta = (h^b_a)'$ by Theorems 7.2.7 and 7.2.12 of [6]. Let $x \in (\sigma_\infty)^b_a$ and $y \in h^b_a$. Because of $y \in h^b_a$, we have $\lim_n \frac{y_n}{a_n b_n} = 0$. By Abel summation

$$\sum_{k=m}^{n} x_k y_k = \sum_{k=m}^{n-1} a_k s_k \frac{1}{a_k} \Delta(\frac{y_k}{b_k}) + a_n s_n \frac{y_n}{a_n b_n} - a_{m-1} s_{m-1} \frac{y_m}{a_{m-1} b_m},$$

where $s_k = \sum_{j=1}^{k} b_j x_j$. Since $\|x\|_{(\sigma_\infty)^b_a} = \sup_k a_k |s_k|$, we have, as $m, n \to \infty$,

$$\left|\sum_{k=m}^{n} x_k y_k\right| \leq \|x\|_{(\sigma_\infty)^b_a} \left(\sum_{k=m}^{n-1} \frac{1}{a_k} \left|\Delta(\frac{y_k}{b_k})\right| + \left|\frac{y_n}{a_n b_n}\right| + \left|\frac{y_m}{a_{m-1} b_m}\right|\right) \to 0.$$

Hence $(\sigma_\infty)^b_a \subset (h^b_a)^\beta$.

Conversely let $u \in (h^b_a)^f$. That is, $u_k = f(\delta^k)$, where $f \in (h^b_a)'$. Then

$$\begin{aligned} a_n \left|\sum_{k=1}^{n} b_k u_k\right| &= \left|a_n \sum_{k=1}^{n} b_k f(\delta^k)\right| = \left|f(a_n \sum_{k=1}^{n} b_k \delta^k)\right| \\ &\leq \|f\| \left\|a_n b^{(n)}\right\|_{h^b_a} = \|f\|. \end{aligned}$$

So $u \in (\sigma_\infty)^b_a$.

For the second part, since h^b_a is an FK-AK-space, we have

$$h^b_a \subset ((\sigma_0)^b_a)^f = ((\sigma_0)^b_a)^\beta = ((\sigma_0)^b_a)'.$$

The reverse inclusion follows from Corollary 1(iii) of [5]. According to that Corollary,

$$((\sigma_0)^b_a)^\beta = \left\{x \in w : \sum_{n=1}^{\infty} \frac{1}{a_n} \left|\Delta(\frac{x_n}{b_n})\right| < \infty,\ \sup \left|\frac{x_n}{a_n b_n}\right| < \infty\right\}.$$

Since we assume $\lim_n a_n b_n = 0$, we have

$$((\sigma_0)^b_a)^\beta \subset \left\{x \in w : \sum_{n=1}^{\infty} \frac{1}{a_n} \left|\Delta(\frac{x_n}{b_n})\right| < \infty,\ \lim |x_n| = 0\right\} = h^b_a.$$

□

Definition 3.10. Let $s = \{s_n\}_{n=1}^{\infty}$ always denote a strictly increasing sequence of integers with $s_1 = 0$. We shall be interested in spaces of the form:

$$c_a^b\,|s| = \left\{ x \in w : \lim_n x_n = 0 \;\text{ and }\; \sup_n \sum_{j=s_n+1}^{s_{n+1}} \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right| < \infty \right\}.$$

Lemma 3.11. *For any strictly increasing sequence* $s = \{s_n\}_{n=1}^{\infty}$*, the space* $c_a^b\,|s|$ *is an* FK*-space with the norm*

$$\|x\|_{c_a^b|s|} = \sup_n \sum_{j=s_n+1}^{s_{n+1}} \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right|. \tag{3.5}$$

Furthermore,

$$h_a^b \subset c_a^b\,|s| \subset c_0 \subset m.$$

Proof. The proof is essentially the same to that of Lemma 3.3 □

Similarly, the proof of the following is like that of Theorem 3.4

Theorem 3.12. *For any strictly increasing sequence* $s = \{s_n\}_{n=1}^{\infty}$*, the space* $c_a^b\,|s|$ *is not an* A*-wedge space.*

Lemma 3.13. *Assume that* $\lim_j a_j z_j^n = 0$, $(n = 1, 2, 3, \ldots)$. *Then there exists* $z \in w$ *with* $\lim_j a_j z_j = 0$ *such that* $\lim_j \frac{z_j^n}{z_j} = 0$, $(n = 1, 2, 3, \ldots)$.

Moreover, for any such z, we have $V_0^b(z) \subset \bigcap_{n=1}^{\infty} V_0^b(z^n)$.

Proof. Since $\lim_j a_j z_j^n = 0$, $(n = 1, 2, 3, \ldots)$, we may choose a sequence $\{j_k\}_{k=1}^{\infty}$ of positive integers such that

$$1 = j_0 < j_1 < j_2 < \cdots < j_k < \cdots$$

and

$$\max_{1 \le n \le k} \left| a_j z_j^n \right| < \frac{1}{4^k}, \; (j \ge j_k\;; k = 1, 2, \ldots).$$

Define $z \in w$ as follows :

$$z_j = \frac{1}{2^k a_j}\;, (\; j_k \le j < j_{k+1};\; k = 0, 1, 2, \ldots).$$

Then $\lim_j a_j z_j = 0$ and, fixing n, we get

$$\left| \frac{z_j^n}{z_j} \right| = \left| \frac{a_j z_j^n}{a_j z_j} \right| < \frac{1}{2^k} \text{ whenever } j_k \le j < j_{k+1} \text{ and } k \ge n. \tag{3.6}$$

Thus $\lim_j \frac{z_j^n}{z_j} = 0$ for each n. The second part of the proof follows from inequality (3.6). □

Remark 3.14. In Lemma 3.13 above, we may choose the sequence $\{j_k\}$ of positive integers such that

$$a_j b_j < \frac{1}{2^k} \text{ for } j \geq j_k$$

and

$$a_{j_{k+1}} < \frac{1}{2} a_{j_k} \text{ for } k = 0, 1, 2, \ldots.$$

Then

$$z_j = \frac{1}{2^k a_j} \text{ for } \; j_k \leq j < j_{k+1}; \; k = 0, 1, 2, \ldots$$

and

$$\frac{z_j}{b_j} = \frac{a_j z_j}{a_j b_j} > 1 \text{ for } \; j_k \leq j < j_{k+1}; \; k = 0, 1, 2, \ldots.$$

Lemma 3.15. *Assume that* $\lim_j a_j z_j^0 = 0$ *for some* $z^0 \in w$. *Then there exists* $z' \in w$ *with* $\lim_j a_j z'_j = 0$ *such that* $V_0^b(z') \subset V_0^b(z^0)$ *and* $V_0^b(z')$ *is an* FK*-space under the norm*

$$\|x\|' = \sum_{j=1}^{\infty} z'_j \left| \Delta(\frac{x_j}{b_j}) \right|.$$

Proof. Let $z^n = z^0$ for $n = 1, 2, 3, \ldots$ and assume the sequence $\{j_k\}$ of positive integers is chosen according to the proof of Lemma 3.13 and Remark 3.14 above. Then, by Lemma 3.13, there exists a sequence $z' \in w$ with $\lim_j a_j z'_j = 0$ such that $V_0^b(z') \subset V_0^b(z^0)$. It is sufficient to show that, for all $x \in V_0^b(z')$, we have

$$\sup_n |x_n| \leq 2 \|x\|'. \tag{3.7}$$

Although the sequence z'_j may not be increasing for all $j = 1, 2, 3 \ldots$, it has an increasing sawtooth shape as observed by (3.8), (3.9), and (3.10) below: Since $\{a_j\}$ is strictly decreasing, for each $i = 0, 1, 2, \ldots$

$$z'_j = \frac{1}{2^i a_j} \text{ is strictly increasing for } j_i \leq j < j_{i+1} \tag{3.8}$$

and for each $j_i \leq n < j_{i+1}$,

$$z'_{j_i} = \frac{1}{2^i a_{j_i}} \leq \frac{1}{2^i a_n} = z'_n < \frac{1}{2^i a_{j_{i+1}}} = \frac{2}{2^{i+1} a_{j_{i+1}}} = 2 z'_{j_{i+1}}. \tag{3.9}$$

Also $\{z'_{j_i}\}_{i=0}^{\infty}$ is positive and strictly increasing because $a_{j_{i+1}} < \frac{1}{2} a_{j_i}$; that is,

$$z'_{j_i} = \frac{1}{2^i a_{j_i}} < \frac{1}{2^{i+1} a_{j_{i+1}}} = z'_{j_{i+1}}. \tag{3.10}$$

Let $x \in V_0^b(z')$ and $j_k \leq n < j_{k+1}$. Define $\Delta_i = \frac{x_{j_i}}{b_{j_i}} - \frac{x_{j_{i+1}}}{b_{j_{i+1}}}$. $(i = 0, 1, 2, \ldots)$. Since $\lim_j \frac{x_j}{b_j} = 0$, and hence $\lim_i \frac{x_{j_i}}{b_{j_i}} = 0$, we have

$$\frac{x_n}{b_n} = \sum_{j=n}^{j_{k+1}-1} \Delta(\frac{x_j}{b_j}) + \sum_{i=k+1}^{\infty} \Delta_i.$$

Thus
$$\frac{|x_n|}{b_n} \leq \sum_{j=n}^{j_{k+1}-1} \left|\Delta(\frac{x_j}{b_j})\right| + \sum_{i=k+1}^{\infty} |\Delta_i| . \qquad (3.11)$$
By (3.8) and (3.10), we have
$$\frac{|x_n|}{b_n} \leq \frac{1}{z'_n} \sum_{j=n}^{j_{k+1}-1} z'_j \left|\Delta(\frac{x_j}{b_j})\right| + \frac{1}{z'_{j_{k+1}}} \sum_{i=k+1}^{\infty} z'_{j_i} |\Delta_i| . \qquad (3.12)$$
By (3.9), we have
$$\frac{|x_n|}{b_n} \leq \frac{1}{z'_n} \sum_{j=n}^{j_{k+1}-1} z'_j \left|\Delta(\frac{x_j}{b_j})\right| + \frac{2}{z'_n} \sum_{i=k+1}^{\infty} z'_{j_i} |\Delta_i| .$$
By Remark 3.14, we have $\frac{z'_n}{b_n} > 1$. Thus
$$|x_n| \leq \frac{z'_n |x_n|}{b_n} \leq \sum_{j=n}^{j_{k+1}-1} z'_j \left|\Delta(\frac{x_j}{b_j})\right| + 2 \sum_{i=k+1}^{\infty} z'_{j_i} |\Delta_i| . \qquad (3.13)$$
Expanding $z'_{j_i} |\Delta_i|$ in (3.13), we have by (3.8),
$$z'_{j_i} |\Delta_i| = z'_{j_i} \left| \frac{x_{j_i}}{b_{j_i}} - \frac{x_{j_{i+1}}}{b_{j_{i+1}}} \right| \leq \sum_{j=j_i}^{j_{i+1}-1} z'_{j_i} \left|\Delta\left(\frac{x_j}{b_j}\right)\right| \leq \sum_{j=j_i}^{j_{i+1}-1} z'_j \left|\Delta\left(\frac{x_j}{b_j}\right)\right| .$$
Inserting this into inequality (3.13),
$$|x_n| \leq \sum_{j=n}^{j_{k+1}-1} z'_j \left|\Delta(\frac{x_j}{b_j})\right| + 2 \sum_{i=k+1}^{\infty} \sum_{j=j_i}^{j_{i+1}-1} z'_j \left|\Delta\left(\frac{x_j}{b_j}\right)\right| \leq 2 \sum_{j=n}^{\infty} z'_j \left|\Delta(\frac{x_j}{b_j})\right| ,$$
gives us the desired result for all n
$$|x_n| \leq 2 \sum_{j=n}^{\infty} z'_j \left|\Delta(\frac{x_j}{b_j})\right| \leq 2 \|x\|' .$$
□

4. A-Wedge FK-Spaces

In this section we study A-wedge FK-space. First we give the fundamental characterization of A-wedge spaces.

Theorem 4.1. *The following conditions are equivalent for an FK-space (X, τ) :*

(i) X is an A-wedge space,

(ii) X contains $V_0^b(z)$ for some $z \in w$ such that $z_j = o(\frac{1}{a_j})$,

(iii) X contains $c_a^b |s|$ for some strictly increasing sequence s and the inclusion mapping $I : (c_a^b |s| , \|.\|_{c_a^b |s|}) \to (X, \tau)$ is compact,

(iv) X contains h_a^b and the inclusion mapping $I : (h_a^b, \|.\|_{h_a^b}) \to (X, \tau)$ is compact.

Proof. $(ii) \Longrightarrow (i)$ follows from Theorems 3.1 and 3.4.

$(i) \Longrightarrow (ii)$. Let $\{q_n\}_{n=1}^{\infty}$ be a defining family of seminorms for the topology τ and let

$$z_j^n := q_n(b^{(j)}) = q_n(\sum_{i=1}^{j} b_i\delta^i);\ j,n = 1,2,\ldots$$

Suppose $x \in \bigcap_{n=1}^{\infty} V_0^b(z^n)$. Then $x \in c_0$ and for all $n = 1,2,3,\ldots$ we have

$$\sum_{j=1}^{\infty} \left|z_j^n\right| \left|\Delta(\frac{x_j}{b_j})\right| = \sum_{j=`1}^{\infty} q_n(b^{(j)}) \left|\Delta(\frac{x_j}{b_j})\right| = \sum_{j=1}^{\infty} q_n((\Delta(\frac{x_j}{b_j}))b^{(j)}) < \infty.$$

Since X complete, $\sum_{j=1}^{\infty}(\Delta(\frac{x_j}{b_j}))b^{(j)}$ converges in (X,τ). Coordinatewise, it converges to x, since, for $k \geq i$, the i^{th} coordinate of $\sum_{j=1}^{k}(\Delta(\frac{x_j}{b_j}))b^{(j)}$ is $x_i - \frac{b_i}{b_{k+1}}x_{k+1}$, which tends to x_i as $k \to \infty$. Since X is a K-space, $x = \sum_{j=1}^{\infty}(\Delta(\frac{x_j}{b_j}))b^{(j)} \in X$. So $\bigcap_{n=1}^{\infty} V_0^b(z^n) \subset X$. Since X is an A-wedge space, $\lim_j q_n(a_n b^{(n)}) = \lim_j a_j z_j^n = 0, (n = 1,2,3,\ldots)$. By Lemma 3.13 we may choose $z \in w$ such that $\lim_j a_j z_j = 0$ and $V_0^b(z) \subset \bigcap_{n=1}^{\infty} V_0^b(z^n) \subset X$.

$(ii) \Longrightarrow (iii)$. Let $V_0^b(z) \subset X$ for some z with $z_j = o(\frac{1}{a_j})$. By Lemma 3.15, $V_0^b(z)$ has a subset $V_0^b(z')$ with norm $\|x\|' = \sum_{j=1}^{\infty} z_j' \left|\Delta(\frac{x_j}{b_j})\right|$. Take $s_0 = 0$ and $\{s_n\}_{n=1}^{\infty}$ denotes a strictly increasing sequence of integers satisfying,

$$\left|a_j z_j'\right| \leq \frac{1}{2^n},\ j \geq s_n;\ (n = 1,2,3,\ldots). \tag{4.1}$$

Let $x \in c_a^b\,|s|$. Suppose $m,p \in N$, $m \leq p$. Then by (4.1) we get

$$\sum_{j=s_m+1}^{s_{p+1}} \left|z_j'\right| \left|\Delta(\frac{x_j}{b_j})\right| \leq \sum_{n=m}^{p}\sum_{j=s_n+1}^{s_{n+1}} a_j \left|z_j'\right| \frac{1}{a_j}\left|\Delta(\frac{x_j}{b_j})\right| \leq \|x\|_{c_a^b|s|} \sum_{n=m}^{p} \frac{1}{2^n} \to 0$$

Thus $x \in V_0^b(z')$ and $c_a^b\,|s| \subset V_0^b(z') \subset X$.

Now assume that $U \subset c_a^b\,|s|$ be such that $\|x\|_{c_a^b|s|} \leq M$ for all $x \in U$. It is clear that $U \subset V_0^b(z')$. For $s_n \leq m < s_{n+1}$ and $x \in U$, by (4.1) we get

$$\begin{aligned}\left\|x - x^{(m)}\right\|' &= \sum_{j=m+1}^{\infty} \left|z_j'\right| \left|\Delta(\frac{x_j}{b_j})\right| \leq \sum_{i=n}^{\infty}\sum_{j=s_i+1}^{s_{i+1}} a_j \left|z_j'\right| \frac{1}{a_j}\left|\Delta(\frac{x_j}{b_j})\right| \\ &\leq \|x\|_{c_a^b|s|} \sum_{i=m}^{\infty} \frac{1}{2^i} \to 0 \text{ (uniformly on } U), (m \to \infty).\end{aligned}$$

Hence $x^{(m)} \to x$, $(m \to \infty)$, in $(V_0^b(z'),\|x\|')$ uniformly on U. Since $V_0^b(z')$ is an AK-space, then by Lemma 2 of [1], U is relatively compact in

$V_0^b(z')$. Since the inclusion mapping $I : V_0^b(z') \to X$ is continuous, $I(U) = U$ is relatively compact in X. Thus the inclusion mapping $I : (c_a^b\,|s|\,, \|.\|_{c_a^b|s|}) \to (X, \tau)$ is compact.

$(iii) \Longrightarrow (iv)$. Because the inclusion mapping $I : h_a^b \to c_a^b\,|s|$ is continuous, the proof is trivial.

$(iv) \Longrightarrow (i)$. Since $\left\|a_n b^{(n)}\right\|_{h_a^b} = 1$, the set $B = \{a_n b^{(n)} : n = 1, 2, \ldots\}$ is a bounded subset of h_a^b. In addition since the inclusion mapping $I : (h_a^b, \|.\|_{h_a^b}) \to (X, \tau)$ is compact, $I(B) = B$ is τ-relatively compact in X. Hence, by Theorem 2.3.11 of [4], since $a_n b^{(n)} \to 0$ in w, we have $a_n b^{(n)} \to 0$ in (X, τ). □

Theorem 4.2. *Suppose $z \in (\sigma_0)_a^b$. Then $z^\beta := \left\{x \in w : \sum_{k=1}^{\infty} z_k x_k \text{ converges}\right\}$ is an A-wedge FK-space.*

Proof. The space z^β is an FK-space with seminorms $p_i(x) = |x_i|$, $(i = 1, 2, \ldots)$, and $P_0(x) = \sup_m \left|\sum_{k=1}^{m} z_k x_k\right|$, by Theorem 4.3.7 in [6]. Let $z \in (\sigma_0)_a^b$. Since, for each $i = 1, 2, 3, \ldots$,

$$p_i(a_n b^{(n)}) = \left\{\begin{array}{ll} a_n b_i & , \text{ if } i \leq n \\ 0 & , \text{ if } i > n \end{array}\right\} \leq a_n b_n \to 0, \ (n \to \infty),$$

it remains to show that $P_0(a_n b^{(n)}) = \max_{1 \leq m \leq n} a_n \left|\sum_{k=1}^{m} z_k b_k\right| \to 0$, $(n \to \infty)$. Since the sequence $\{a_n\}$ is decreasing and $\lim_n a_n = 0$, we choose a sequence $\{\zeta_N\}$ of natural numbers for which $\frac{a_{\zeta_{N-1}}}{a_{\zeta_N}} \geq 2^N$ and $a_\zeta \left|\sum_{i=1}^{\zeta} z_i b_i\right| \leq 2^{-N}$, $(\forall \zeta \geq \zeta_N)$.

Then for any $N > 2$, take $n \geq \zeta_N$. We have

(i) $a_n \left|\sum_{k=1}^{m} z_k b_k\right| = \frac{a_n}{a_m} a_m \left|\sum_{k=1}^{m} z_k b_k\right| \leq 2^{-(N-1)}$, for $\zeta_{N-1} \leq m \leq n$

(ii) $a_n \left|\sum_{k=1}^{m} z_k b_k\right| \leq \frac{a_{\zeta_N}}{a_{\zeta_{N-1}}} a_m \left|\sum_{k=1}^{m} z_k b_k\right| \leq 2^{-N} \sup_m a_m \left|\sum_{k=1}^{m} z_k b_k\right|$, for $m < \zeta_{N-1}$,

Hence

$$P_0(a_n b^{(n)}) = \max\left\{2^{-(N-1)}, \quad 2^{-N} \sup_{m < \zeta_{N-1}} a_m \left|\sum_{k=1}^{m} z_k b_k\right|\right\}$$

which tends to zero as $n \to \infty$. □

Now we give the following result.

Corollary 4.3. *The intersection of all A-wedge FK-spaces is h_a^b.*

Proof. Let the intersection of all A-wedge FK-spaces be Y. By Theorem 4.1 $(i) \Longrightarrow (iv)$, Theorem 4.2, and Lemma 3.9 we have

$$h_a^b \subset Y \subset \bigcap \{ z^\beta : z \in (\sigma_0)_a^b \} = \{ (\sigma_0)_a^b \}^\beta = h_a^b.$$

Hence the result. □

Corollary 4.4. $\bigcap_{z_n = o(\frac{1}{a_n})} V_0^b(z) = h_a^b.$

Proof. By Theorem 3.4, if $z_n = o(\frac{1}{a_n})$, then $V_0^b(z)$ is an A-wedge FK-space. Also by Theorem 4.1(iv), $h_a^b \subset \bigcap_{z_n = o(\frac{1}{a_n})} V_0^b(z)$. The reverse inclusion is obtained by Theorem 4.1(ii) and Corollary 4.3. □

Remark 4.5. By Corollary 4.3, there is no smallest A-wedge space.

5. Weak A-Wedge FK-Spaces

In this section, we deal with weak A-wedge FK-spaces

Theorem 5.1. *An FK-space (X, τ) is a weak A-wedge space if and only if X contains h_a^b and the inclusion mapping $I : (h_a^b, \|.\|_{h_a^b}) \to (X, \tau)$ is weakly compact.*

Proof. **Necessity:** Let (X, τ) be a weak A-wedge space. Then for all $f \in X'$,

$$f\left(a_n b^{(n)}\right) = f\left(a_n \sum_{k=1}^{n} b_k \delta^k\right) = a_n \sum_{k=1}^{n} b_k f(\delta^k) \to 0, (n \to \infty), \tag{5.1}$$

and thereby $\{f(\delta^k)\} \in (\sigma_\infty)_a^b$. Thus $X^f \subset (\sigma_\infty)_a^b$. Since $(\sigma_\infty)_a^b = (h_a^b)^f$ and h_a^b is an AD-space, then $h_a^b \subset X$ by Theorem 8.6.1 in [6]. This inclusion requires that the inclusion mapping $I : h_a^b \to X$ is continuous. Because h_a^b is an AK-space, we have for all $x \in h_a^b$ and $f \in X'$ that

$$f\left(\sum_{k=1}^{\infty} x_k \delta^k\right) = \sum_{k=1}^{\infty} x_k f(\delta^k) = \langle I(x),\ f\ \rangle = \langle\ x,\ f(\delta^k)\ \rangle.$$

On the other hand, $\{f(\delta^k)\} \in (\sigma_0)_a^b$ for all $f \in X'$ by (5.1).

Thus, since $\sigma\left(((\sigma_0)_a^b)', (\sigma_0)_a^b\right) = \sigma\left(h_a^b, (\sigma_0)_a^b\right)$, then the mapping $I : (h_a^b, \sigma\left(h_a^b, (\sigma_0)_a^b\right)) \to (X, \sigma\left(X, X'\right))$ is continuous. By the Banach-Alaoğlu Theorem (Theorem 1, Section 13.3 of [7]), the set $B = \left\{x \in h_a^b :\ \|.\|_{h_a^b} \le 1\right\}$ is $\sigma\left(h_a^b, (\sigma_0)_a^b\right)$-compact and hence $I(B) = B$ is $\sigma\left(X, X'\right)$-compact. Consequently the inclusion mapping $I : (h_a^b, \|.\|_{h_a^b}) \to (X, \tau)$ is weakly compact.

Sufficiency: Let $h_a^b \subset X$ and the inclusion mapping $I : (h_a^b, \|.\|_{h_a^b}) \to (X, \tau)$ be weakly compact. Then $B = \left\{x \in h_a^b :\ \|.\|_{h_a^b} \le 1\right\}$ is $\sigma\left(X, X'\right)$-relatively compact. Hence, by Theorem 2.3.11 of [4], $a_n b^{(n)} \to 0, (n \to \infty)$, in $\sigma\left(X, X'\right)$ since it converges to zero in w. □

Now we have the following

Corollary 5.2. *The intersection of all weak A-wedge FK-spaces is h_a^b.*

Proof. The proof is like that of Corollary 4.3 by using Theorems 5.1 and 4.2. □

Using Theorem 3.1 for weak A-wedge FK-spaces, we obtain the following.

Remark 5.3. There is no smallest weak A-wedge FK- space.

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