STRONG SUMMABILITY IN FRÉCHET SPACES
WITH APPLICATIONS TO FOURIER SERIES

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STRONG SUMMABILITY IN FRÉCHET SPACES

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Abstract. This paper examines strong Cesàro summability and strong Cesàro sectional boundedness of order $1 \leq r < \infty$ in Banach and Fréchet spaces $E$. The major result shows these topological properties of $E$ to be equivalent to multiplier properties of the form $E = (dv_r \cap c_0) \cdot E$ and $E = dv_r \cdot E$, where $dv_r$ is the space of sequences of dyadic variation of order $r$ defined in this paper. These multiplier results show that several classical spaces of Fourier series have these properties. This introduces a new form of convergence in norm for Fourier series. The space $L^1_{2\pi}$, for example, has strong Cesàro summability of all orders $1 \leq r < \infty$. Fejér’s Theorem states that for all $f \in L^1_{2\pi}$,

$$\frac{1}{n+1} \left\| \sum_{k=0}^{n} s^k f - f \right\|_{L^1} = o(1), \quad (n \to \infty),$$

where $s^k f$ is the $k$th partial sum of the Fourier series of $f$; since the dual of $L^1_{2\pi}$ is $L^\infty_{2\pi}$, this is equivalent to

$$\sup_{\| g \|_{L^\infty} \leq 1} \frac{1}{n+1} \left| \sum_{k=0}^{n} \int_0^{2\pi} g \cdot (s^k f - f) \right| = o(1), \quad (n \to \infty).$$

As a consequence of strong Cesàro summability, the absolute value can be taken inside the summation and raised to any power $1 \leq r < \infty$. Namely, for all $f \in L^1_{2\pi}$,

$$\sup_{\| g \|_{L^\infty} \leq 1} \frac{1}{n+1} \left( \sum_{k=0}^{n} \left[ \int_0^{2\pi} g \cdot (s^k f - f) \right]^r \right) = o(1), \quad (n \to \infty).$$

The supremum, however, cannot be taken inside the summation.

1. Introduction. A Fréchet space is a complete metrizable locally convex space; for example, every Banach space is a Fréchet space. Consider a Fréchet space $E$ with a total biorthogonal sequence $\{e^k, f_j\}$, [1]. That is,

\begin{align}
(1.A) & \quad e^k \in E \text{ for all } k; \\
(1.B) & \quad f_j \in E' \text{ (the space of continuous linear functionals) for all } j; \\
(1.C) & \quad f_j(e^k) = \delta_{jk} \text{ (Kronecker } \delta) \text{ for all } k \text{ and } j; \\
(1.D) & \quad f_j(x) = 0 \text{ for all } j \text{ implies } x = 0.
\end{align}

Generally we assume that the indices $k$ and $j$ range over the nonnegative integers but when discussing Fourier series of $2\pi$–periodic functions the indices will range over all integers. Each $x$ in $E$ can be identified with the sequence $\hat{x} = (x_0, x_1, x_2, \ldots)$ where $x_j = f_j(x)$. Let $\hat{E} = \{\hat{x} \mid x \in E\}$. $E$ and $\hat{E}$ are isometric and isomorphic if, for each defining seminorm $p_E$ on $E$, we define $p_E(\hat{x}) = p_E(x)$. If $E$ is a Banach space we define $\|\hat{x}\|_{\hat{E}} = \|x\|_E$. By conditions (1.B) and (1.C), $\hat{E}$ has continuous coordinate functionals. Such a Fréchet (respectively Banach) sequence space is called an FK–space (respectively, BK–space). By condition (1.A) $\hat{E}$ contains the space of finite sequences

$$
\phi := \{x = (x_k) \mid x_k = 0 \text{ except for finitely many } k\}.
$$

For simplicity, most theorems in this paper will be stated for FK–spaces (that is, $E = \hat{E}$ where, for each $k$, $e^k$ is the sequence with 1 in the $k^{th}$ position and 0 elsewhere); however, when considering function spaces it will often be more convenient to work directly on $E$ instead of the corresponding FK–space $\hat{E}$.

An element $x$ in $E$ has the property of sectional convergence (denoted $AK$) in $E$ if the sections $s^n x := x_0 e^0 + x_1 e^1 + \ldots + x_n e^n$ converge to $x$ (as $n \to \infty$) with respect to the topology of $E$. In case the biorthogonal sequence ranges over all integers, we define $s^n x := \sum_{|k| \leq n} x_k e^k$ for $n = 0, 1, 2, \cdots$. More generally, an element $x$, not necessarily in $E$, has the property of sectional boundedness (denoted $AB$) in $E$ if the sections $s^n x$
are bounded in $E$. Similarly an element $x$ in $E$ has the property of Cesàro sectional convergence (denoted $\sigma K$) in $E$ if the Cesàro sections $\sigma^n x := \frac{s^0 x + \cdots + s^n x}{n+1}$ converge to $x$ (as $n \to \infty$), with respect to the topology of $E$. This is equivalent to

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} (s^k x - x) = 0.$$ 

An element $x$, not necessarily in $E$, has the property of Cesàro sectional boundedness (denoted $\sigma B$) in $E$ if $\sup_n p(\sigma^n x) < \infty$ for all continuous seminorms $p$ on $E$.

Let $1 \leq r < \infty$. Section 2 contains basic definitions and introduces the properties of strong Cesàro summability of order $r$ (denoted $[\sigma K]_r$) and strong Cesàro boundedness of order $r$ (denoted $[\sigma B]_r$) in Fréchet spaces. These properties are stronger than $\sigma K$ and $\sigma B$, respectively, but are weaker than $AK$ and $AB$, respectively. Section 3 contains general results on strong Cesàro summability and strong Cesàro boundedness in Fréchet spaces. In section 4 specific spaces are considered; namely the convergence fields $H_r$ and boundedness domains $B_r$ of the strong Cesàro summability methods, and their spaces of convergence factors $dv_r$ and $dv_r \cap c_0$. In section 5 we show the equivalence of the properties $[\sigma B]_r$ and $[\sigma K]_r$ to multiplier properties with respect to the spaces $dv_r$ and $dv_r \cap c_0$. In particular, a Fréchet space $E$ containing $\phi$ has the property $[\sigma B]_r$ if and only if $E = dv_r \cdot E$, and it has the property $[\sigma K]_r$ if and only if $E = (dv_r \cap c_0) \cdot E$. In section 6 we consider function spaces and show how these multiplier results can be used to obtain a new form of convergence for Fourier series. For example, we show that the spaces $L^p_{2\pi}$ ($1 \leq p < \infty$) and $C_{2\pi}$ ($2\pi$–periodic continuous functions) have the property $[\sigma K]_r$ for all $r$ and the spaces $L^\infty_{2\pi}$ and $M_{2\pi}$ ($2\pi$–periodic Radon measures) have the property $[\sigma B]_r$ for all $r$. Fejér’s theorem for $L^1_{2\pi}$ is equivalent to the property $\sigma K$ but, for all $r$, the property $[\sigma K]_r$, is stronger.

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2. Definitions. Let $E$ be an FK–space containing $\phi$. By the Hahn–Banach theorem each continuous seminorm $p$ can be expressed in the form

\begin{equation}
(2.A) \quad p(x) = \sup_{f \in \Lambda_p} |f(x)|
\end{equation}
for some subset $A_p$ of $E'$. Thus an element $x$ of $E$ has the property $\sigma K$ if
\[
\lim_{n \to \infty} \sup_{f \in A_p} \left| f \left( \frac{1}{n+1} \sum_{k=0}^{n} (s^k x - x) \right) \right| = 0
\]
for every continuous seminorm $p$. Let $1 \leq r < \infty$. We define the property of strong Cesàro summability of order $r$ (denoted $[\sigma K]_r$) for $x \in E$ by
\[
(2.B) \quad \lim_{n \to \infty} \sup_{f \in A_p} \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k x - x) \right|^r = 0
\]
for every continuous seminorm $p$. Similarly a formal expansion $x = \sum_{k=0}^{\infty} x_ke^k$ with $\hat{x} = (x_0, x_1, \cdots)$, but $\hat{x}$ not necessarily in $\hat{E}$, has strong Cesàro boundedness of order $r$ (denoted $[\sigma B]_r$) in $E$ if
\[
(2.C) \quad p_r(x) := \sup_n \sup_{f \in A_p} \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k x) \right|^r \right\}^\frac{1}{r} < \infty
\]
for each continuous seminorm $p$. If $E$ is a Banach space, we write $\|x\|_r$ instead of $p_r(x)$.

A continuous seminorm $p$ on $E$ does not uniquely determine the set $A_p$. For now, we will assume that
\[
(2.D) \quad A_p = \{ f \in E' | |f| \leq p \};
\]
however, a proper subset of (2.D) is often more natural. For example, if $E = C_{2\pi}$ with norm $\|g\|_{\infty} = \sup_y |g(y)|$, it is natural to choose $A_\|\cdot\|_{\infty}$ to be the set of $f_y$ defined by $f_y(g) = g(y)$, $0 \leq y \leq 2\pi$. Theorems (3.1) and (3.3) (or more explicitly, (5.1)) show that the choice of $A_p$ does not affect the definitions of $[\sigma K]_r$ and $[\sigma B]_r$.

We write $E_{[\sigma K]_r} := \{ x \in E | x has the property $[\sigma K]_r$ in $E$ \}$,
$E_{[\sigma B]_r} := \{ x = \sum_{k=0}^{\infty} x_ke^k | x has the property $[\sigma B]_r$ in $E$ \}$,
$E_{AD} := \{ x \in E | x \ is \ in \ the \ closure \ of \ \phi \ in \ E \}$,
and similarly for $E_{AK}$, $E_{AB}$, $E_{\sigma K}$ and $E_{\sigma B}$. Note that the spaces $E_{[\sigma B]_r}$, $E_{\sigma B}$, and $E_{AB}$ need not be subsets of $E$. 

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We say that $E$ has the property $[\sigma K]_r$ if $E = E_{[\sigma K]_r}$; $E$ has the property $[\sigma B]_r$ if $E \subset E_{[\sigma B]_r}$; $E$ has the property $AD$ if $E = E_{AD}$; etc.

Hölder’s inequality applied to (2.C) shows that $p_r(x) \leq p_s(x)$ for $1 \leq r \leq s < \infty$. Similarly it can be shown that $[\sigma K]_s$ implies $[\sigma K]_r$ and $[\sigma B]_s$ implies $[\sigma B]_r$, if $1 \leq r \leq s < \infty$. It is clear from the definitions that $[\sigma K]_r$ implies both $\sigma K$ and $[\sigma B]_r$. By an argument similar to that showing that ordinary summability implies Cesàro summability, it can be shown that, for all $r$, the property $AK$ in $E$ implies

\[(2.E) \quad \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} p(s^k x - x)^r = 0\]
for all continuous seminorms $p$. This in turn implies $[\sigma K]_r$. Similarly the property $AB$ implies

\[(2.F) \quad \sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} p(s^k x)^r \right\}^{\frac{1}{r}} < \infty\]
for all continuous seminorms $p$

which implies $[\sigma B]_r$. The converse implications are not true as shown by examples in (4.4) and (6.3).

It will be shown in Theorem 3.1 that an FK–space $E$ has the property $[\sigma K]_r$ if and only if it has the properties $AD$ and $[\sigma B]_r$.

The convergence field for strong Cesàro summability of order $r$ is denoted by

\[H_r := \left\{ x = (x_k) | \exists s \in C \exists \lim_n \frac{1}{n+1} \sum_{k=0}^{n} |X_k - s|^r = 0 \right\}\]
where $X_k = x_0 + \ldots + x_k$ (or $X_k = \sum_{|j| \leq k} x_j$ if the index $k$ ranges over all integers) and the boundedness domain for strong Cesàro summability of order $r$ is

\[B_r := \left\{ x = (x_k) | \| x \|_{B_r} := \sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{\frac{1}{r}} < \infty \right\}\]

Below we list some useful BK–spaces and their norms. We use the notation $\Delta y_k = y_k - y_{k+1}$, $\Delta^2 y_k = \Delta y_k - \Delta y_{k+1}$, $\max_{2^n} = \max_{2^n \leq k < 2^{n+1}}$, and $\sum_{2^n} = \sum_{k=2^n}^{2^{n+1}-1}$. If the index
$k$ ranges over the integers we assume that $y_k = y_{-k}$ for the spaces $bv$, $dv_r$ and $q$ listed below. The \textit{space of bounded sequences} is

$$\ell^\infty = \{ x = (x_k) \mid x = (x_k) \mid \|x\|_\infty := \sup |x_k| < \infty \}$$

and the \textit{space of null sequences}

$$c_0 = \{ x = (x_k) \mid \lim x_k = 0 \}$$

is a closed subspace of $\ell^\infty$. The \textit{space of sequences of bounded variation} is

$$bv = \{ y = (y_k) \mid \|y\|_{bv} := \sum_{k=0}^\infty |\Delta y_k| + \sup |y_k| < \infty \}$$

The \textit{spaces of sequences of dyadic variation} are defined as follows:

$$dv = dv_1 = \{ y = (y_k) \mid \|y\|_{dv} := \sum_{n=0}^\infty 2^n \max 2^n |\Delta y_k| + \sup |y_k| < \infty \}$$

$$dv_r = \{ y = (y_k) \mid \|y\|_{dv_r} := \sum_{n=0}^\infty 2^{n/r} \left( \sum_{2^n} |\Delta y_k - 1| \right)^{1/s} + \sup |y_k| < \infty \}$$

for $1 < r < \infty$ and $\frac{1}{r} + \frac{1}{s} = 1$. It is natural to define $dv_\infty = bv$. The \textit{space of bounded quasiconvex sequences} is

$$q = \{ y = (y_k) \mid \|y\|_q := \sum_{k=0}^\infty (k+1) |\Delta^2 y_k| + \sup |y_k| < \infty \}$$

We have $q \subset dv \subset dv_r \subset dv_s \subset bv$ for $1 \leq r \leq s \leq \infty$, ([6], [9]).

\textbf{3. General results.} Zeller has shown that an FK–space $E$ containing $\phi$ has the property $AK$ if and only if it has the properties $AB$ and $AD$. A corresponding theorem for $\sigma K$ was proved in [3]. Using a similar $\epsilon/3$ argument we obtain it for the property $[\sigma K]_r$:

\textbf{(3.1) Theorem.} Let $1 \leq r < \infty$. An FK–space $E$ containing $\phi$ has the property $[\sigma K]_r$ if and only if it has the properties $[\sigma B]_r$ and $AD$.

\textbf{Proof.} Let $x \in E$ have the property $[\sigma K]_r$ and let $p$ be a continuous seminorm on $E$. The property $[\sigma B]_r$ follows immediately. For each $n$, $\sigma^n x \in \phi$ and $p(\sigma^n x - x) = \ldots$
\[ \sup_{f \in A_p} \left| f \left( \frac{1}{n+1} \sum_{k=0}^{n} (s^k x - x) \right) \right| \leq \sup_{f \in A_p} \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k x - x) \right| \to 0 \quad \text{as} \quad n \to \infty. \]

Thus \( x \) has the property \( AD \). Conversely, suppose \( E \) has the properties \( AD \) and \([\sigma B]_r\). Since for all \( x \in E \) we have \( p_r(x) = \sup_{n} \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k x) \right| \), the seminorm \( p_r \) is lower semicontinuous and hence continuous on \( E \) (Theorem 7.1.1 of [7]). Let \( x \in E \) and \( \epsilon > 0 \). Since \( E \) has the property \( AD \) we can find \( y \in \phi \) such that \( p_r(x - y) < \epsilon/3 \) and \( p_r(x - y) < \epsilon/3 \). We can choose \( n \) sufficiently large so that \( \left\{ \frac{1}{n+1} \sum_{k=0}^{n} p(s^k y - y)^r \right\}^{\frac{1}{r}} < \epsilon/3 \). Let \( |f| \leq p \). Then by Minkowski’s inequality

\[
\left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k x - x) \right|^r \right\}^{\frac{1}{r}} \leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k x - s^k y) \right|^r \right\}^{\frac{1}{r}}
\]

\[
+ \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| f(s^k y - y) \right|^r \right\}^{\frac{1}{r}} + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| f(y - x) \right|^r \right\}^{\frac{1}{r}}
\]

\[
< p_r(x - y) + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} p(s^k y - y)^r \right\}^{\frac{1}{r}} + p(y - x) < \epsilon.
\]

Taking the supremum over all \( |f| \leq p \), we see that \( x \) has the property \( [\sigma K]_r \).

(3.2) Corollary. Let \( 1 \leq r < \infty \). If an FK–space \( E \) containing \( \phi \) has the property \([\sigma B]_r\), then \( E_{[\sigma K]_r} = E_{AD} \).

Proof. \( E_{AD} \), being a closed subspace of \( E \), is an FK–space with the properties \( AD \) and \([\sigma B]_r\). By (3.1) we have \( E_{AD} \subseteq E_{[\sigma K]_r} \). The inclusion \( E_{[\sigma K]_r} \subseteq E_{AD} \) is shown in the proof of (3.1).

For FK–spaces \( E \) and \( F \), define

\[ E^\phi = \{ y = (y_k) \mid \text{for some } f \in E', \ y_k = f(e^k) \} \]

and

\[ E^F = \{ y = (y_k) \mid x \cdot y = (x_k y_k) \in F \ \forall \ x \in E \} \).

If \( E \) is a BK–space, then \( E^\phi \) can be identified with the dual space of \( E_{AD} \) [5] and is thus a BK–space under the dual norm of \( E_{AD} \). However, for an FK–space \( E \), the
space $E^\varphi$ need not be an FK–space (The space $E = \omega$ of all complex valued sequences is such an example). The proof of the following theorem is consequently more difficult than in the BK–space case.

**(3.3) Theorem.** Let $1 \leq r < \infty$ and let $E$ be an FK–space containing $\phi$. A sequence $x$ has the property $[\sigma B]_r$ with respect to $E$ if and only if

$$s_f(x) := \sup_n \frac{1}{n+1} \sum_{k=0}^{n} |f(s^kx)|^r < \infty$$

for every $f \in E'$.

**Proof.** The proof of $(\Rightarrow)$ is obvious. For the converse, we use a uniform boundedness argument. Suppose $s_f(x) < \infty$ for all $f \in E'$. Let $p$ be a continuous seminorm on $E$ and let $E_p$ be the space $E$ under the locally convex topology defined by the single seminorm $p$. Then $E_p'$ is a Banach space with unit ball $U = \{ f \in E_p' \mid |f| \leq p \}$. (Actually $E_p^\varphi$ is a BK–space with $E_p^\varphi \subset E^\varphi$.) Let $B = \{ f \in E_p' \mid s_f(x) \leq 1 \}$. $B$ is clearly absolutely convex and closed. $B$ is radial (absorbing) since $s_f(x) < \infty$ for all $f \in E_p'$. Thus $B$ is a barrel in $E_p'$. Since $E_p'$ is barreled, $B$ is a neighborhood of $0$. Thus there exists an $N > 0$ such that $\frac{1}{N} U \subset B$. Then $\sup_{|f| \leq p} s_f(x) \leq N$. 

**(3.4) Corollary.** Let $1 \leq r < \infty$ and let $E$ be an FK–space containing $\phi$. Then

$$E_{[\sigma B]_r} = (E^\varphi)^{B_r}.$$

**(3.5) Corollary.** Let $1 \leq r < \infty$ and let $E$ be an FK–space containing $\phi$. Then

(a) $E$ has the property $[\sigma B]_r$ if and only if $E^\varphi = E^{B_r}$;

(b) $E$ has the property $[\sigma B]_r$ if and only if $E_{[\sigma B]_r} = (E^{B_r})^{B_r}$;

(c) if $E$ has the property $[\sigma K]_r$, then $E^\varphi = E^{H_r}$.

**Proof.** Let $(X \cdot Y)_k = x_0y_0 + \cdots + x_ky_k$ and define

$$E^{\sigma b} := \{ y = (y_k) \mid \sup_n \frac{1}{n+1} \left| \sum_{k=0}^{n} (X \cdot Y)_k \right| < \infty \text{ for all } x \in E \}.$$
It was shown in [4, Theorem 3] that for any FK–space \( E \) containing \( \phi \) we have \( E^{\sigma b} \subset E^\varphi \). Clearly \( E^{B r} \subset E^{\sigma b} \subset E^\varphi \). Thus (a) follows immediately from (3.3). Statement (b) follows from (a), (3.4) and \( E \subset (E^{B r})^{B r} \). Statement (c) is a consequence of \( E^{H r} \subset E^{B r} \subset E^\varphi \) and the definition of \( [\sigma K]_r \). 

**(3.6) Theorem.** Let \( 1 \leq r < \infty \). If \( E \) is an FK–space containing \( \phi \) generated by a set of seminorms \( P \), then \( E_{[\sigma B]_r} \) is an FK–space under the topology generated by the set of seminorms \( \{p_r | p \in P\} \).

**Proof.** It is shown in [3] that \( E_{\sigma B} \) is an FK–space under the seminorms \( q_p(x) := \sup_n \left( \frac{1}{n+1} \sum_{k=0}^{n} s^k x \right) \). We have \( q_p \leq p_1 \) by the definition of \( p_1 \); also \( p_1 \leq p_r \) by Hölder’s inequality. Since \( E_{[\sigma B]_r} = \{x \in E_{\sigma B} | p_r(x) < \infty \ \forall \ p \in P\} \) it follows from Garling’s Theorem [10, p.998] that \( E_{[\sigma B]_r} \) is an FK–space.

**(3.7) Remark.** We will show in (5.3) that for every FK–space \( E \) containing \( \phi \), the space \( E_{[\sigma B]_r} \) always has the property \( [\sigma B]_r \). However, an FK–space \( E \) (containing \( \phi \)) with the property \( [\sigma B]_r \) need not be a closed subspace of \( E_{[\sigma B]_r} \). Thus if the topology of \( E \) is generated by a set of seminorms \( P \), the topology generated by the set of seminorms \( \{p_r | p \in P\} \) need not make \( E \) an FK–space.

**(3.8) Remark.** If \( E \) is an FK–space containing \( \phi \), it can be shown that the set of all sequences \( x \) (not restricted to those belonging to \( E \) ) which satisfy condition (2.F) forms an FK–space. The proof is similar to that of (3.3). It can even be shown that this FK–space satisfies the condition (2.F).

**4. Convergence Fields and Convergence Factors.** In this section we look at the convergence fields \( H_r \) and boundedness domains \( B_r \) of the strong Cesàro summability methods, and at their spaces of convergence factors \( dv_r \) and \( dv_r \cap c_0 \). These spaces are of interest in themselves. Moreover, the properties \( [\sigma B]_r \) and \( [\sigma K]_r \) of these spaces are important in the proofs of multiplier results in the next section.
Maddox [13, p. 101] has observed that $H_r$ is a BK–space under the norm

$$\|x\|_{B_r} = \sup_m \left\{ \frac{1}{m+1} \sum_{j=0}^{m} |X_j|^r \right\}^{\frac{1}{r}}$$

where $X_j = x_0 + \ldots + x_j$. Under the same norm $B_r$ is also a BK–space; this can easily be shown using Garling’s Theorem [10, p. 998].

(4.1) Theorem. Let $1 \leq r < \infty$. Consider the BK–spaces $H_r$ and $B_r$ under the norm $\| \cdot \|_{B_r}$.

(a) $H_r$ satisfies condition (2.E);
(b) $B_r$ satisfies condition (2.F);
(c) $H_r = (B_r)_{[\sigma K]_r}$;
(d) $(H_r)_{[\sigma B]_r} = (B_r)_{[\sigma B]_r} = (H_r)_{\sigma B} = (B_r)_{\sigma B} = B_r$.

Proof. (a): Let $x \in H_r$ and $\epsilon > 0$. We show that

$$\frac{1}{n+1} \sum_{k=0}^{n} \| s^k x - x \|_{B_r}^r \to 0$$

Since $x \in H_r$, there exists a complex number $s$ such that

$$\frac{1}{n+1} \sum_{k=0}^{n} |X_k - s|^r \to 0.$$  

By changing the value of $x_0$ we may assume $s = 0$. Choose $N$ such that, for all $n > N^2$,

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{\frac{1}{r}} < \frac{\epsilon}{3} \quad \text{and} \quad \left( \frac{N+1}{n+1} \right)^{\frac{1}{r}} \| x \|_{B_r} < \frac{\epsilon}{3}.$$  

Using Minkowski’s inequality we have for $n > N^2$

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} \| s^k x - x \|_{B_r}^r \right\}^{\frac{1}{r}} \leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=k+1}^{m} |X_j|^r \right\}^{\frac{1}{r}} + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=k+1}^{m} |X_k|^r \right\}^{\frac{1}{r}}$$

$$\leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \| x \|_{B_r}^r \right\}^{\frac{1}{r}} + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=0}^{m} |X_j|^r \right\}^{\frac{1}{r}}$$

$$\leq \left\{ \frac{1}{n+1} \sum_{k=0}^{N} \| x \|_{B_r}^r \right\}^{\frac{1}{r}} + \left\{ \frac{1}{n+1} \sum_{k=N+1}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=0}^{m} |X_j|^r \right\}^{\frac{1}{r}}$$

$$+ \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{\frac{1}{r}} \leq \left( \frac{N+1}{n+1} \right)^{\frac{1}{r}} \| x \|_{B_r} + \left\{ \frac{1}{n+1} \sum_{k=N+1}^{n} \left( \frac{\epsilon}{3} \right)^r \right\}^{\frac{1}{r}} + \frac{\epsilon}{3} < \epsilon.$$
(b): It is readily shown that
\[
\frac{1}{n+1} \sum_{k=0}^{n} \| s^k x \|_{B_r}^r \leq \frac{1}{n+1} \sum_{k=0}^{n} \sup_{h<\infty} \left\{ \| x \|_{B_r}^r \cdot \frac{k+1}{m+1} \| x \|_{B_r}^r + \left(1 - \frac{k+1}{m+1}\right) |X_k|^r \right\} \leq 2 \| x \|_{B_r}^r.
\]

(c): Since \( H_r \) is a closed subspace of \( B_r \) with the property \([\sigma K]_r, H_r = (B_r)_{AD} \). By (3.2) we have \( H_r = (B_r)_{[\sigma K]_r} \).

(d): By (b) we have \( B_r \subset (B_r)_{[\sigma B]_r} \). Since \( H_r \) is a closed subspace of \( B_r \), we have \((H_r)_{[\sigma B]_r} = (B_r)_{[\sigma B]_r} \subset (H_r)_{\sigma B} = (B_r)_{\sigma B} \). Finally we show \((B_r)_{\sigma B} \subset B_r \) by showing \( \| x \|_{B_r} \leq \sup_m \| \sigma^m x \|_{B_r} \). Let \( \epsilon > 0 \). For each \( n \), we can find \( m \) such that
\[
\left| \sum_{j=0}^{k} x_j \right| < \left| \sum_{j=0}^{k} \left(1 - \frac{j}{m+1}\right) x_j \right| + \epsilon \text{ for } k = 0, \ldots, n. \text{ Then }
\]
\[
\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{\frac{1}{r}} < \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \left(1 - \frac{j}{m+1}\right) x_j \right) + \epsilon \right\}^{\frac{1}{r}}
\]
\[
\leq \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{k} \left(1 - \frac{j}{m+1}\right) x_j \right\}^{\frac{1}{r}} + \epsilon \leq \| \sigma^m x \|_{B_r} + \epsilon \leq \sup_{m} \| \sigma^m x \|_{B_r} + \epsilon.
\]
Thus \( \| x \|_{B_r} = \sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{\frac{1}{r}} \leq \sup_m \| \sigma^m x \|_{B_r}. \)

Let \( \sigma s = \left\{ x = (x_k) \left| \lim_n \frac{1}{n+1} \sum_{k=0}^{n} X_k \text{ exists} \right. \right\} \) be the series-sequence convergence field of the Cesàro method of order 1. Jackson [11] has shown that
\[
(4.A) \quad H_r^{H_r} = H_r^{\sigma s} = d\sigma d_r
\]
as well as some more general multiplier results. Maddox [12] had earlier shown that \( H_r^\varphi = d\sigma d_r \). The identities (4.A) thus also follow from (4.1) and (3.5).

(4.2) Theorem. Let \( 1 \leq r < \infty \). Then \( (d\sigma d_r)_{[\sigma B]_r} = (d\sigma d_r)_{\sigma B} = d\sigma d_r \).

Proof. Since \( H_r \) is a closed subspace of \( B_r \), we have \( H_r^\varphi = B_r^\varphi = d\sigma d_r \). Since \( H_r \) has the property \( \sigma K \), we have \( H_r^\varphi = H_r^{\sigma s} \) [3]. Thus \( d\sigma d_r = (B_r^\varphi)_{[\sigma B]_r} = (B_r)_{\sigma B} = B_r \) and \( d\sigma d_r = (B_r^\varphi)_\varphi = (B_r)_{\sigma B} = B_r \) [5]. Then \( d\sigma d_r \cdot d\sigma d_r \subset B_r \) (actually \( d\sigma d_r \cdot d\sigma d_r \subset B_r \)).
\( dv_r^\varphi = B_r \) since \((1,1,\ldots) \in dv_r\). Hence \( dv_r \subset (dv_r^\varphi)^B_r = (dv_r)_{[\sigma B]} \subset (dv_r)_{\sigma B} \). Conversely, \( (dv_r)_{\sigma B} = (H_r^\varphi)_{\sigma B} \subset H_r^\varphi \subset dv_r \) by [4, Prop. 1].

Using (3.5)(b) we can now add the following to the multiplier identities (4.A):

\[(4.B) \quad H_r^B = B_r^B = dv_r.\]

\[(4.3) \textbf{Theorem.} \text{ Let } 1 \leq r < \infty. \text{ Then } (dv_r)_{[\sigma K]} = dv_r \cap c_0.\]

\textbf{Proof.} By (3.2) \((dv_r)_{[\sigma K]} = (dv_r)_{AD}\). Since \(dv_r \subset \ell^\infty\), we have \((dv_r)_{AD} \subset \ell^\infty_{AD} = c_0\).

Thus \((dv_r)_{[\sigma K]} \subset dv_r \cap c_0\). Conversely, let \(y \in dv_r \cap c_0\) and \(\sigma^n y = \frac{1}{n+1} \sum_{k=0}^{n} s^k y = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) y_k e^k\). We show \(y \in (dv_r)_{AD}\) by showing \(\lim_n \| \sigma^n y - y \|_{dv_r} = 0\).

Let \(\epsilon > 0\) and choose \(M\) such that \(|y_k| < \epsilon\) whenever \(k > 2^M\) and

\[\sum_{N=M+1}^{\infty} 2^{N/r} \left(\sum_{2^N}^{2^{N+1}} |\Delta y_k|^r \right)^{1/r} < \epsilon.\]

Let \(t = 2^n\) and \(n > M\). Then \(\| \sigma^t y - y \|_{dv_r} = \| \sigma^t y - y \|_\infty + \sum_{N=0}^{M-1} 2^{N/r} \left(\sum_{2^N}^{2^{N+1}} |\frac{k}{t+1} y_{k-1} - \frac{k+1}{t+1} y_k|^r \right)^{1/r} + \sum_{N=M}^{n-1} 2^{N/r} \left(\sum_{2^N}^{2^{N+1}} |y_{k-1} - y_k|^r \right)^{1/r} = S_1 + S_2 + S_3 + S_4.\)

Since \(y \in c_0\) we have \(S_1 = o(1)\). Clearly \(S_2 = o(1)\) and \(S_4 < \epsilon\). Finally \(S_3 = \sum_{M}^{n-1} 2^{N/r} \left(\sum_{2^N}^{2^{N+1}} |\Delta y_{k-1} + \frac{1}{t+1} y_k|^r \right)^{1/r} \leq \sum_{M}^{n-1} 2^{N/r} \left(\sum_{2^N}^{2^{N+1}} |\Delta y_{k-1}|^r \right)^{1/r} + \epsilon \sum_{M}^{n-1} 2^{N/r} \geq 2^{(N+1)/r} / r < \epsilon + \epsilon \sum_{M}^{n-1} 2^{N+1} < \epsilon + \epsilon \cdot 2^2 < 2\epsilon.\]

\[(4.4) \textbf{Remark.} \text{ The spaces } dv_r \text{ do not satisfy (2.F) as can be seen by considering the sequence } (1,1,1,\ldots). \text{ It can be shown that } y \text{ in } dv_r \text{ satisfies (2.F) if and only if}

\[(4.C) \quad \sum_{2^n} |y_{k-1}| = O(2^{n/s}), \quad \frac{1}{r} + \frac{1}{s} = 1.\]

This is equivalent to the condition

\[(4.D) \quad \frac{1}{n+1} \sum_{k=0}^{n} (k+1)^{1/s} |y_k| = O(1).\]
Similarly, a sequence \( y \) in \( dv_r \) satisfies the condition (2.E) if and only if

\[(4.E) \quad \sum_{2^n} |y_{k-1}| = o(2^{n/s}) , \quad \frac{1}{r} + \frac{1}{s} = 1 , \]

which is equivalent to

\[(4.F) \quad \frac{1}{n+1} \sum_{k=0}^{n} (k+1)^{1/s} |y_k| = o(1) . \]

5. Multiplier results. If \( E \) and \( F \) are FK–spaces, we write

\[ E \cdot F = \{ x \cdot y := (x_ky_k) \mid x \in E , \ y \in F \} , \]

\[ (E \rightarrow F) = \{ y = (y_k) \mid x \cdot y \in F \text{ for all } x \in E \} . \]

The set \( E \cdot F \) need not be a linear space. The set \( (E \rightarrow F) \) is a sequence space but it need not be an FK–space. However if \( E \) and \( F \) are BK–spaces, then \( (E \rightarrow F) \) is a BK–space under the norm \( \| y \| = \sup_{\|x\| \leq 1} \| x \cdot y \|_F . \)

An FK–space \( E \) containing \( \phi \) has the property \( AB \) if and only if \( E = bv \cdot E \) and it has the property \( AK \) if and only if \( E = (bv \cap c_0) \cdot E [10] \). Similarly \( E \) has the property \( \sigma B \) if and only if \( E = q \cdot E \) and it has the property \( \sigma K \) if and only if \( E = (q \cap c_0) \cdot E [3] \). Now we show that strong Cesàro summability and strong Cesàro boundedness for an FK–space are also equivalent to multiplier statements.

(5.1) Theorem. Let \( E \) be an FK–space containing \( \phi \) with a defining family of continuous seminorms \( p^1 \leq p^2 \leq p^3 \leq \cdots \). Let \( A_N \subset E' \) such that \( p^N(x) = \sup_{f \in A_N} |f(x)| \) for all \( x \in E \). For \( 1 \leq r < \infty \) the following statements are equivalent:

(a) \[ x \in E_{[\sigma B]}_{r} ; \]

(b) \[ \sup_n \sup_{f \in A_N} \frac{1}{n+1} \sum_{k=0}^{n} |f(s^kx)|^r < \infty \text{ for all } N ; \]

(c) \[ dv_r \cdot x \subset E . \]
Proof. (a) $\implies$ (b) is immediate. (b) $\implies$ (c): Suppose (b) and let $y \in dv_r$. We show $y \cdot x \in E$ by showing that $\sigma^t(y \cdot x) = \sum_{k=0}^{s} k^t y^t x$ is a Cauchy sequence in $E$ for $t = 2^m$. Using summation by parts we obtain $\sigma^t(y \cdot x) = \sum_{k=0}^{s} \left\{ \left( 1 - \frac{k}{t+1} \right) \Delta y_k + \frac{1}{t+1} y_{k+1} \right\} s^k x$. For $s = 2^m < 2^n = t$ and $f \in A_N$ we have

$$| f(\sigma^t(y \cdot x) - \sigma^s(y \cdot x)) | \leq \left( \frac{1}{s+1} - \frac{1}{t+1} \right) \sum_{k=0}^{s} |k\Delta y_k - y_{k+1}| |f(s^k x)|$$

$$+ \sum_{k=s+1}^{t} \left\{ \left( 1 - \frac{k}{t+1} \right) \Delta y_k + \frac{1}{t+1} y_{k+1} \right\} |f(s^k x)|$$

$$\leq \frac{1}{2^m} \sum_{j=0}^{n} \sum_{j=0}^{j+1} \left( \left( 1 - \frac{k}{t+1} \right) |\Delta y_{k-1}| + |y_k| \right) |f(s^k x)|$$

Using Hölder’s inequality on the sums $\sum_{j}$ we obtain

$$\frac{1}{2^m} \sum_{j=0}^{n} \left\{ \sum_{j=0}^{j+1} \left( \left( 1 - \frac{k}{t+1} \right) |\Delta y_{k-1}| + |y_k| \right) \right\} \frac{1}{p'} \left\{ \sum_{j=0}^{m} \left\{ \sum_{j=0}^{n} \left( \sum_{j=0}^{j+1} \left( |\Delta y_{k-1}| + |y_k| \right) \right) \right\} \frac{1}{p'} \right\}$$

$$\leq \frac{1}{2^m} \sum_{j=0}^{n} \left\{ \sum_{j=0}^{j+1} \left( \sum_{j=0}^{n} \left( |\Delta y_{k-1}| + |y_k| \right) \right) \right\} \frac{1}{p'} \left\{ \sum_{j=0}^{m} \left\{ \sum_{j=0}^{n} \left( |\Delta y_{k-1}| + |y_k| \right) \right\} \right\} \frac{1}{p'} \left\{ \sum_{j=0}^{n} \left( \sum_{j=0}^{j+1} \left( |\Delta y_{k-1}| + |y_k| \right) \right) \right\}$$

$$\leq p^N_r(x) \left\{ \frac{2^{M+2}}{2^m} \left( \sum_{j=0}^{M} 2^{j/p} \left( \sum_{j=0}^{j+1} |\Delta y_{k-1}| + \sup_{k>2^M} |y_k| \right) \right) \frac{1}{p'} + \sup_{k>2^M} |y_k| \right\}$$

$$+ \left( 4 \sum_{j=0}^{M+1} 2^{j/p} \left( \sum_{j=0}^{j+1} |\Delta y_{k-1}| + \sup_{k>2^M} |y_k| \right) \right) \frac{1}{p'} + \sup_{k>2^M} |y_k| \right\}$$

$$+ \left( 4 \sum_{j=0}^{n} 2^{j/p} \left( \sum_{j=0}^{n} |\Delta y_{k-1}| + \sup_{k>2^M} |y_k| \right) \right) \frac{1}{p'} + \sup_{k>2^M} |y_k| \right\}.$$
This can be made arbitrarily small by choosing \( M \) and \( m \) sufficiently large. Thus 
\[
p^N(\sigma^t(y \cdot x) - \sigma^s(y \cdot x)) = \sup_{f \in AN} |f(\sigma^t(y \cdot x) - \sigma^s(y \cdot x))| \to 0 \text{ as } n, m \to \infty.
\]
Since \( E \) is complete and has continuous coordinate functionals, \( \sigma^t(y \cdot x) \) converges to \( y \cdot x \).
This shows \( dv_r \cdot x \subset E \).

\[(c) \implies (a): \text{Suppose } dv_r \cdot x \subset E \text{. Then by the closed graph theorem, } T_x(y) := y \cdot x \text{ is a continuous map from } dv_r \text{ to } E \text{ [14]. Let } p \text{ be a continuous seminorm on } E \text{. Then } p \circ T_x \text{ is a continuous seminorm on } dv_r \text{. Thus } p(T_x(y)) \leq K_p \| y \|_{dv_r} \text{ for some constant } K_p \text{. Hence for } f \in E' \text{ with } |f| \leq p \text{ we have } f \circ T_x \in dv_r' \text{ and } |f(T_x(y))| \leq p(T_x(y)) \leq K_p \| y \|_{dv_r} \text{. Since } dv_r \text{ has the property } [\sigma B]_r, \text{ we have}
\]
\[
\sup_{|f| \leq p} \frac{1}{n+1} \sum_{k=0}^n |f(s^n(x \cdot y))|^r = \sup_{|f| \leq p} \frac{1}{n+1} \sum_{k=0}^n |f \circ T_x(s^n y)|^r < \infty.
\]
This shows that every sequence in \( dv_r \cdot x \) has the property \( [\sigma B]_r \).

\[\text{(5.2) Theorem. Let } E \text{ be an FK–space containing } \phi \text{ and let } 1 \leq r < \infty \text{. Then } E \text{ has the property } [\sigma B]_r \text{ if and only if } E = dv_r \cdot E \text{.}
\]

**Proof.** If \( E \) has the property \( [\sigma B]_r \), then by (5.1) (a) \( (c) \implies (a) \) we have \( dv_r \cdot E \subset E \).

Since the sequence \((1, 1, \ldots)\) is in \( dv_r \), the opposite inclusion is immediate. The converse follows from (5.1) \( (c) \implies (a) \).

\[\text{(5.3) Theorem. Let } E \text{ be an FK–space containing } \phi \text{ and let } 1 \leq r < \infty \text{. Then } E \text{ has the property } [\sigma K]_r \text{ if and only if } E = (dv_r \cap c_0) \cdot E \text{.}
\]

**Proof.** Suppose \( E \) has the property \( [\sigma K]_r \). Then \( E \) has the property \( \sigma K \). It was shown in [3] that if \( E \) has \( \sigma B \) then \( E_{\sigma K} = (q \cap c_0) \cdot E \).
For the same reasons 
\[
(dv_r)_{[\sigma K]_r} = (dv_r)_{\sigma K} = (q \cap c_0) \cdot dv_r = dv_r \cap c_0.
\]
Thus since \( E = dv_r \cdot E \) by (5.2), we have \( E = E_{\sigma K} = (q \cap c_0) \cdot E = (q \cap c_0) \cdot (dv_r \cdot E) = (dv_r \cap c_0) \cdot E \).

Conversely suppose \( E = (dv_r \cap c_0) \cdot E \). For each \( x \in E \), \( T_x(y) = y \cdot x \) is a continuous map from \( dv_r \cap c_0 \) into \( E \). Let \( p \) be a continuous seminorm on \( E \). As in the proof of (5.1), there exists a constant \( K_p \) such that 
\[
\sup_{|f| \leq p} \frac{1}{n+1} \sum_{k=0}^n |f(s^n(x \cdot y) - x \cdot y)|^r = \sup_{|f| \leq p} \frac{1}{n+1} \sum_{k=0}^n |f \circ T_x(s^n y - y)|^r \leq \sup_{|g| \leq K_p} \frac{1}{n+1} \sum_{k=0}^n |g(s^n y - y)|^r. \text{ Since } dv_r \cap c_0
\]

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has the property \([\sigma K]_r\), this tends to 0.

The following is a consequence of (3.4), (3.5) and (5.2).

**5.4 Proposition.** Let \(E\) be an FK–space containing \(\phi\) and let \(1 \leq r < \infty\). Then the space \(E[\sigma B]_r\) is an FK–space having the property \([\sigma B]_r\).

**5.5 Remark.** Multiplier statements corresponding to (5.2) (respectively (5.3)) do not hold for spaces satisfying condition (2.F) (respectively condition (2.E)) with respect to the multipliers \(dv_r\) (respectively \(dv_r \cap c_0\)) or with respect to the sequences satisfying conditions (4.C) (respectively condition (4.E)). The spaces \(dv_r\) and \(H_r\) serve as counterexamples. However, we can obtain a partial result which we give without proof.

**5.6 Proposition.** Let \(E\) be an FK–space containing \(\phi\), let \(1 \leq r < \infty\), and let \(x\) be a sequence. If \(x \cdot y \in E\) for every sequence \(y\) in \(dv_r\) satisfying condition (2.E), then the sequences \(x \cdot y\) satisfy (2.F).

### 6. Function spaces

We now consider spaces of \(2\pi\)–periodic functions or distributions \(g\) for which Fourier coefficients \(\hat{g}(k)\) are defined [8]. Sequences will be defined on the integers, and the sequences in \(q\), \(dv_r\) and \(bv\) will be assumed to be symmetric (that is, \(y_k = y_{-k}\)). Here \(e^k\) is the function \(e^k(x) = e^{ikx}\) and \(s^n g(x) = \sum_{|k| \leq n} \hat{g}(k)e^{ikx}\).

Zygmund [15, Theorem XIII.7.3] shows that for every function \(g\) in \(C_{2\pi}\) and \(1 \leq r < \infty\) we have

\[
(6.A) \quad \frac{1}{n+1} \sum_{k=0}^{n} |s^k g(x) - g(x)|^r \to 0 \text{ uniformly for all } x.
\]

Since the norm on \(C_{2\pi}\) is \(\| g \|_{\infty} = \sup_x |g(x)|\), this is equivalent to saying that \(C_{2\pi}\) has the property \([\sigma K]_r\), where \(A_{\| \cdot \|} = \{ F_x \in C_{2\pi} \mid F_x(g) := g(x), \ 0 \leq x \leq 2\pi \}\) as defined by (2.A). This shows by (5.2) and (5.3) that for all \(1 \leq r < \infty\), we have

\[
(6.B) \quad dv_r \cdot \widehat{\mathcal{C}}_{2\pi} = \widehat{\mathcal{C}}_{2\pi} = (dv_r \cap c_0) \cdot \widehat{\mathcal{C}}_{2\pi}.
\]
Since \((\hat{C}_{2\pi} \to \hat{C}_{2\pi}) = (\hat{L}^1_{2\pi} \to \hat{L}^1_{2\pi}) = (\hat{M}_{2\pi} \to \hat{M}_{2\pi}) = (\hat{L}^\infty_{2\pi} \to \hat{L}^\infty_{2\pi}) = \hat{M}_{2\pi}\) [8, vol. 2, p. 246], an immediate consequence is the result \(dv_r \subset \hat{M}_{2\pi}\) for all \(1 \leq r < \infty\) [6]. We also obtain Fomin’s integrability result \(dv_r \cap c_0 = (dv_r)_{AD} \subset (\hat{M}_{2\pi})_{AD} = \hat{L}^1_{2\pi}\) [6], [9]. Since \(e = (\cdots, 1, 1, 1, \cdots) \in dv_r\), we have also \(dv_r \cdot \hat{L}^1_{2\pi} = \hat{L}^1_{2\pi}, dv_r \cdot \hat{M}_{2\pi} = \hat{M}_{2\pi}\), and \(dv_r \cdot \hat{L}^\infty_{2\pi} = \hat{L}^\infty_{2\pi}\).

Conversely, our multiplier results show that (6.A) can be obtained from Fomin’s integrability result.

Furthermore, \((\hat{M}_{2\pi})_{AD} = \hat{L}^1_{2\pi}\) and \((\hat{L}^\infty_{2\pi})_{AD} = \hat{C}_{2\pi}\). By (5.2), (3.2) and (5.3) we have the following.

(6.1) Theorem. Let \(1 \leq r < \infty\). The spaces \(C_{2\pi}\) and \(L^1_{2\pi}\) have the property \([\sigma K]_r\). The spaces \(L^\infty_{2\pi}\) and \(M_{2\pi}\) have the property \([\sigma B]_r\).

(6.2) Remark. Theorem 6.1 for \(L^1_{2\pi}\) is stronger than the Theorem of Fejér’s which states for \(f \in L^1_{2\pi}\),

\[
\frac{1}{n+1} \left\| \sum_{k=0}^{n} (s^k f - f) \right\|_{L^1} = o(1), \quad (n \to \infty).
\]

This is equivalent to

\[
\sup_{F \in A} \frac{1}{n+1} \left| \sum_{k=0}^{n} F \cdot (s^k f - f) \right| = o(1), \quad (n \to \infty)
\]

for some subset \(A\) of the dual of \(L^1_{2\pi}\). Since the dual of \(L^1_{2\pi}\) is \(L^\infty_{2\pi}\) and the continuous linear functionals on \(L^1_{2\pi}\) are of the form \(F_g(f) = \int_0^{2\pi} g \cdot f\) for \(g \in L^\infty_{2\pi}\), we have

\[
\sup_{\|g\|_\infty \leq 1} \frac{1}{n+1} \left| \sum_{k=0}^{n} \int_0^{2\pi} g \cdot (s^k f - f) \right| = o(1), \quad (n \to \infty).
\]

Theorem 6.1 shows that the absolute value can be taken inside the summation and raised to any power \(1 \leq r < \infty\) to obtain

\[
(6.C) \quad \sup_{\|g\|_\infty \leq 1} \frac{1}{n+1} \sum_{k=0}^{n} \left| \int_0^{2\pi} g \cdot (s^k f - f) \right|^r = o(1), \quad (n \to \infty).
\]
One could consider direct proofs of (6.C) from (6.A) but the main idea here is the equivalence of convergence theorems and multiplier theorems.

(6.3) Remark. The following example due N. Tanović–Miller shows that (6.C) cannot be further strengthened by taking the supremum inside the summation. That is, the property \([\sigma K]_r\) cannot be strengthened to the property (2.E). The example shows that for each \(1 \leq r < \infty\), there exist \(f \in L^1_{2\pi}\) such that

\[
\frac{1}{n+1} \sum_{k=0}^{n} \sup_{\|g\|_{\infty} \leq 1} \left| \int_{0}^{2\pi} g \cdot (s^k f - f) \right|^r = \frac{1}{n+1} \sum_{k=0}^{n} \|s^k f - f\|_{L^1}^r \neq O(1), \quad (n \to \infty).
\]

It is sufficient to let \(r = 1\), since by Hölder’s inequality

\[
\frac{1}{n+1} \sum_{k=0}^{n} \|s^k f - f\|_{L^1} \leq \left( \frac{1}{n+1} \sum_{k=0}^{n} \|s^k f - f\|_{L^1}^r \right)^{1/r}.
\]

Consider the cosine series \(\frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx\), where \(a_0 = a_1 = 0\), and \(a_k = \frac{1}{\sqrt{\log k}}\), \((k \geq 2)\). We have \(a_k \downarrow 0\), \(k \Delta a_k \to 0\) and \(\sum_k (k+1) |\Delta^2 a_k| < \infty\) since \(\Delta a_k \sim \frac{1}{k \log^{3/2} k}\), and \(\Delta^2 a_k \sim \frac{1}{k^2 \log^{5/2} k}\). By a classical result of Kolmogorov ([7], vol. 1, 7.3.1 and 7.3.2) the cosine series converges to

\[
(6.D) \quad f(x) = \frac{1}{2} \sum_{k=0}^{\infty} (k+1) \Delta^2 a_k F_k(x)
\]

(\(F_k\) denotes the Fejér kernel) pointwise for \(x \neq 0 \mod 2\pi\), \(f \in L^1_{2\pi}\), and (6.D) is the Fourier series of \(f\) (moreover, \(f \geq 0\)). By partial summation and by (6.D), we have

\[
s^n f(x) - f(x) = \frac{1}{2} n \Delta a_{n-1} F_{n-1}(x) + \frac{1}{2} a_n D_n(x) - \frac{1}{2} \sum_{k=n-1}^{\infty} (k+1) \Delta^2 a_k F_n(x)
\]

(\(D_n\) denotes the Dirichlet kernel) for \(x \neq 0 \mod 2\pi\). Thus

\[
\|s^n f - f\|_{L^1} \geq \frac{1}{2} |a_n| \|D_n\|_{L^1} - \frac{1}{2} n |\Delta a_{n-1}| \|F_{n}\|_{L^1} - \frac{1}{2} \sum_{k=n-1}^{\infty} (k+1) |\Delta^2 a_k| \|F_{n}\|_{L^1}.
\]
Since \( \| F_n \|_{L^1} = 1 \), given \( \epsilon > 0 \), there exists \( N \) such that for \( n > N \),
\[
\| s^n f - f \|_{L^1} \geq \frac{1}{2} |a_n| \| D_n \|_{L^1} - \frac{\epsilon}{2}.
\]
Hence \( \frac{1}{n+1} \sum_{k=0}^{n} \| s^n f - f \|_{L^1} \geq \frac{1}{n+1} \sum_{k=N}^{n} |a_k| \| D_k \|_{L^1} - \frac{\epsilon}{2} \). But \( \| D_k \|_{L^1} \sim \frac{4}{\pi} \log k \), and consequently \( \frac{1}{n+1} \sum_{k=0}^{n} \| s^k f - f \|_{L^1} \to \infty \), \((n \to \infty)\).

Since \( L^\infty_{2\pi} = (L^\infty_{2\pi})_{\sigma B} \) and \( M^2_{2\pi} = (M^2_{2\pi})_{\sigma B} \), we have
\[(6.E) \quad L^\infty_{2\pi} = (C_{2\pi})_{[\sigma B]} \quad \text{and} \quad M^2_{2\pi} = (L^1_{2\pi})_{[\sigma B]}.
\]
From (3.6) and the first identity in (6.E) we obtain the following.

\[ (6.4) \text{ Theorem. Let } g \in L^1_{2\pi} \text{ and } 1 \leq r < \infty. \text{ Then } g \in L^\infty_{2\pi} \text{ if and only if}
\]
\[
\| g \|_{L^\infty_{2\pi}} = \sup_{n,x} \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s^k g(x)|^r \right\}^{\frac{1}{r}} < \infty.
\]
Furthermore \( \| \cdot \|_{L^\infty_{2\pi}} \) is a defining norm on \( L^\infty_{2\pi} \).

We can obtain a similar result for the space \( M^2_{2\pi} \) from the second identity in (6.E).

Since the continuous linear functionals on \( L^1_{2\pi} \) are of the form \( F_f(g) = \int_0^{2\pi} f \cdot g \) for \( f \in L^\infty_{2\pi} \), we have \( \| g \|_{L^1_{2\pi}} = \sup_{\| f \|_{L^\infty_{2\pi}} \leq 1} \sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| \int_0^{2\pi} s^k (f \cdot g) \right|^r \right\}^{\frac{1}{r}} \) for \( g \in L^1_{2\pi} \). Consequently we obtain the following.

\[ (6.5) \text{ Theorem. For each } g \in L^1_{2\pi} \text{ and } 1 \leq r < \infty \text{ we have}
\]
\[
\sup_{\| f \|_{L^\infty_{2\pi}} \leq 1} \sup_n \frac{1}{n+1} \sum_{k=0}^{n} \left| \int_0^{2\pi} s^k (f \cdot g) \right|^r < \infty.
\]

Finally, since \( (\hat{L}^\infty_{2\pi})^\varphi = (\hat{C}_{2\pi})^\varphi = \hat{M}^2_{2\pi} \) and \( (\hat{M}^2_{2\pi})^\varphi = (\hat{L}^1_{2\pi})^\varphi = \hat{L}^\infty_{2\pi} \) we obtain the following from (3.4).

\[ (6.6) \text{ Theorem. For each } 1 \leq r < \infty, \quad \hat{L}^\infty_{2\pi} = (\hat{L}^1_{2\pi} \to B_r) \quad \text{and} \quad \hat{M}^2_{2\pi} = (\hat{L}^\infty_{2\pi} \to B_r).
\]
REFERENCES


