Analysis **7**, 293-304 (1987)

PRODUCTS OF SEQUENCE SPACES

Martin Buntinas*
Loyola University of Chicago

1. Introduction. Let E and F be FK-spaces (respectively BK-spaces) of complex sequences $x = (x_k)_{k=1}^{\infty}$. The coordinatewise product $E \cdot F$ is the set of all sequences $x \cdot y = (x_k y_k)$ with $x \in E$ and $y \in F$. The set $E \cdot F$ is not generally a vector space. It is shown in [2] that the sequence space spanned by $E \cdot F$, under a certain topology, has a completion which can be identified with an FK-space (respectively BK-space). This space is called the *FK-product* $E \widehat{\otimes} F$ and it is shown to be important in the characterization of multiplier spaces. It is also shown there that the FK-product $E \widehat{\otimes} F$ is the intersection of all FK-spaces (respectively BK-spaces) containing $E \cdot F$. In the present paper, we show the same using a more explicit definition of the FK-product. This then leads to a shorter proof of a key result that the FK-product of FK-spaces is an FK-space. Multiplier results are given in Section 4 which unify and generalize those in [2]. Examples of FK-products of l^p spaces and mixed norm spaces $l^{p,q}$ are given in Section 5. These spaces then lead in Section 6 to a counterexample of a conjecture of G. Goes.

2. New Definition of $E \widehat{\otimes} F$. We give here a definition of the FK-product $E \widehat{\otimes} F$ of two FK-spaces (respectively BK-spaces) E and F different from that in [2]. We show that it is the smallest FK-space (respectively BK-space) containing $E \cdot F$ (Theorem 3). Since this is a characterization of the FK-product given in [2] (Theorem 3.3), the definitions are equivalent.

Let ω be the space of all complex sequences $x = (x_k)$ with the topology of coordinatewise convergence generated by the seminorms $p_k : x \to |x_k|$ $(k = 1, 2, \ldots)$. Let E and F be FK-spaces (respectively BK-spaces); that is, subspaces of ω which are locally convex Fréchet spaces (respectively Banach spaces) on which the coordinate functionals $x \to x_k$ $(k = 1, 2, \ldots)$ are continuous. The *FK-product* $E \widehat{\otimes} F$ is the set of all sequences in ω for which there exists a representation

$$u = \sum_{j=1}^{\infty} x^j \cdot y^j$$

with $x^j \in E, y^j \in F$ where convergence of the series is coordinatewise (that is, with respect to the topology of ω) and where $\sum_{j=1}^{\infty} p(x^j) q(y^j) < \infty$ for every pair of continuous seminorms p on E and q on F.

Throughout this paper, for every *representation* of $u \in E \widehat{\otimes} F$ as $u = \sum_{j=1}^{\infty} x^j \cdot y^j$ we assume that the series converges coordinatewise, $x^j \in E$, $y^j \in F$ for all j, and $\sum_{j=1}^{\infty} p(x^j) q(y^j) < \infty$ for *all* continuous seminorms p on E and q on F.

It is not difficult to show that $E \widehat{\otimes} F$ is a linear space.

We define the topology on $E \widehat{\otimes} F$ to be generated by the seminorms $r = p \otimes q$ given by

$$r(u) = \inf \sum_{j=1}^{\infty} p(x^j) q(y^j)$$

where p and q are continuous seminorms on E and F respectively and the infimum is taken over all representations of $u \in E \widehat{\otimes} F$ as $u = \sum_j x^j \cdot y^j$. Clearly if $\{p_k\}$ and $\{q_k\}$ are increasing generating families of seminorms on E and F respectively, then the seminorms $r_k = p_k \otimes q_k$ are increasing and generate the topology of $E \widehat{\otimes} F$.

---

* Presented to the St. Lawrence University Conference on Sequence Spaces, Canton, N.Y., July 1985. Presented to the American Mathematical Society, Laramie, WY, August 1985.

Theorem 1. *If E and F are FK-spaces (respectively BK- spaces), then $E\widehat{\otimes}F$ is an FK-space (respectively BK- space).*

Proof. Let E and F be FK-spaces. Then for each k, $p(x) = |x_k|$ and $q(y) = |y_k|$ are continuous seminorms on E and F respectively. Let $r = p \otimes q$. Then r is a continuous seminorm on $E\widehat{\otimes}F$ and, as $u \in E\widehat{\otimes}F$ ranges over representations $u = \sum_j x^j \cdot y^j$, we have

$$|u_k| = |\sum_{j=1}^{\infty} x_k^j y_k^j| \leq \inf \sum_{j=1}^{\infty} p(x^j)q(y^j) = r(u).$$

Thus $E\widehat{\otimes}F$ has continuous coordinate functionals. To show that $E\widehat{\otimes}F$ is complete, let $\{u^n\}$ be a Cauchy sequence in $E\widehat{\otimes}F$ and let u be the coordinatewise limit of $\{u^n\}$. Let p_m, q_m be increasing sequences of seminorms generating the topologies of E and F, respectively, and let $r_m = p_m \otimes q_m$. If E and F are BK-spaces, we may choose $r_m = r_{m+1}$. Choose $N_i < N_{i+1}$ such that $r_i(u^{N_i} - u^n) < 1/2^{i+1}$ for all $n > N_i$ and let $w^i = u^{N_{i+1}} - u^{N_i}$. Then there exist representations of $w^i \in E\widehat{\otimes}F$ as $w^i = \sum_j x^{ij} \cdot y^{ij}$ such that $r_i(w^i) \leq \sum_j p_i(x^{ij})q_i(y^{ij}) < 1/2^i$. Also let $u^{N_1} = \sum_j x^{0j} \cdot y^{0j}$ be any representation of $u^{N_1} \in E\widehat{\otimes}F$. If p and q are any continuous seminorms on E and F respectively, choose N such that $p \leq p_N$ and $q \leq q_N$. Then

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} p(x^{ij})q(y^{ij}) &\leq \sum_{i=0}^{N-1}\sum_{j=1}^{\infty} p_N(x^{ij})q_N(y^{ij}) + \sum_{i=N}^{\infty}\sum_{j=1}^{\infty} p_i(x^{ij})q_i(y^{ij}) \\ &< \sum_{i=0}^{N-1}\sum_{j=1}^{\infty} p_N(x^{ij})q_N(y^{ij}) + \sum_{i=N}^{\infty} \frac{1}{2^i} < \infty.\end{aligned}$$

In particular, since $\sum_i \sum_j |x_k^{ij}||y_k^{ij}| < \infty$ for $k = 1, 2, \ldots$, $\sum_i \sum_j x^{ij} \cdot y^{ij}$ converges coordinatewise to u. This shows that $u = \sum_i \sum_j x^{ij} \cdot y^{ij}$ is a representation of u in $E\widehat{\otimes}F$ and hence $u \in E\widehat{\otimes}F$. It remains to be shown that $\{u^n\}$ converges to u in $E\widehat{\otimes}F$. For $r = p \otimes q$ and $\epsilon > 0$, let $r_m \geq r$, $1/2^{i-2} < \epsilon$, $i > m$ and $n > N_i$. Since

$$u - u^{N_i} = \sum_{k=i}^{\infty}\sum_{j=1}^{\infty} x^{kj} \cdot y^{kj}$$

is a representation of $u - u^{N_i}$, we have

$$\begin{aligned} r(u - u^n) &\leq r_i(u - u^{N_i}) + r_i(u^{N_i} - u^n) \leq r_i(\sum_{k=i}^{\infty}\sum_{j=1}^{\infty} x^{kj} \cdot y^{kj}) + \frac{1}{2^{i+1}} \\ &\leq \sum_{k=i}^{\infty}\sum_{j=1}^{\infty} p_k(x^{kj})q_k(y^{kj}) + \frac{1}{2^{i+1}} < \sum_{k=i}^{\infty} \frac{1}{2^k} + \frac{1}{2^{i+1}} < \frac{1}{2^{i-2}} < \epsilon. \end{aligned}$$

∎

The vector space spanned by $E \cdot F$ under the topology induced by $E\widehat{\otimes}F$ we denote by $E \otimes F$.

Theorem 2. *If E and F are FK-spaces, then $E \otimes F$ is a dense subspace of $E\widehat{\otimes}F$.*

Proof. Let $u = \sum_j x^j \cdot y^j$ be a representation of $u \in E\widehat{\otimes}F$. Then clearly $u^n = \sum_{j=1}^n x^j \cdot y^j \in E \otimes F$ defines a Cauchy sequence which converges coordinatewise to u. Since $E\widehat{\otimes}F$ is an FK-space, $u^n \to u$ as $n \to \infty$. ∎

The proof of the following theorem is similar to that of Theorem 3.3 in [2].

Theorem 3. *If E and F are FK-spaces (respectively BK- spaces), then $E\widehat{\otimes}F$ is the smallest FK-space (respectively BK-space) containing $E \cdot F$.*

Proof. $E\widehat{\otimes}F$ is an FK-space (respectively BK- space) containing $E \cdot F$. Suppose H is an FK-space (respectively BK-space) containing $E \cdot F$. We show that $E\widehat{\otimes}F \subset H$. Define the map f from $E \times F$ to

H by $f(x,y) = x \cdot y$. The linear maps $x \to x \cdot y$ and $y \to x \cdot y$ are continuous from E to H and F to H, respectively, since they are multiplier maps which are special cases of matrix maps [6]. Since every separately continuous bilinear map on $E \times F$ to H is continuous ([5], p. 88), f is continuous. Then for each continuous seminorm ρ on H, there exist continuous seminorms p on E and q on F such that for all $x \in E$, $y \in F$, we have $\rho(x \cdot y) = \rho(f(x,y)) \le p(x)q(y)$, ([5], p. 74). Let $u = \sum_j x^j \cdot y^j$ be a representation of $u \in E \widehat{\otimes} F$. Clearly the truncations $u^n = \sum_{j=1}^n x^j \cdot y^j$ belong to H since it is a linear space containing $E \cdot F$. We have $\rho(u^m - u^n) = \rho(\sum_{j=m+1}^n x^j \cdot y^j) \le \sum_{j=m+1}^n \rho(x^j \cdot y^j) \le \sum_{j=m+1}^n p(x^j)q(y^j)$ which tends to 0 as $m, n \to \infty$. Thus u^n is a Cauchy sequence in H which converges coordinatewise to u. Since H is an FK-space, $u^n \to u$ and $u \in H$. ∎

3. Zero-neighborhood base of $E \widehat{\otimes} F$. To further examine the topology of the FK-product, we consider two theorems that reveal the structure of the neighborhoods of zero. The closed convex hull W of a subset of an FK-space E is denoted by $\widehat{W}$.

Theorem 4. *Let p and q be continuous seminorms on FK- spaces E and F, respectively. If $U = \{ x \in E \mid p(x) \le 1 \}$ and $V = \{ y \in F \mid q(y) \le 1 \}$, then $\{ w \in E \widehat{\otimes} F \mid p \otimes q(w) \le 1 \}$ is the closed convex hull of $U \cdot V$.*

Proof. Let w be in the closed convex hull of $U \cdot V$. Then for each $n = 1, 2, \ldots$, there exists a w^n in the convex hull of $U \cdot V$ such that $w^n \to w$ (as $n \to \infty$). For each n, w^n is a convex linear combination of $U \cdot V$. That is, we can find $0 \le \lambda_{nj} \le 1$ and $u^{nj} \in U$, $v^{nj} \in V$ such that, $\sum_{j=1}^{N(n)} \lambda_{nj} = 1$ and $w^n = \sum_{j=1}^{N(n)} \lambda_{nj} u^{nj} \cdot v^{nj}$. Then $p \otimes q(w^n) \le \sum_j \lambda_{nj} p(u^{nj}) q(v^{nj}) \le \sum_j \lambda_{nj} = 1$. Thus $p \otimes q(w) = \lim_n p \otimes q(w^n) \le 1$.

Conversely, suppose $p \otimes q(w) \le 1$. Let $w^n = \frac{n-1}{n} w$. There exists a representation $w^n = \sum_j x^{nj} \cdot y^{nj}$ in $E \widehat{\otimes} F$ such that $\sum_j p(x^{nj}) q(y^{nj}) < 1$. Let $\{r_n\}$ be increasing generating seminorms of $E \widehat{\otimes} F$. Consider the truncations $t^n = \sum_{j=1}^{N(n)} x^{nj} \cdot y^{nj}$ where $N(n)$ are chosen so that $r_n(t^n - w^n) < 1/n$ and let $\alpha_n = \sum_{j=1}^{N(n)} p(x^{nj}) q(y^{nj})$. If we let $u^{nj} = (\sqrt{\alpha_n}/p(x^{nj})) x^{nj}$, $v^{nj} = (\sqrt{\alpha_n}/q(y^{nj})) y^{nj}$ and $\lambda_{nj} = p(x^{nj}) q(y^{nj}) / \alpha_n$, then $t^n = \sum_{j=1}^{N(n)} \lambda_{nj} u^{nj} \cdot v^{nj}$ is a convex linear combination of $U \cdot V$. Also t^n converges to w since for each m and for all $n > m$ we have $r_m(t^n - w) \le r_n(t^n - w^n) + r_m(w^n - w) < (1/n) + (1/n) r_m(w)$, which tends to 0 as n tends to ∞. Hence w is in the closed convex hull of $U \cdot V$. ∎

The following theorem is an easy consequence of the above. The proof is omitted.

Theorem 5. *Let E and F be FK-spaces and let $\{U\}$ and $\{V\}$ be 0-neighborhood bases of E and F, respectively. Then $\{\widehat{U \cdot V}\}$ is a 0-neighborhood base of $E \widehat{\otimes} F$.*

4. FK-products and multipliers. Here we consider results that relate multiplier spaces and FK-products. For FK-spaces E and F, the multiplier space from E to F is defined by

$$(E \to F) = \{ y \in \omega \mid x \cdot y \in F \quad \text{for all} \quad x \in E \}.$$

Let T be a row finite infinite matrix with columns converging to 1. If the summability field of T is $c_T = \{x \in \omega \mid \lim_{n \to \infty} \sum_k t_{nk} x_k \text{ exists}\}$ and the boundedness domain of T is $m_T = \{x \in \omega \mid \sup_n |\sum_k t_{nk} x_k| < \infty\}$, we define the β_T- and γ_T-duals of E by $E^{\beta_T} = (E \to c_T)$ and $E^{\gamma_T} = (E \to m_T)$. We say that an element x of an FK-space E has the property TK if the sequences $t^n x = (t_{n1} x_1, \ldots, t_{nk} x_k, \ldots)$ belong to E and converge to x (as $n \to \infty$) in the topology of E. If for some $x \in \omega$ and all continuous linear functionals f on E, $f(t^n x)$ converges, we say that x has the property FTK in E and if the set $\left\{ t^n x \right\}_{n=1}^{\infty}$ is a bounded subset of E, we say that x has the property TB. The sequence x need not belong to E for FTK and TB. The properties TK, FTK, and TB and their relationship to β_T- and γ_T-duality are considered in [1] and [4].

The set of all $x \in E$ with the property TK is denoted by E_{TK}, the set of all $x \in \omega$ with the property FTK is denoted by E_{FTK} and the set of all $x \in \omega$ with the property TB is denoted by E_{TB}. If $E = E_{TK}$ (respectively, $E \subset F_{FTK}, E \subset E_{TB}$) we say that E is a TK-space (respectively, FTK-space, TB- space).

If T is the matrix with 1 on and below the diagonal and 0 elsewhere, the properties TK, FTK, and TB correspond to AK, FAK, and AB considered in [2]. If T is the matrix with $t_{nk} = 1 - \frac{k-1}{n}$ for $k \le n$ and 0 elsewhere, we get the properties σK, $F\sigma K$ and σB of [2]. The properties TK, FTK, and TB thus unify and generalize properties considered in [2].

Theorem 6. *(cf 4.1 and 4.2 of [2]). Let E and F be FK- spaces. If E is a TK-, FTK- or TB-space, then so is $E\widehat{\otimes}F$.*

Proof. Suppose $E = E_{TK}$. For each $y \in F$, define $f_y(x) = x \cdot y$. The multiplier map f_y is a special case of a matrix map from the FK-space E into the FK-space $E\widehat{\otimes}F$. Matrix maps between FK-spaces are continuous [5]. Thus f_y maps the convergent sequence $t^n x$ into a convergent sequence $t^n x \cdot y = t^n(x \cdot y)$ in $E\widehat{\otimes}F$. This shows that $E \cdot F \subset (E\widehat{\otimes}F)_{TK}$. By an argument similar to the proof of Theorem 7 of [1], $(E\widehat{\otimes}F)_{TK}$ is an FK-space. By Theorem 3 above, $E\widehat{\otimes}F \subset (E\widehat{\otimes}F)_{TK}$ and hence $E\widehat{\otimes}F = (E\widehat{\otimes}F)_{TK}$. The proof is similar for the properties FTK and TB. ∎

Remark. An FK-space E is a TK-space if and only if the space of finite sequences $\phi = \{\, x \in \omega \,|\, x_k = 0 \quad \text{except for finitely many} \quad k \,\}$ is dense in E (the property AD) and E is a TB-space [4]. Since the property AD is also inherited by the FK-product, the product $E\widehat{\otimes}F$ is a TK-space whenever E has the property AD and F is a TB-space.

Theorem 7. *Let E and F be FK-spaces. If $E\widehat{\otimes}F$ is a TK-space, then the topological dual of $E\widehat{\otimes}F$ can be identified with $\big(E \to F^{\beta_T}\big)$ in the sense that every continuous linear functional f on $E\widehat{\otimes}F$ can be represented by*

$$f(z) = \lim_n \sum_{k=1}^{\infty} t_{nk}\, f_k\, z_k \quad \text{for some unique} \quad (f_k) \in \big(E \to F^{\beta_T}\big).$$

In this case $\big(E \to F^{\beta_T}\big) = \big(E \to F^{\gamma_T}\big)$.

Proof. For an FK-space which is a TK-space, the β_T- and γ_T-duals are equal and can be identified with the topological dual (Theorem 3.2 of [4]). If $E\widehat{\otimes}F$ is a TK-space, then $(E\widehat{\otimes}F)' = (E\widehat{\otimes}F)^{\beta_T} = (E\widehat{\otimes}F)^{\gamma_T}$. By 5.6 of [2], $((E\widehat{\otimes}F) \to G) = (E \to (F \to G))$ for any FK space G. Thus $(E\widehat{\otimes}F)' = \big(E \to F^{\beta_T}\big) = \big(E \to F^{\gamma_T}\big)$. ∎

Theorem 8. *(cf 5.4 and 5.6 of [2]). Let E and F be FK- spaces. If $F \subset F_{FTK}$, then*

$$\big(E \to F^{\beta_T}\big)^{\beta_T} = (E\widehat{\otimes}F)_{FTK}.$$

If in addition F^{β_T} is an FK-space, then

$$(E \to F_{FTK}) = \big(E\widehat{\otimes}F^{\beta_T}\big)^{\beta_T}.$$

The statement remains true if FTK is replaced by TB and β_T is replaced by γ_T.

Proof. As in the above proof, by 5.6 of [2], we have $\big(E \to F^{\beta_T}\big) = (E\widehat{\otimes}F)^{\beta_T}$. Thus $\big(E \to F^{\beta_T}\big)^{\beta_T} = (E\widehat{\otimes}F)^{\beta_T\beta_T}$. By Theorem 6, $E\widehat{\otimes}F$ is an FTK- space. Thus by Theorem 4 of [1], we have $(E\widehat{\otimes}F)^{\beta_T\beta_T} = (E\widehat{\otimes}F)_{FTK}$. Similarly, if F^{β_T} is an FK-space, we have by Theorem 4 of [1], $(E \to F_{FTK}) = \big(E \to F^{\beta_T\beta_T}\big)$ and by 5.6 of [2], $\big(E \to F^{\beta_T\beta_T}\big) = \big(E\widehat{\otimes}F^{\beta_T}\big)^{\beta_T}$. The proof of TB and γ_T is similar. ∎

5. FK-products of mixed norm spaces $l^{p,q}$. For $1 \leq p, q < \infty$ the *mixed norm spaces* $l^{p,q}$ are defined by

$$l^{p,q} = \{\, x \in \omega \mid \|x\|_{p,q} = (\sum_{n=0}^{\infty}(\sum_{2^n \leq k < 2^{n+1}} |x_k|^p)^{\frac{q}{p}})^{\frac{1}{q}} < \infty \,\},$$

for $q = \infty$ they are defined by

$$l^{p,\infty} = \{\, x \in \omega \mid \|x\|_{p,\infty} = \sup_n (\sum_{2^n \leq k < 2^{n+1}} |x_k|^p)^{\frac{1}{p}} < \infty \,\},$$

and for $p = \infty$ they are defined by

$$l^{\infty,q} = \{x \in \omega \mid \|x\|_{\infty,q} = (\sum_{n=0}^{\infty}(\sup_{2^n \leq k < 2^{n+1}} |x_k|)^q)^{\frac{1}{q}} < \infty \}.$$

These spaces were introduced by Kellogg [3]. The spaces $l^{p,q}$, under the norms $\|\cdot\|_{p,q}$, are BK-spaces that lie between l^p and l^q. Since $l^{p,p} = l^p$, mixed norm spaces include the l^p spaces.

A sequence space E is said to be *solid* if $x \in E$ and $|y_k| \le |x_k|$ for all k, implies $y \in E$. Clearly the mixed norm spaces $l^{p,q}$ are solid. Also it follows from our definition of the FK-product that the FK-product of solid FK-space and any other FK-space is again solid.

For each sequence $x \in \omega$, we define $s^n x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$. For each FK-space E, we define, as in section 4,

$$E_{AB} = \{\, x \in \omega \mid \{s^n x\} \text{ is a bounded subset of } E \,\}$$

and we say that E is an *AB-space* if $E \subset E_{AB}$. E is *AB-perfect* if $E = E_{AB}$. Clearly the mixed norm spaces $l^{p,q}$ are AB-perfect for all $1 \le p, q \le \infty$. By Theorem 4.1 of [2] or Theorem 6, the FK-product of AB-spaces is an AB-space. In the next section we show that the FK-product of AB- perfect spaces need not be AB-perfect.

We define $E_{AK} = \{\, x \in E \mid \lim_n s^n x = x \;$ with respect to the topology of $E \,\}$.We say that E is an *AK-space* if $E = E_{AK}$. Clearly the mixed norm spaces $l^{p,q}$ are AK-spaces for $1 \le q < \infty$ but $(l^{p,\infty})_{AK} = \{\, x \in \omega \mid \lim_{n\to\infty} \sum_{2^n \le k < 2^{n+1}} |x_k|^p = 0 \,\}$.

Theorem 9. *$(l^{p,q} \widehat{\otimes} l^{r,s})_{AB} = l^{u,v}$, where $u = \max(1, \frac{pr}{p+r})$ and $v = \max(1, \frac{qs}{q+s})$ with the convention $\frac{ab}{a+b} = a$ whenever $b = \infty$.*

Proof. Let E be an FK-spaces. By Theorem 4 of [1], E is an AB-space if and only if $E_{AB} = (E^\gamma)^\gamma$, where the γ-dual is defined by $E^\gamma = \{\, y \in \omega \mid \sup_n |\sum_k^n x_k y_k| < \infty$ for all $x \in E \,\}$. It can easily be shown that if E is solid, then $E^\gamma = E^\alpha$, where the α-dual is defined by $E^\alpha = \{\, y \in \omega \mid \sum_k^\infty |x_k y_k| < \infty$ for all $x \in E \,\}$. Thus $(l^{p,q} \widehat{\otimes} l^{r,s})_{AB} = ((l^{p,q} \widehat{\otimes} l^{r,s})^\gamma)^\gamma = ((l^{p,q} \widehat{\otimes} l^{r,s})^\alpha)^\alpha$. By Theorem 5.6 of [2], $(E \widehat{\otimes} F)^\alpha = (E \to F^\alpha)$. Thus $(l^{p,q} \widehat{\otimes} l^{r,s})_{AB} = (l^{p,q} \to (l^{r,s})^\alpha)^\alpha$. Kellogg ([3], Theorem 1) gave the characterization $(l^{u_1,u_2} \to l^{v_1,v_2}) = l^{w_1,w_2}$, where $\frac{1}{w_i} = \frac{1}{v_i} - \frac{1}{u_i}$ if $u_i > v_i$ and $w_i = \infty$ if $u_i \le v_i$, $(i = 1, 2)$. Thus $(l^{r,s})^\alpha = (l^{r,s} \to l^1) = (l^{r,s} \to l^{1,1}) = l^{r',s'}$, where $'$ denotes the conjugate exponent. Hence $(l^{p,q} \to (l^{r,s})^\alpha)^\alpha = (l^{p,q} \to l^{r',s'})^\alpha = (l^{u',v'})^\alpha = l^{u,v}$, where $\frac{1}{u'} = \frac{1}{r'} - \frac{1}{p}$ if $p > r'$ and $u' = \infty$ if $p \le r'$, and where $\frac{1}{v'} = \frac{1}{s'} - \frac{1}{q}$ if $q > s'$ and $v' = \infty$ if $q \le s'$. It follows that $u = \frac{pr}{p+r}$ if $p > r'$ and $u = 1$ if $p \le r'$. But $\frac{pr}{p+r} > 1$ if and only if $p > r'$. Thus $u = \max(1, \frac{pr}{p+r})$. Similarly $v = \max(1, \frac{qs}{q+s})$. ∎

Theorem 10. *If q and s are not both infinite, then $l^{p,q} \widehat{\otimes} l^{r,s}$ is an AK-space which is AB-perfect and $l^{p,q} \widehat{\otimes} l^{r,s} = l^{u,v}$, where u and v are defined as in Theorem 9.*

Proof. If either q or s is finite, then either $l^{p,q}$ or $l^{r,s}$ is an AK-space. Then $(l^{p,q} \widehat{\otimes} l^{r,s})$ is an AK-space by 4.1 of [2] or Theorem 6. Then also, with u and v as in Theorem 9,v is finite and $(l^{p,q} \widehat{\otimes} l^{r,s})_{AB} = l^{u,v} = (l^{u,v})_{AK} = ((l^{p,q} \widehat{\otimes} l^{r,s})_{AB})_{AK} = (l^{p,q} \widehat{\otimes} l^{r,s})_{AK} = l^{p,q} \widehat{\otimes} l^{r,s}$. ∎

An example where both q and s are infinite is considered in the next section.

6. The FK-product of AB-perfect spaces need not be AB-perfect. In 1977, G. Goes conjectured that the FK-product of AB-perfect spaces is AB-perfect. The various examples of FK-products given in [2] support this conjecture. If E and F are AB-perfect FK-spaces, then $E \widehat{\otimes} F \subset (E \widehat{\otimes} F)_{AB}$. However, the following shows that the inclusion can be strict, even if E and F are solid BK-spaces.

Theorem 11. $l^{1,\infty}\widehat{\otimes}l^{1,\infty} \subsetneqq (l^{1,\infty}\widehat{\otimes}l^{1,\infty})_{AB} = l^{1,\infty}$.

Proof. The equality follows from Theorem 9. Define the subspace $F = \{ x \in l^{1,\infty} \mid \lim_n \frac{N_n}{2^n} = 0$, where N_n is the number of nonzero x_k for $2^n \leq k < 2^{n+1} \}$ and let G be the closure of F in $l^{1,\infty}$. G is a BK-space under the norm of $l^{1,\infty}$ and it contains all of the sequences with only a finite number of nonzero entries. We show that G is a proper subspace of $l^{1,\infty}$ which contains $l^{1,\infty}\widehat{\otimes}l^{1,\infty}$.

G is a proper subspace of $l^{1,\infty}$: Consider the sequence $d = (d_k)$, where $d_k = 2^{-n}$ whenever $2^n \leq k < 2^{n+1}$. Clearly $d \in l^{1,\infty}$ since $\|d\|_{1,\infty} = 1$. But $d \notin G$ since for every $x \in F$, we have

$$\|d - x\|_{1,\infty} = \sup_n \sum_{2^n \leq k < 2^{n+1}} |2^{-n} - x_k| \geq \sup_n \sum_{\substack{2^n \leq k < 2^{n+1} \\ x_k = 0}} 2^{-n}$$

$$= \sup_n 2^{-n}(2^n - N_n) = \sup_n (1 - \frac{N_n}{2^n}) \geq \lim_{n\to\infty} (1 - \frac{N_n}{2^n}) = 1.$$

$l^{1,\infty}\widehat{\otimes}l^{1,\infty} \subset G$: We show $l^{1,\infty} \cdot l^{1,\infty} \subset G$ and use Theorem 3. Let $x, y \in l^{1,\infty}$ and let $\epsilon > 0$. If $y = 0$, then $x \cdot y \in G$. Otherwise construct x' by defining, for all n and $2^n \leq k < 2^{n+1}$,

$$x'_k = \begin{cases} x_k & \text{if } |x_k| > \dfrac{\epsilon n}{2^n \|y\|_{1,\infty}} \\ 0 & \text{otherwise} \end{cases} .$$

Clearly

$$\infty > \|x\|_{1,\infty} \geq \|x'\|_{1,\infty} \geq \sup_n \sum_{\substack{2^n \leq k < 2^{n+1} \\ x'_k \neq 0}} \frac{\epsilon n}{2^n \|y\|_{1,\infty}} = \sup_n (\frac{\epsilon n}{\|y\|_{1,\infty}})(\frac{N_n}{2^n}),$$

where N_n is the number of nonzero entries of x' in $2^n \leq k < 2^{n+1}$. This inequality shows that $\frac{N_n}{2^n} \to 0$ as $n \to \infty$. Hence $x' \in F$. Since F is solid and $|x'_k y_k| \leq |x'_k| \cdot \|y\|_{1,\infty}$, we have also $x' \cdot y \in F$. Further

$$\|x \cdot y - x' \cdot y\|_{1,\infty} = \sup_n \sum_{2^n \leq k < 2^{n+1}} |x_k y_k - x'_k y_k| = \sup_n \sum_{\substack{2^n \leq k < 2^{n+1} \\ x'_k = 0}} |x_k y_k|$$

$$\leq \sup_n \frac{\epsilon n}{2^n \|y\|_{1,\infty}} \sum_{2^n \leq k < 2^{n+1}} |y_k| \leq \epsilon \sup_n \frac{n}{2^n} < \epsilon$$

Thus, we have found an element $x' \cdot y \in F$ arbitrarily close to $x \cdot y$. Since G is the closure of F, it follows that $x \cdot y \in G$. Hence $l^{1,\infty} \cdot l^{1,\infty} \subset G$. By Theorem 3, $l^{1,\infty}\widehat{\otimes}l^{1,\infty} \subset G$. ∎

REFERENCES

[1] M. Buntinas, "On Toeplitz sections in sequence spaces," *Math. Proc. Cambridge Philosophical Soc.* **121** (1975), 451– 460.

[2] M. Buntinas and G. Goes, "Product of sequence spaces and multipliers." To appear.

[3] C.N.Kellogg, "An extension of the Hausdorff-Young Theorem," *Michigan Math. J.* **18** (1971), 121–127.

[4] G. Meyers, "On Toeplitz sections in FK-spaces," *Studies Math.* **51** (1974), 23–33.

[5] H.H.Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.

[6] K. Zeller, "Allgemeine Eigenschaften von Limitierungsverfahren," *Math. Zeitschrift* **55** (1951), 55–70.

Department of Mathematical Sciences, Loyola University of Chicago, Chicago, Illinois 60626.