INTEGRABILITY CLASSES AND SUMMABILITY*

by M. BUNTINAS and N. TANOVIC-MILLER

Dept. of Mathematical Sciences, Loyola University Chicago, Chicago, IL 60626, USA
Dept. of Mathematics, University of Sarajevo, 71000 Sarajevo, Yugoslavia

1980 Mathematics Subject Classification (1985 Revision): Primary 46A45; Secondary 42A16, 42A24.

This paper shows the relationship between problems of integrability and problems of the summability of Fourier series. The largest known family of integrability classes $H^{p}(p > 1)$ for even trigonometric series is introduced.

1. Integrability classes. Let $L^{1}$ denote the Banach space of $2\pi$-periodic integrable functions $f$ with the usual norm

$$\|f\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, dx.$$ 

The Fourier series of a function $f \in L^{1}$ is the series

$$\sum_{k=-\infty}^{\infty} c_{k} e^{ikx},$$

where

$$c_{k} = \widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{ikx} \, dx, \quad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

The sequence $\widehat{f} = (c_{k})_{k=-\infty}^{\infty}$ is called the sequence of Fourier coefficients of $f \in L^{1}$. Let $\widehat{L^{1}} = \{\widehat{f} \mid f \in L^{1}\}$. If one identifies functions that differ only on a set of measure zero,
then the association of $f$ to $\hat{f}$ is a one to one correspondence between $L^1$ and $\hat{L}^1$. Many questions of Fourier series can thus be interpreted as questions about sequences. Furthermore, since $L^1$ is a Banach space under the above norm, $\hat{L}^1$ is a BK–space (a Banach space of sequences with continuous coordinate functionals) under the induced norm

$$||\hat{f}|| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, dx.$$ 

The same can be done with other spaces of periodic functions or distributions such as $C$ (2\pi–periodic continuous functions), $M$ (2\pi–periodic Radon measures), $BV$ (2\pi–periodic functions of bounded variation), and $L^p$ (2\pi–periodic $p$–integrable functions) for $1 < p \leq \infty$. Günther Goes pioneered the study of function spaces $E$ by using BK–space techniques on spaces of Fourier coefficients $\hat{E}$.

The general integrability problem is to characterize those sequences that are Fourier coefficients of integrable functions. Although there are some known characterizations of $\hat{L}^1$ (for example [12], [21], [23] [24]), they are too complicated to be useful and they involve properties of functions such as the pointwise limit of the Fourier series or of the Dirichlet kernel. Unfortunately, there is no known characterization of $\hat{L}^1$ given in terms of properties of sequences alone.

The Riemann–Lebesgue theorem states that for each $f \in L^1$, we have $\hat{f}(n) \to 0$ (as $|n| \to \infty$). This can be written as the inclusion

$$\hat{L}^1 \subset c_0,$$

where $c_0 = \{ x = (x_k)_{k=-\infty}^{\infty} | x_k \to 0 \text{ as } |k| \to \infty \}$ is the space of two–way null sequences. Essentially, there is no known improvement of this inclusion in this direction.

Concerning inclusions in the other direction, a subset of $\hat{L}^1$ is called an integrability class. For a sequence $(c_k)$ let $\Delta c_k = c_k - c_{k+1}$ and $\Delta^2 c_k = \Delta c_k - \Delta c_{k+1}$. A sequence $(c_k)$ is even if $c_k = c_{-k}$ ($k = 1, 2, \ldots$) and it is convex if

$$\Delta^2 c_k \geq 0, \text{ for } k = 0, 1, 2, \ldots.$$ 

In 1913 W.H. Young [30] showed that the even convex null sequences are Fourier coefficient of integrable functions. If we write

$$K = \{ (c_k)_{k=-\infty}^{\infty} | c_k = c_{-k}, \Delta^2 c_k \geq 0 \text{ (} k = 0, 1, 2, \ldots \text{), } c_k \to 0 \text{ (} k \to \infty \} \},$$
then
\[ K \subset \hat{L}^1 \subset c_o. \]

In 1922 S. Sidon [25] (see also [2, vol I, p. 239]) gave an example of an even monotone null sequence which is not in \( \hat{L}^1 \).

An even sequence \((c_k)\) is of *bounded variation* if
\[
\sum_{k=0}^\infty |\Delta a_k| < \infty,
\]
and is *quasiconvex* if
\[
\sum_{k=0}^\infty (k+1)|\Delta^2 c_k| < \infty.
\]

In 1923 A.N. Kolmogorov [19] extended Young’s integrability class to the class of even quasiconvex null sequences \( q_o \). In 1934 J. Pfleger [20] showed that every real quasiconvex sequence is a difference of two convex sequences (this is an extension of the result that every real sequence of bounded variation is a difference of two monotone sequences). Consequently, Pfleger’s result shows that Kolmogorov’s class \( q_o \) is the linear span of Young’s class \( K \). The above mentioned result of Sidon shows that the space of even null sequences of bounded variation \( bv_o \) is not an integrability class.

In 1973 S.A. Telyakovskii [29], using a 1939 result of Sidon [26], extended Kolmogorov’s integrability class to the space \( ST \) of all even null sequences \((c_k)\) for which there exists a positive sequence \((A_k)\) satisfying

\[
\begin{align*}
(1) & \quad A_k \downarrow 0 \quad (k \to \infty) \\
(2) & \quad \sum_{k=0}^\infty A_k < \infty \\
(3) & \quad |\Delta c_k| \leq A_k.
\end{align*}
\]

The Sidon–Telyakovskii class \( ST \) is an extension of Young’s class \( K \). Clearly a sequence \((c_k)\) is convex if and only if \( \Delta c_k \) is monotonically decreasing. Thus if we let \( A_k = \Delta c_k \), every even convex null sequence satisfies (1), (2) and (3). Furthermore, since \( ST \) is a linear space, we have the chain
\[
K \subset q_o \subset ST \subset \hat{L}^1 \subset c_o.
\]
More recently, using three different approaches, the largest known integrability class was extended by G.A. Fomin [15], C.V. Stanojević [27], and Buntinas [7] to the classes $\mathcal{F}_p$ ($p > 1$). There are different formulations of the classes $\mathcal{F}_p$. We can define them as the set of all even null sequences $(c_k)$ for which

$$\sum_{j=0}^{\infty} 2^j \left\{ \sum_{k=2^j}^{2^{j+1}-1} |\Delta c_k|^p \right\}^{1/p} < \infty,$$

where $1 < p < \infty$ and $1/p + 1/q = 1$. Using Hölder’s inequality it can be shown that these Fomin classes $\mathcal{F}_p$ become smaller as $p$ increases. The limiting case $\mathcal{F}_1$ ($p = 1$ and $q = \infty$) is the space of even null sequences of bounded variation, which by Sidon’s example is too large to be an integrability class.

The smallest Fomin class $\mathcal{F}_\infty$ is defined by the condition

$$\sum_{j=0}^{\infty} 2^j \max_{2^j \leq k < 2^{j+1}} |\Delta c_k| < \infty. \quad (4)$$

**PROPOSITION 1.** $ST = \mathcal{F}_\infty$.

**PROOF.** There is no loss of generality if we set $A_j = \sup \{ \Delta c_k \}$ in (1), (2), (3). Thus the Sidon–Telyakovskii conditions are equivalent to

$$\sum_{j=0}^{\infty} \sup_{k \geq j} |\Delta c_k| < \infty. \quad (5)$$

By a theorem of Cauchy, if $A_j \downarrow 0$ ($j \to \infty$), then $\sum_{j=0}^{\infty} A_j < \infty$ if and only if $\sum_{j=0}^{\infty} 2^j A_{2j} < \infty$. Thus (5) is equivalent to

$$\sum_{j=0}^{\infty} 2^j \sup_{k \geq 2^j} |\Delta c_k| < \infty. \quad (6)$$

Clearly (6) implies (4). Conversely, suppose $(c_k)$ satisfies (4). Let $D_j = \max_{2^j \leq k < 2^{j+1}} |\Delta c_k|$. Then

$$2^n \sum_{j=n}^{\infty} D_j \leq \sum_{j=n}^{\infty} 2^j D_j = o(1) \quad \text{as} \quad (n \to \infty).$$

Hence for all $n > 1$, using summation by parts, we have

$$\sum_{j=0}^{n} 2^j \sup_{k \geq 2^j} |\Delta c_k| \leq \sum_{j=0}^{n} 2^j \sum_{i=j}^{\infty} D_i = \sum_{j=0}^{n} (2^{j+1} - 2^j) \sum_{i=j}^{\infty} D_i$$

$$= 2^{n+1} \sum_{j=n}^{\infty} D_j + \sum_{j=1}^{n} 2^j D_{j-1} - \sum_{j=0}^{\infty} D_j \leq 2 \sum_{j=0}^{\infty} 2^j D_j$$
which implies (6).

The conditions (4), (5) and (6) are each equivalent to the Sidon–Telyakovskii conditions. For any \( p > 1 \), we have the chain

\[
K \subset q_o \subset ST = \mathcal{F}_{\infty} \subset \mathcal{F}_p \subset \hat{L}^1 \subset c_o.
\]

2. Summability in \( L^1 \). In this section we consider the connection between integrability classes and summability in \( L^1 \). Generally, to each summability method on \( L^1 \), there corresponds an integrability class. For example, it was shown in [3] that Kolmogorov’s integrability class \( q_o \) can be obtained from Fejér’s theorem on Cesàro summability of \( L^1 \), and conversely Fejér’s theorem follows from Kolmogorov’s result. We consider this type of equivalence in general BK–spaces.

Let \( e^k \) be the sequence with 1 in the \( k \)th position and 0 elsewhere. The \( n \)th section of a sequence \( c \) is defined to be the sequence \( s^nc = \sum_{k=-n}^{k=n} c_k e^k \) in the case of two–way sequences, and \( s^nc = \sum_{k=0}^{k=n} c_k e^k \) in the case of one–way sequences. A BK–space \( E \) is said to have sectional density if the linear span of the sequences \( e^k \) is dense in \( E \). For sets of sequences \( A \) and \( B \), we define the coordinatewise product \( A \cdot B \) as the set of all \( x \cdot y = (x_k y_k) \) with \( x \in A \) and \( y \in B \).

In 1968 D.J.H. Garling [17] showed that a sectionally dense BK–space \( E \) has sectional convergence, that is

\[
\|s^nc - c\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad c \in E,
\]

if and only if \( bv_o \cdot E \subset E \).

This is an interesting result since it equates an approximation statement with a multiplier statement. For Fourier series, its value is limited since \( \hat{L}^1 \) does not have sectional convergence. Actually, sectional convergence in \( \hat{L}^1 \) is equivalent to the convergence of Fourier series in norm. There is no known characterization of the subspace of \( \hat{L}^1 \) with the property of sectional convergence.

In 1904 L. Fejér [14] showed that the Fourier series of every \( f \in L^1 \) is Cesàro summable in norm. In 1970 the first author [3] showed that a sectionally dense BK–space \( E \) has Cesàro summability, that is

\[
\frac{1}{n+1} \sum_{k=0}^{n} (s^kc - c) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad c \in E,
\]
if and only if \( q_o \cdot E \subset E \). When applied to \( \hat{L}^1 \), the first statement is Fejér’s theorem and the second is \( q_o \cdot \hat{L}^1 \subset \hat{L}^1 \). The latter is equivalent to Kolmogorov’s integrability result. For the space of all multipliers from \( \hat{L}^1 \) into \( \hat{L}^1 \) is \( \hat{M} \), the space of Fourier coefficients of Radon measures [13, vol. 2, p. 255]. Thus \( q_o \subset \hat{M} \). Since the BK–spaces \( q_o \) has sectional density and \( \hat{L}^1 \) is the closed subspace of \( \hat{M} \) with sectional density, we have \( q_o \subset \hat{L}^1 \), which is Kolmogorov’s result. Reversing the argument, Kolmogorov’s result implies the multiplier statement.

In 1972 the first author [4] [5] showed that results equating summability statements and multiplier statements hold for other matrix methods of summability. Let \( T \) be a row finite reversible Toeplitz matrix in series–sequence form, let \( c_T = \{ x = (x_k) | \lim(Tx)_n \text{ exists} \} \) be its summability field, and let \( q_T \) be the intersection of \( c_o \) and the space of multipliers from \( c_T \) into \( c_T \). If \( c_T \) is a sum space in the sense of W.H. Ruckle [22], then a sectionally dense BK–space \( E \) has Toeplitz summability, that is

\[
\left\| \sum_{k=0}^{\infty} t_{nk} c_k e^k - c \right\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad c \in E,
\]

if and only if \( q_T \cdot E \subset E \). In 1976 W. Balser [1] obtained an improvement by eliminating the condition that \( c_T \) be a sum space and enlarged \( q_T \) to the sequence space associated with the topological dual of \( c_T \).

Theorems which show the equivalence of summability statements and multiplier statements were later found for certain non-matrix methods such as Abel summability [6], absolute convergence [9], and strong convergence [10].

In 1913 G.H. Hardy [18] showed that the Fourier series of every \( f \in L^1 \) is strongly Cesàro summable of order \( q \geq 1 \), a.e. That is,

\[
\frac{1}{n + 1} \sum_{k=0}^{n} |s_k f(x) - f(x)|^q \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad f \in L^1,
\]

where \( s_k f(x) = \sum_{j=0}^{k} \hat{f}(j)e^{i j x} \). For \( f \in C \), this can be improved to uniform convergence.

Strong summability methods are non-matrix methods. The definition of strong summability in norm is not obvious. In [8] the first author defined it as follows: A
BK–space $E$ has strong Cesàro summability of order $q \geq 1$ if for continuous linear functionals $g \in E'$

$$\sup_{\|g\|_{E'} \leq 1} \frac{1}{n+1} \sum_{k=0}^{n} |g(s^k c - c)|^q \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad c \in E.$$ 

The statement about $C$ above is equivalent to saying that the BK–space $\hat{C}$ has strong Cesàro summability of every order $q \geq 1$. The following result was obtained in [8]: A sectionally dense BK–space $E$ has strong Cesàro summability of order $q \geq 1$ if and only if $\mathcal{F}_p \cdot E \subset E$, where $1/p + 1/q = 1$. When applied to $\hat{L}^1$, Fomin’s integrability result implies strong Cesàro summability of every order $q \geq 1$ for $\hat{L}^1$ and visa versa. By using known multiplier results, one can also obtain strong Cesàro summability in other spaces.

In 1989 the authors [11] found the largest then known integrability classes $hv^p$ ($p > 1$) for even series which are defined in terms of coefficients only. For each $p > 1$ the class $hv^p$ is defined to be the set of all even null sequences $(c_k)$ for which there exist an increasing sequence $(k_j)$ and a non-decreasing sequence $(n_j)$ such that $1 \leq n_j \leq k_{j+1}$, $1/p + 1/q = 1$ and

$$\sum_{j=0}^{\infty} n_j^{1/q} \left\{ \sum_{k=k_j}^{k_{j+1}-1} |\Delta c_k|^p \right\}^{1/p} + \sum_{j=0}^{\infty} \log\left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta c_k| < \infty.$$ 

This condition may appear complicated but it is easily useable. If we set $k_j = 2^j$ and $n_j = 2^{j+1}$, then the second term disappears and we obtain a subclass equal to $\mathcal{F}_p$. If $n_j = 1$, then the first term can be ignored and we obtain a subclass $av^1$ which can be defined by the condition

$$\sum_{k=1}^{\infty} \log k |\Delta c_k| < \infty. \quad (7)$$

It is shown in [11] that $av^1$ is a subclass of every $hv^p$ but it is not a subclass of any of the other previously mentioned integrability classes.

### 3. The spaces $HV^p$.

The integrability classes $hv^p$ are not linear spaces. In this section we enlarge the classes $hv^p$ ($p > 1$) to BK–spaces $HV^p$. We show in Theorem 1 that the classes $HV^p$ are proper extensions of $hv^p$. Then we show in Theorems 2 and 3 that these spaces are integrability classes and BK–spaces.
For each $p > 1$ we define the space $HV^p$ to be the set of all sequences $c$ for which there exist $c^i \in hv^p$, $i = 1, 2, 3, \ldots$, such that $c = \sum_{i=1}^{\infty} c^i$ coordinatewise and $\sum_{i=1}^{\infty} v(c^i) < \infty$, where

$$v(c^i) = \inf \left\{ \sum_{j=0}^{\infty} n_j^{1/q} \left\{ \sum_{k=k_j}^{k_{j+1}-1} |\Delta c^i_k|^p \right\}^{1/p} + \sum_{j=0}^{\infty} \log \left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta c^i_k| \right\},$$

the infimum ranging over all increasing sequences $(k_j)$ and all non-decreasing sequences $(n_j)$ with $1 \leq n_j \leq k_{j+1}$. The spaces $HV^p$ are generated by infinite sums of elements in $hv^p$. They are clearly linear spaces.

**PROPOSITION 2.** The classes $hv^p$ are not linear spaces for any $p > 1$.

**PROOF.** We construct sequences $a$ and $b$ such that $a, b \in hv^p$ for all $p > 1$ but such that $a + b \notin hv^p$ for all $p > 1$. Let $a$ and $b$ be defined so that

$$\Delta a_k = \frac{1}{(j+1)^3} \text{ for } k = 2^j (j = 0, 1, 2, \ldots) \text{ and } \Delta a_k = 0 \text{ otherwise;}$$

$$\Delta b_k = \frac{1}{k \log^2(k+1)} \text{ for } k = 1, 2, 3, \ldots.$$

Then, for appropriate values of $a_0$ and $b_0$, we have $a, b \in c_0$. Clearly

$$\sum_{k=2}^{\infty} \log k |\Delta a_k| = \log 2 \sum_{j=1}^{\infty} j \frac{1}{(j+1)^3} < \infty$$

so that by (7) we have $a \in av^1$. Since by [11, Theorem 7] $av^1 \subset hv^p$ for all $p > 1$, we see that $a \in hv^p$ for all $p > 1$.

Let $p > 1$ and $1/p + 1/q = 1$. Then for the second sequence we have

$$\sum_{j=0}^{\infty} 2^{j/q} \left\{ \sum_{k=2^j}^{2^{j+1}-1} |\Delta b_k|^p \right\}^{1/p} = \sum_{j=0}^{\infty} 2^{j/q} \left\{ \sum_{k=2^j}^{2^{j+1}-1} \frac{1}{k \log^2(k+1)} \right\}^{1/p} \leq \sum_{j=0}^{\infty} \frac{1}{(j+1)^2 \log^2 2} < \infty.$$

Hence, for all $p > 1$, $b \in F_p \subset hv^p$.

We now verify that $a + b \notin hv^p$ for all $p > 1$. Suppose to the contrary that $a + b \in hv^p$ for some $p > 1$. Then there exist sequences $(k_j)$ increasing and $(n_j)$ non-decreasing such that $1 \leq n_j \leq k_{j+1}$ (we may assume $k_0 = 0$) and

$$\sum_{j=0}^{\infty} n_j^{1/q} \left\{ \sum_{k=k_j}^{k_{j+1}-1} |\Delta a_k + \Delta b_k|^p \right\}^{1/p} + \sum_{j=0}^{\infty} \log \left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta a_k + \Delta b_k| < \infty.$$
Observing that \(|\Delta a_k + \Delta b_k| \geq \Delta a_k \geq 0\) and \(|\Delta a_k + \Delta b_k| \geq \Delta b_k \geq 0\) for all \(k\), we have

\[
\sum_{j=0}^{\infty} n_j^{1/q} \left( \sum_{k=k_j}^{k_{j+1}-1} |\Delta a_k|^p \right)^{1/p} < \infty
\]

and

\[
\sum_{j=0}^{\infty} \log\left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta b_k| < \infty.
\]

We shall show that both cannot hold.

For each \(m\) let \(j_m\) be that integer for which \(k_{j_m} \leq 2^m < k_{j_m+1}\). Clearly \((j_m)\) is non-decreasing. Using Minkowski’s inequality for the space \(\ell^p\) we have

\[
\sum_{j=j_m}^{\infty} n_j^{1/q} \left( \sum_{k=k_j}^{k_{j+1}-1} |\Delta a_k|^p \right)^{1/p} \geq n_j^{1/q} \left( \sum_{i=m}^{\infty} \frac{1}{(i+1)^{3p}} \right)^{1/p} \geq Kn_j^{1/q} \frac{1}{m^{2+1/q}}
\]

for some constant \(K\). From this and (8) it follows that \(n_{j_m} = o(1)m^{2q+1} (m \to \infty)\). Letting \((j_{m_r})\) be the increasing subsequence of \((j_m)\) obtained by dropping the repeating terms we have \(n_{j_{m_r}} = o(1)m_r^{2q+1} (r \to \infty)\). Then

\[
n_{j_{m_r}} < 2^{m_r-1}
\]

for sufficiently large \(r\), say \(r \geq N\). Considering the sum in (9) we see that

\[
\sum_{j=0}^{\infty} \log\left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta b_k| \geq \sum_{r=1}^{j_{m_r+1}} \sum_{j=j_{m_r+1}}^{j_{m_r+1}-1} \log\left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta b_k| \\
\geq \sum_{r=1}^{\infty} \log\left( \frac{k_{j_{m_r+1}}}{n_{j_{m_r+1}}} \right) \sum_{k=k_{j_{m_r}}}^{k_{j_{m_r+1}}-1} |\Delta b_k|.
\]

Since \(j_m = j_{m_r}\) for \(m \leq m_r < m_{r+1}\), it follows that \(k_{j_{m_r}} \leq 2^{m_r+1-1} < k_{j_{m_r}+1}\). Hence from (10) we obtain

\[
\sum_{j=0}^{\infty} \log\left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta b_k| \geq \sum_{r=N}^{\infty} \log\left( \frac{2^{m_r+1-1}}{2^{m_{r+1}+1-1}} \right) \sum_{k=k_{j_{m_r}}}^{k_{j_{m_r}+1}-1} |\Delta b_k| \\
\geq \frac{1}{2} \sum_{r=N}^{\infty} \sum_{k=k_{j_{m_r}}}^{k_{j_{m_r}+1}-1} \log k |\Delta b_k| = \frac{1}{2} \sum_{k=k_{j_{m_N}}}^{\infty} \log k |\Delta b_k|.
\]
Observing that by definition of $b$,

\[ \sum_{k=2}^{\infty} \log k |\Delta b_k| = \sum_{k=2}^{\infty} \frac{1}{k \log k} = \infty, \]

the above inequality contradicts (9). Hence $a + b \not\in hv^p$. \hfill \blacksquare

**THEOREM 1.** For every $p > 1$ we have $hv^p \subset HV^p$ properly.

**PROOF.** Clearly the classes $HV^p$ are linear spaces containing $hv^p$. Since the classes $hv^p$ are not linear spaces, the inclusion is proper. \hfill \blacksquare

**THEOREM 2.** For every $p > 1$ we have $HV^p \subset \hat{L}^1$.

**PROOF.** In [11, Theorem 3] it is shown that if $c \in hv^p$, then

\[ \sum c_k e^{ikx} \]

is the Fourier series of a function $f$ in $L^1$ and $\|f\|_{L^1} \leq K v(c)$ for some constant $K$. Now suppose that $c = \sum_i c^i$ with $c^i \in hv^p$, $\sum_i v(c^i) < \infty$, and $\hat{f}^i = c^i$. Then clearly $\sum_i f^i$ converges absolutely to a function $f \in L^1$ and $\hat{f} = c$. \hfill \blacksquare

Putting all of the results together we have for any $p > 1$,

\[ K \subset q_0 \subset ST = F_\infty \subset F_p \subset hv^p \subset HV^p \subset \hat{L}^1 \subset c_0. \]

Now it is important to find summability methods corresponding to this new family of integrability classes $HV^p$ ($p > 1$). These would be the strongest summability methods known for $L^1$. The authors are considering a family of candidate summability methods which have not been investigated before. As a step in this direction, we show that the spaces $HV^p$ are BK–spaces.

Let $p > 1$ and $1/p + 1/q = 1$. We first consider $hl^p$ defined as the set of all sequences $(x_k)_{k=0}^{\infty}$ for which there exist increasing sequences $(k_j)$ and non-decreasing sequences $(n_j)$ such that $1 \leq n_j \leq k_{j+1}$ and

\[ \sum_{j=0}^{\infty} n_j^{1/q} \left\{ \sum_{k=k_j}^{k_{j+1}-1} |x_k|^p \right\}^{1/p} + \sum_{j=0}^{\infty} \log \left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |x_k| < \infty. \]

If we let $c$ be an even null sequence with $\Delta c_k = x_k$, then $c \in hv^p$ if and only if $x \in hl^p$. Define a real valued function $r$ on $hl^p$ by

\[ r(x) = \inf \left\{ \sum_{j=0}^{\infty} n_j^{1/q} \left\{ \sum_{k=k_j}^{k_{j+1}-1} |x_k|^p \right\}^{1/p} + \sum_{j=0}^{\infty} \log \left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |x_k| \right\}, \]

(11)
where the infimum ranges over all increasing sequences \((k_j)\) and non-decreasing sequences \((n_j)\) with \(1 \leq n_j \leq k_{j+1}\).

Then we can define the normed space \(HL^p\) as the set of all sequences \(x\) for which there is a coordinatewise sum \(x = \sum x^i\) with \(x^i \in hl^p\) such that \(\sum r(x^i) < \infty\). We define the norm on \(HL^p\) by

\[
\|x\| = \inf \sum_{i=1}^{\infty} r(x^i),
\]

where the infimum ranges over all coordinatewise representations \(x = \sum x^i, \ x^i \in hl^p\).

**PROPOSITION 3.** For every \(x \in HL^p\) and \(n = 0, 1, 2, \ldots\), we have \(\|s^nx\| \leq \|x\|\).

**PROOF.** This is obvious from the definition of the norm since both sums of the form \(\sum_{k=k_j}^{k_{j+1}-1} \sum_{k=k_j}^{k_{j+1}-1} x^i k_j \in k_{j+1} \) in (11) are smaller for \(s^nx\) than for \(x\).

**PROPOSITION 4.** The spaces \(HL^p\) have the property of sectional convergence; that is, for every \(x \in HL^p\), we have \(\|x - s^nx\| \to 0\) as \(n \to \infty\).

**PROOF.** Let \(\epsilon > 0, x \in HL^p\), and \(x = \sum x^i\) with \(x^i \in hl^p\) as in (12). There exists \(N\) such that \(\|\sum_{i=N+1}^{\infty} x^i\| < \epsilon/4\). By Proposition 3, we have for all \(n\), \(\|x - s^nx\| \leq \|\sum_{i=1}^{N} (x^i - s^nx^i)\| + \epsilon/2\). We can find a \(J\) such that for each \(i = 1, 2, \ldots, N\) there exists an increasing sequences \((k_j)\) and non-decreasing sequences \((n_j)\) such that

\[
\sum_{j=J}^{\infty} n_j^{1/q} \{ \sum_{k=k_j}^{k_{j+1}-1} |x^i_k|^p \}^{1/p} + \sum_{j=J}^{\infty} \log \left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |x^i_k| < \epsilon/2N.
\]

Then for each \(n > J\) and \(i = 1, 2, \ldots, N\) we have \(r(x^i - s^nx^i) < \epsilon/2N\). Thus

\[
\|x - s^nx\| < \sum_{i=1}^{N} r(x^i - s^nx^i) + \epsilon/2 < \epsilon
\]

whenever \(n > J\).

From Propositions 3 and 4, for every \(x \in HL^p\), we have \(\|x\| = \sup_n \|s^nx\|\).

**PROPOSITION 5.** The spaces \(HL^p\) with \(p > 1\) are BK-spaces.

**PROOF.** Clearly, for each \(p > 1\), \(HL^p\) is a linear space with continuous coordinate functionals. It is sufficient to show completeness. Suppose that \(x^1, x^2, x^3, \ldots\) is a Cauchy sequence in \(HL^p\). This Cauchy sequence has a coordinatewise limit, say \(x\). Find \(N_1 < N_2 < N_3 < \cdots\) such that \(\|x^n - x^m\| < 1/2^i\) for all \(n, m \geq N_i\). Then \(x = x^{N_i} +\)
\[ \sum_{i=1}^{\infty} (x^{N_{i+1}} - x^{N_i}) \] coordinatewise and \[ \|x\| \leq \|x^{N_1}\| + \sum_{i=1}^{\infty} \|x^{N_{i+1}} - x^{N_i}\| < \infty. \] Thus \[ x \in H L^p. \] Furthermore, the Cauchy sequence \[ x^1, x^2, x^3, \ldots \] converges to \[ x : \] If \[ N_i < n \leq N_{i+1}, \] then \[ x - x^n = (x^{N_i} - x^n) + \sum_{k=i}^{\infty} (x^{N_{k+1}} - x^{N_k}) \] coordinatewise and

\[ \|x - x^n\| \leq \|x^{N_i} - x^n\| + \sum_{k=i}^{\infty} \|x^{N_{k+1}} - x^{N_k}\| < 1/2^i + \sum_{k=i}^{\infty} 1/2^k = 3/2^i. \]

Thus \[ x = \lim x^n \] in \[ H L^p. \]

**THEOREM 3.** The spaces \[ H V^p \] with \[ p > 1 \] are BK–space under the norms \[ \|c\|_{H V^p} = \|x\|_{H L^p}, \] where \[ x_k = \Delta c_k \] for \[ k = 0, 1, 2, \ldots. \]

**PROOF.** Clearly the transformation \[ T(c) = x, \] where \[ x_k = \Delta c_k, \] for all \[ k, \] is an isometry and isomorphism between \[ H V^p \] and \[ H L^p. \]

**4. General trigonometric series.** So far all integrability classes we have considered have consisted of even sequences. In 1987 Fournier and Self [16] extended Fomin’s family of integrability classes to general trigonometric series as follows. The pointwise sum of the series \[ \sum_{k=-\infty}^{\infty} c_k e^{ikx} \] belongs to \[ L^1 \] whenever

\[ \sum_{j=0}^{\infty} 2^{j/q} \left\{ \sum_{2^j \leq |k| < 2^{j+1}} |\Delta c_k|^p \right\}^{1/p} + \sum_{j=1}^{\infty} \frac{|c_j - c_{-j}|}{j} < \infty, \]

where \[ 1 < p < \infty \] and \[ 1/p + 1/q = 1. \]

In 1990 the second author [28] also extended the family of integrability classes \[ h v^p \] to general trigonometric series as follows. A null sequence \( (c_k) \) belongs to \( \hat{L}^1 \) whenever

\[ \sum_{j=0}^{\infty} \frac{1}{n_j} \left\{ \sum_{k=k_j}^{k_{j+1}-1} |\Delta c_k|^p \right\}^{1/p} + \sum_{j=0}^{\infty} \log \left( \frac{k_{j+1}}{n_j} \right) \sum_{k=k_j}^{k_{j+1}-1} |\Delta c_k| + \sum_{j=0}^{\infty} \log \left( \frac{k_{j+1}}{k_j} \right) |c_{k_j} - c_{-k_j}| < \infty \]

for some increasing sequence \( (k_j) \) and non-decreasing \( (n_j) \) with \( 1 \leq n_j \leq k_{j+1}. \)

The next goal is to use the connection between summability methods and integrability classes to find even larger integrability classes and corresponding summability methods. Although this paper has been limited to a discussion of \( L^1 \), the same problems discussed exist for most other spaces of functions such as \( C, M \) and \( L^p \) \( (p > 1) \) (the space \( L^2 \) is a notable exception for which there exists a complete characterization). The approach of relating summability statements to multiplier statements in sequence spaces can be applied to these spaces as well.
REFERENCES


