SEQUENCES AND SERIES FROM ALGEBRAIC UNITS

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The main purpose of this paper is to show the applications that flow from the theorem that all of the coefficients of a unit in an Algebraic number field satisfy the same linear recurrence. While this is true for fields of any degree, for simplicity it is stated here for a cubic field.

Theorem 1. Let $e = a_1 + b_1w + c_1w^2$ be a unit in the algebraic number field Q(w) where $w = a + bw + cw^2$, $a, b, c, \in Q$, Q the rational numbers. Let $e^n = a_n + b_nw + c_nw^2$ $n = 0,1,2,...$ Then the coefficients a_n, b_n, c_n, all satisfy the same linear recurrence of the form $T_{n+3} = xT_{n+2} + yT_{n+1} + zT_n$, $x,y,z, \in Q$.

From this theorem follow several results for sequences and series. For sequences we have the following corollary that shows the equivalence between zeros of a linear recurrence and solutions of a diophantine equation.

Corollary 1. Suppose $e = a_1 + b_1w + c_1w^2$ is the single fundamental unit of Z[w] with positive norm where $w^3 - aw^2 + bw - c = 0$ (Z rational integers). Then all units of Z[w] have the form
$e^n = a_n + b_nw + c_nw^2$ $n = \pm0,\pm1,\pm2,...$ Hence, $x^3 + ax^2y + bxy^2 + cy^3 = 1$, has a solution $x = a_n$, $y = b_n$ if and only if $e^n = a_n + b_nw$ is a binary unit, i.e., $c_n = 0$, in other words, c_n is a zero of a linear recurrence.

From the corollary we have the following example. The homogeneous diophantine equation $x^3 - x y^2 + y^3 = 1$ has exactly five solutions, (1,0), (0,1), (1,1), (-1,1), (4,-3) which is equivalent to the result that there are exactly five binary units

$w^0 = 1 + 0w + 0w^2$
$w^1 = 0 + 1w + 0w^2$
$w^3 = 1 + 1w + 0w^2$
$w^{-4} = -1 + 1w + 0w^2$
$w^{-13} = 4 + -3w + 0w^2$

in the field Q(w), where $w^3 = 1 + w$, which is equivalent to the result that there are exactly five zeros $a_1 = a_2 = a_4 = r_3 = r_{12} = 0$ in the linear recurrences $a_{n+3} = a_n + a_{n+1}$ ($a_0 = 1$, $a_1 = a_2 = 0$); $r_{n+3} = r_n - r_{n+2}$ ($r_0 = 1$, $r_1 = -1$, $r_2 = 1$).

As an application to series, we can obtain explicit sums from algebraic units in fields of any degree. We illustrate the methods for a quadratic field, Q(w), where $w^2 - w - 1 = 0$. Here w is itself a unit and all positive powers are given by $w^k = F_{k-1} + F_k w$, $k = 1,2,...$ where the F_k are the Fibonacci numbers. We now have $(w^k)^{-1} = = (F_{k-1} + F_k w)^{-1}$. We now expand the right side of the equation as an infinite series, and compute the inverse unit for the left side, yielding $(-1)^k F_{k+1} + (-1)^{k+1} F_k w$ = infinite series. By careful comparison of coefficients on both sides we get, for example, the following explicit sum of an infinite series.

$$F_{2kl-1} = \left\{\frac{1}{F_{2l+1}}\right\}^k \left[1 + \sum_{n=1}^{\infty} \binom{k+n-1}{n} \left\{\frac{F_{2l}}{F_{2l+1}}\right\}^n F_{n-1}\right].$$

References

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