Degree of approximation in Besov space

by

* G.Das, Harachand Mohanty and † B.K.Ray

* 177, Dharma Vihar,Khandagiri,Bhubaneswar-751030,Orissa,India

e-mail : gdas100@yahoo.com

G.C.college , Ramchandrapur , Jajpur

† Plot.No-102, Saheed Nagar, Bhubaneswar-751007, Orissa,India

e-mail:raykbraja@yahoo.co.in

1. Definition: Modulus of smoothness. Let $A = \mathbb{R}, \mathbb{R}_+, [a, b] \subset \mathbb{R}$ or T (which is usually taken to be $\mathbb{R}$ with identification of points modulo 2π). The k^{th} order modulus of smoothness [4] of a function $f : A \to \mathbb{R}$ is defined by

$$w_k(f, t) = \sup_{0 \le h \le t} \{\sup\{|\Delta_h^k f(x)| : x, x + kh \in A\}\}, t \ge 0 \tag{1.1}$$

where

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih), k \in N \tag{1.2}$$

For $k = 1, w(f, t)$ is called the modulus of continuity of f. The function w is continuous at $t = 0$ if and only if f is uniformly continuous on A, that is , $f \in \widetilde{C}(A)$. The k^{th} order modulus of smoothness of $f \in L_p(A), 0 < p < \infty$, or of $f \in \widetilde{C}(A)$, if $p = \infty$ is defined by

$$w_k(f, t)_p = \sup_{0 < h \le t} \|\Delta_h^k(f, .)\|_p, t \ge 0 \tag{1.3}$$

If $p \ge 1, k = 1$ then $w(f, t)_p = w(f, t)_p$ is a modulus of continuity (or integral modulus of continuity).If $p = \infty, k = 1$ and f is continuous the $w_k(f, t)_p$ reduces to modulus of continuity $w_1(f, t)$ (or $w(f, t)$).

Lipschitz spaces: If $f \in \widetilde{C}(A)$ and

$$w(f, t) = O(t^\alpha), 0 < \alpha \le 1 \tag{1.4}$$

then we write $f \in Lip\alpha$. If $w(f,t) = o(t)$ as $t \to 0+$ (in particular (1.4) holds for $\alpha > 1$) then f reduces to a constant.

If $f \in L_p(A), 0 < p < \infty$ and

$$w(f,t)_p = O(t^\alpha), 0 < \alpha \leq 1 \tag{1.5}$$

then we write $f \in Lip(\alpha, p), 0 < p < \infty, 0 < \alpha \leq 1$. The case where $\alpha > 1$ is of no-interest as the function reduces to a constant whenever

$$w(f,t)_p = o(t) \text{ as } t \to 0+ \tag{1.6}$$

We note that if $p = \infty$ and $f \in C(A)$ then $Lip(\alpha, p)$ class reduces to $Lip\alpha$ class.

Generalised Lipschitz space: Let $\alpha > 0$ and suppose that $k = [\alpha] + 1$. For $f \in L_p(A), 0 < p \leq \infty$, if

$$w_k(f,t) = O(t^\alpha), t > 0 \tag{1.7}$$

then we write

$$f \in Lip^*(\alpha, p), \alpha > 0, 0 < p \leq \infty \tag{1.8}$$

and say that f belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha, L_p)} = \sup_{t>0}(t^{-\alpha} w_k(f,t)_p).$$

It is known ([4], p.52) that the space $Lip^*(\alpha, L_p)$ contains $Lip(\alpha, L_p)$. For $0 < \alpha < 1$ the spaces coincide, (for $p = \infty$, it is necessary to replace L_∞ by $\widetilde{C}$ of uniformly continuous functions on A). For $0 < \alpha < 1$ and $p = \infty$ the space $Lip^*(\alpha, L_p)$ coincide with $Lip\alpha$. For $\alpha = 1, p = \infty$, we have

$$Lip(1, \widetilde{C}) = Lip\ 1, \tag{1.9}$$

but

$$Lip^*(1, \widetilde{C}) = Z \tag{1.10}$$

is the Zygmund space [8], which is characterized by (1.7) with $k = 2$.

Hölder space H_α [6]. For $0 < \alpha \leq 1$, let

$$H_\alpha = \{f \in C_{2\pi} : w(f,t) = O(t^\alpha)\}. \tag{1.11}$$

It is known [6] that H_α is a Banach space with the norm $\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{t>0} t^{-\alpha} w(t), 0 < \alpha \leq 1 \tag{1.12}$$

$$\|f\|_0 = \|f\|_c$$

and

$$H_\alpha \subseteq H_\beta \subset C_{2\pi}, 0 < \beta \leq \alpha \leq 1. \tag{1.13}$$

Hölder space H_α **[3]**. For $0 < \alpha \leq 1$, we write

$$H(\alpha,p) = \{f \in L_p[0,2\pi] : 0 < p \leq \infty : w(f,t)_p = O(t^\alpha)\} \tag{1.14}$$

and introduce the norms $\|.\|_{(\alpha,p)}$ as follows:

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha \leq 1 \tag{1.15}$$

$$\|f\|_{(0,p)} = \|f\|_p.$$

It is known [3] that $H(\alpha,p)$ is a Banach space for $p \geq 1$ and a complete p-normed space for $0 < p < 1$. Also

$$H(\alpha,,p) \subseteq H(\beta,p) \subseteq L_p, 0 < \beta \leq \alpha \leq 1. \tag{1.16}$$

Note that $H(\alpha,\infty)$ is the space H_α defined above.

For the study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in H_α and $H_{(\alpha,p)}$ spaces. As we have seen above, only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on α. We require a finer scale of smoothness than is provided by Lipschitz classes. For each $\alpha > 0$, Besov developed a reasonable technique for restricting moduli of smoothness by introducing a third parameter q (in addition to p and α) and applying α, q norms (rather than α, ∞ norms) to the modulus of smoothness $w_k(f, .)_p$.

Besov space: Let $\alpha > 0$ be given and let $k = [\alpha]+1$. For $0 < p, q \leq \infty$ the Besov space ([4], p.54) $\alpha_q(L_p)$ is defined as follows:

$$B_q^\alpha(L_p) = \{f \in L_p : |f|_{B_q^\alpha(L_p)} = \|w_k(f, .)\|_{\alpha,q} \text{ is finite } \} \tag{1.17}$$

where

$$\|w_k(f, .)\|_{\alpha,q} = \begin{cases} \left(\int_0^\infty \left(t^{-\alpha} w_k(f, t)_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{t>0} t^{-\alpha} w_k(f, t)_p, & q = 0 \end{cases} \tag{1.18}$$

It is known ([4], p.55) that $\|w_k(f, .)_p\|_{\alpha,q}$ is a semi norms if $1 \leq p, q \leq \infty$ and a quasi seminorm in other cases. The Besov norm for $B_q^\alpha(L_p)$ is

$$\|f\|_{B_q^\alpha(L_p)} = \|f\|_p + \|w_k(f, .)_p\|_{\alpha,q} \tag{1.19}$$

It is known ([7], p.237) that for 2π-periodic function f the integral $\int_0^\infty$ in (1.18) is replaced by $\int_0^\pi$.We know ([4], p.56),([7], p.236) the following inclusion relations :

(I) For fixed α and p

$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), q < q_1$

(II) For fixed p and q

$B_q^\alpha(L_p) \subset B_q^\beta(L_p), \beta < \alpha$

(III) For fixed α and q

$B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), p_1 < p$

Special cases of Besov space.

For $q = \infty, B_\infty^\alpha(L_p), \alpha > 0, p \geq 1$ is same as $Lip^*(\alpha, L_p)$ the generalised Lipschitz class and the corresponding norm $\|.\|_{B_\infty^\alpha(L_p)}$ is given by

$$\|f\|_{\beta_\infty^\alpha(L_p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w_k(f, t)_p \tag{1.20}$$

for every $\alpha > 0$ with $k = [\alpha] + 1$. In the special case when $0 < \alpha < 1, B_\infty^\alpha(L_p)$ space reduces to $H(\alpha, p)$ space due to Das, Ghosh and Ray[3] and the corresponding norm is given by

$$\|f\|_{B_\infty^\alpha(L_p)} = \|f\|_{(\alpha,p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha < 1 \tag{1.21}$$

For $\alpha = 1$, the norm is given by

$$\|f\|_{B'_\infty(L_p)} = \|f\|_p + \sup_{t>0} t^{-1} w_2(f,t)_p. \tag{1.22}$$

Note that $\|f\|_{B^1_\infty(L_p)}$ is not same as $\|f\|_{(1,p)}$ and the space $B^1_\infty(L_p)$ includes the space $H(1,p), p \geq 1$.If we further specialize by taking $p = \infty, B_\infty^\alpha(L_\infty), 0 < \alpha < 1$ coincides with H_α space due to Prossdorf [6] and the norm is given by

$$\|f\|_{B_\infty^\alpha(L_\infty)} = \|f\|_\alpha = \|f\|_c + \sup_{t>0} t^{-\alpha} w(f,t), 0 < \alpha < 1. \tag{1.23}$$

For $\alpha = 1, p = \infty$, the norm is given by

$$\|f\|_{B^1_\infty(L_\infty)} = \|f\|_c + \sup_{t>0} t^{-1} w_2(f,t), \alpha = 1 \tag{1.24}$$

which is different from $\|f\|_1$ and $B^1_\infty(L_\infty)$ includes the H_1 space.

2 Introduction:

Let f be a 2π-periodic function and let $f \in L_p[0, 2\pi], p \geq 1$. The Fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{2.1}$$

In the case $0 < p < 1$, we can still regard (2.1) as the Fourier series of f by further assuming $f(t)\cos nt$ and $f(t)\sin nt$ are integrable.

Prossdorf [6] first obtained the following on the degree of approximation of functions in H_α using Fejer's mean of Fourier series.

Theorem A Let $f \in H_\alpha(0 < \alpha \leq 1)$ and $0 \leq \beta < \alpha \leq 1$. Then

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} \dfrac{1}{n^{\alpha-\beta}}, & 0 < \alpha < 1 \\ \\ \frac{(\log n)^{1-\beta}}{n^{1-\beta}}, & \alpha = 1, \end{cases}$$

where $\sigma_n(f)$ is the Fejer means of the Fourier series of f.The case $\beta = 0$ of Theorem A is due to Alexists [1]. Chandra [2] obtained a generalisation of Theorem A in the Nörlund (N, p) and $(\overline{N}, p)$ transform set up. Later Mohapatra and Chandra [5] studied the problem by matrix means and obtained the above results as corollaries. Das, ghosh and Ray[3] further generalised the work by studying the problem for functions in $H(\alpha, p)$ space $(0 < \alpha \leq 1, p \geq 1)$ by matrix means of their Fourier series in the generalised Hölder metric.

In the present work we propose to study the degree of approximation of functions in Besov space which is a generalisation of $H(\alpha, p)$ space.

In what follows, we present some notations which we need in the sequel

We write

$$\phi_x(u) = f(x+u) + f(x-u) - 2f(x) \tag{2.2}$$

Let $S_n(f; x)$ denote the n th partial sum of the Fourier series. It is known ([9], vol.I.p.50) that

$$S_n(f; x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \phi_x(u) D_n(u) du \tag{2.3}$$

where the Dirichlet's kernel

$$D_n(x) = \frac{1}{2} + \sum_{\nu=0}^{n} \cos \nu u = \frac{\sin\left(n + \frac{1}{2}\right) u}{2 \sin u/2} \tag{2.4}$$

Let σ_n^γ denote the Cesàro mean $(C, \gamma), \gamma > 0$ of the Fourier series. Then

$$\sigma_n^\gamma(f; x) = \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} S_\nu(f; x) \tag{2.5}$$

where A_n^γ is given by the formula ([9], vol.I, p.76)

$$\sum_{n=0}^{\infty} A_n^\gamma x^n = (1-x)^{-\gamma-1}, \gamma > -1, |x| < 1 \tag{2.6}$$

We know ([9] , Vol. I,p. 94) that

$$l_n^\gamma(x) \equiv \sigma_n^\gamma(f; x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \phi_x(u) K_n^\gamma(u) du \tag{2.7}$$

where

$$K_n^\gamma(u) = \frac{1}{A_n^\gamma}\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} D_\nu(u) \tag{2.8}$$

3 Main Theorem

We prove the following theorem

Theorem Let $0 < \alpha < 2$ and $0 < \beta < \alpha$. If $f \in B_q^\alpha(L_p), p \geq 1$ and $1 < q \leq \infty$, then

$$\|l_n^\gamma(.)\|_{B_q^\beta(L_p)} = O(1)\begin{cases} \frac{1}{n^\gamma}, \alpha - \beta > \gamma, & 0 < \gamma \leq 1 \\ \frac{1}{n^{\alpha-\beta}}, \alpha - \beta < \gamma, & 0 < \gamma \leq 1 \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^\gamma}, \alpha - \beta = \gamma, & 0 < \gamma \leq 1 \end{cases}$$

and

$$\|l_n^\gamma(.)\|_{B_q^\beta(L_p)} = O(1)\begin{cases} \frac{1}{n}, \alpha - \beta > 1, & 1 \leq \gamma < 2 \\ \frac{1}{n^{\alpha-\beta}}, \alpha - \beta < 1, & 1 \leq \gamma < 2 \\ \frac{(\log n)^{1-\frac{1}{q}}}{n^\gamma}, \alpha - \beta = 1, & 1 \leq \gamma < 2 \end{cases}$$

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