

**Summer Analysis Seminar
Some series of Euler.**

First the series $(1 + 1/2^2 + 1/3^2 + \dots) = (1/6)\pi^2$

We start with $0 = 1/2 + \cos(t) + \cos(2t) + \dots$ for $(0 < t < 2\pi)$ as we obtained in class.

(This follows from the series $1 + x + x^2 + \dots = \frac{1}{1-x}$ when $x = e^{it} = \cos t + i \sin t$. Take the real part of the series.)

Then we change t to $t + \pi$ to get

$$0 = 1/2 - \cos(t) + \cos(2t) - \dots \quad (-\pi < t < \pi) \text{ or}$$
$$1/2 = \cos(t) - \cos(2t) + \dots$$

Integrate from $t = 0$ to $t = x$ to get

$$(1/2)x = \sin(x) - (1/2)\sin(2x) + (1/3)\sin(3x) - \dots \quad (-\pi < x < \pi)$$

This series is convergent.

Integrate again from $x = 0$ to $x = t$ to get

$$(1/4)t^2 = (1 - \cos(t)) - (1 - \cos(2t))/2^2 + (1 - \cos(3t))/3^2 - \dots$$

This series is uniformly convergent so we may include the limit $t = \pi$ to get

$$(1/4)\pi^2 = (2) - (0) + (2)/3^2 - (0) + \dots \text{ or}$$
$$(1/8)\pi^2 = 1 + 1/3^2 + 1/5^2 + \dots$$

Since $1 + 1/3^2 + 1/5^2 + \dots = (1 + 1/2^2 + 1/3^2 + \dots) - (1/2^2 + 1/4^2 + \dots)$, we have

$$1 + 1/3^2 + 1/5^2 + \dots = (1 - (1/4))(1 + 1/2^2 + 1/3^2 + \dots)$$

$$\text{So we get } (1 + 1/2^2 + 1/3^2 + \dots) = (4/3)((1/8)\pi^2) = (1/6)\pi^2.$$

Done.

Next, we get $1 - 1/2^2 + 1/3^2 - \dots = (1/12)\pi^2$

We obtain this sum by the decomposition

$$1 + 1/3^2 + 1/5^2 + \dots = (1 - 1/2^2 + 1/3^2 - \dots) + (1/2^2 + 1/4^2 + \dots).$$

$$(1/8)\pi^2 = (1 - 1/2^2 + 1/3^2 - \dots) + (1/4)(1/6)\pi^2, \text{ or}$$

$$1 - 1/2^2 + 1/3^2 - \dots = (1/12)\pi^2$$