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for the special tangent vector $X_l = -(1-f/f)\nabla_l f$ at ξ. Combining all of these we had done, at (ξ, τ), implies that

$$0 \geq \left[f(1+f)+\frac{2}{t}(1-f)\right]\varepsilon e^{kt}\varphi g_{ij}+fY\left(1+\frac{1}{t}\right)g_{ij}+\left[\frac{2}{f}Y+\frac{2}{t}(1-f)^2\right]\nabla_i f\nabla_j f,$$

which is a contradiction. Therefore, for all $\varepsilon > 0$, we have $Z_{ij} + \varepsilon e^{kt}\varphi g_{ij} > 0$. This implies

$$Z_{ij} \geq 0,$$

which completes the proof of Theorem 5.1. □

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SEQUENTIAL DEFINITIONS OF CONTINUITY FOR REAL FUNCTIONS

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ABSTRACT. A function $f : \mathbf{R} \to \mathbf{R}$ is continuous at a point u if, given a sequence $\mathbf{x} = (x_n)$, $\lim \mathbf{x} = u$ implies that $\lim f(\mathbf{x}) = f(u)$. This definition can be modified by replacing lim with an arbitrary linear functional G. Generalizing several definitions that have appeared in the literature, we say that $f : \mathbf{R} \to \mathbf{R}$ is G-continuous at u if $G(\mathbf{x}) = u$ implies that $G(f(\mathbf{x})) = f(u)$. When $G(\mathbf{x}) = \lim_n n^{-1} \sum_{k=1}^{n} x_k$, Buck showed that if a function f is G-continuous at a single point then f is linear, that is, $f(u) = au + b$ for fixed a and b. Other authors have replaced convergence in arithmetic mean with A-summability, almost convergence and statistical convergence. The results in this paper include a sufficient condition for G-continuity to imply linearity and a necessary condition for continuous functions to be G-continuous, thereby generalizing several known results in the literature. It is also shown that, in many situations, the G-continuous functions must be either precisely the linear functions or precisely the continuous functions. However, examples are found where this dichotomy fails, which, in particular, leads to a counterexample to a conjecture of Spigel and Krupnik.

1. Introduction. The typical 'advanced calculus' student is often relieved to find that the standard $\varepsilon - \delta$ definition of continuity for real-valued functions of a real variable can be replaced by a sequential definition of continuity. That many of the properties of continuous functions can be easily derived using sequential arguments has also been, no doubt, a source of relief to the occasional advanced calculus instructor.

In this paper we investigate the impact of changing the definition of the convergence of sequences on the structure of the set of continuous functions. This continues a line of research initiated with a 1946 American Mathematical Monthly problem. Robbins [**24**] asked readers

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to show that a function $f : \mathbf{R} \to \mathbf{R}$ which exhibits the property

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = u \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = f(u),$$

even at a single point $u = u_0$, has to be linear, that is, of the form $f(u) = au + b$ for all u, where a, b are fixed real numbers. In other words, when in the usual definition of continuity sequential convergence is replaced by convergence in arithmetic mean then only the linear functions remain continuous in the new sense. In the same problem, Robbins coined the term 'Cesàro continuous' for functions satisfying (1). Buck's solution was published in 1948 [**7**]; the problem was also solved by five others.

Since then, there have been a number of similar investigations that replace the usual definition of sequential convergence with one of a variety of other definitions that are typically related to matrix summability, almost convergence or statistical convergence (these will be discussed below). In all of these investigations, the resulting continuous functions were either precisely the linear functions or precisely the functions which are continuous in the ordinary sense. This paper shows that, as long as the new definition of convergence satisfies certain restrictions, this is always the case. But we will also find examples where this dichotomy does not hold. In particular, we will give a counterexample to a conjecture of Spigel and Krupnik [**28**].

Before we can begin, it will be necessary to introduce some definitions and notation. We will use boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}, \ldots$ of real numbers. If f is a real-valued function of a real variable then we define $f(\mathbf{x}) = (f(x_n))_n$. By a *method of sequential convergence*, or briefly a *method*, we mean a linear functional G defined on a linear subspace c_G of the vector space of all real-valued sequences. A sequence $\mathbf{x} = (x_n)$ is said to be *G-convergent* to l if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = l$. In particular, lim denotes the limit functional $\lim \mathbf{x} = \lim_n x_n$ on the space c of convergent sequences. A method G is called *regular* if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. Throughout this paper we shall denote by I a nondegenerate interval in $\mathbf{R}$. We are now ready to define the G-continuity of a function.

Definition. Let G be a method of sequential convergence and $f : I \to \mathbf{R}$ a function. Then f is *G-continuous* at $u \in I$ provided that whenever an I-valued sequence $\mathbf{x} = (x_n)$ is G-convergent to u then the sequence $f(\mathbf{x}) = (f(x_n))$ is G-convergent to $f(u)$. For a subset D of I, f is called *G-continuous* on D if it is G-continuous at every $u \in D$, and f is *G-continuous* if it is G-continuous on its domain I.

The G-continuity of f (on its domain) can also be expressed briefly as follows: If $\mathbf{x}$ is an I-valued G-convergent sequence then

$$G(f(\mathbf{x})) = f(G(\mathbf{x})).$$

It is important to distinguish between the G-continuity of f on a subinterval J and the G-continuity of $f|_J$. Clearly, if f is G-continuous on J then $f|_J$ is G-continuous, but the converse is not necessarily true because in the latter case the sequences $\mathbf{x}$ are restricted to J. We demonstrate this by an example.

Example 1. A regular method G and a function $f : \mathbf{R} \to \mathbf{R}$ so that $f|_{[a,b]}$ is G-continuous for every interval $[a, b]$ but f itself is not G-continuous. It suffices to consider $c_G = c + \operatorname{span}\{(2^n)\}$ with $G(\mathbf{x} + \lambda(2^n)) = \lim \mathbf{x}$ for $\lambda \in \mathbf{R}$. Then the function $f : u \mapsto u^2$ clearly has the desired properties.

Another phenomenon that is unknown for ordinary continuity is that G-continuity need not be a local property. Antoni and Šalát [**2**] have given an example of a regular method G and a function $f : \mathbf{R} \to \mathbf{R}$ that is G-continuous only at 0 although f coincides on $[-1, 1]$ with a linear and hence G-continuous function.

We now discuss some special classes of methods of sequential convergence that have been studied in the literature. Probably the most important class is the class of matrix methods. Consider an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ of real numbers. Then, for any sequence $\mathbf{x} = (x_n)$ the sequence $A\mathbf{x}$ is defined as

$$A\mathbf{x} = \left(\sum_{k=1}^{\infty} a_{nk} x_k \right)_n ,$$

provided that each of the series exists. A sequence $\mathbf{x}$ is *A-convergent* (or *A-summable*) to l if $A\mathbf{x}$ exists and is convergent with

$$\lim A\mathbf{x} = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}x_k = l.$$

Then l is called the *A-limit of* $\mathbf{x}$. We have thus defined a method of sequential convergence, called a *matrix method* (or an *ordinary matrix method* to distinguish it from the strong matrix methods considered below), and A is called a *summability matrix*. For matrix methods the notion of regularity introduced above coincides with the classical notion of regularity for matrices. See [5], [16], [22], [20] and [31] for an introduction to regular summability matrices.

A number of authors (Posner [23], Iwiński [17], Srinivasan [29], Antoni and Šalát [2], [1], Spigel and Krupnik [28]) have studied G-continuity defined by a regular summability matrix A. In this case G-continuity is usually called *A-continuity*. Note that Buck's original result is for the Cesàro matrix $C_1 = (a_{nk})$ with $a_{nk} = 1/n$ if $k \le n$ and 0 otherwise.

The Hahn-Banach theorem can be used to define methods which are not generated by a regular summability matrix. Banach used this theorem to show that the limit functional can be extended from the convergent sequences to the bounded sequences while preserving linearity, positivity and translation invariance [3]; these extensions have come to be known as Banach limits. If a bounded sequence is assigned the same value l by each Banach limit, the sequence is said to be *almost convergent to* l. Lorentz [18] proved that a sequence $\mathbf{x} = (x_n)$ is almost convergent to l if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+j} = l \quad \text{uniformly in } j.$$

Observe that a sequence is almost convergent if and only if it is 'uniformly Cesàro convergent'; replacing the Cesàro matrix by a regular summability matrix A (or even a family of general regular methods) leads to another collection of methods of sequential convergence. For an introduction to the theory of almost convergence (with respect to the Cesàro matrix), see [18], [22] and [4].

Some authors (Öztürk [21], Savaş and Das [25], [26], Borsík and Šalát [6]) have studied G-continuity for the method of almost convergence or for related methods. In particular, Borsík and Šalát have obtained the analogue of Buck's result for almost convergence.

Next we consider a class of methods that is unrelated to the preceding two classes. Fast [13] introduced the definition of *statistical convergence* in 1951. Recall that for subsets A of $\mathbf{N}$ the *asymptotic density* of A, denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in A\}\right|,$$

if this limit exists, where $|B|$ denotes the cardinality of the set B. A sequence $\mathbf{x} = (x_n)$ is *statistically convergent to* l if

$$\delta(\{n : |x_n - l| > \varepsilon\}) = 0 \quad \text{for every } \varepsilon > 0.$$

In this case l is called the *statistical limit of* $\mathbf{x}$. The statistically convergent sequences form a linear subspace of the space of all real-valued sequences and the statistical limit is a linear functional on this space. Note that convergent sequences are statistically convergent to the same limit and that if a sequence is statistically convergent to l, then the sequence has a subsequence which converges to l in the ordinary sense, cf. [9].

The notion of statistical convergence can be generalized to *μ-statistical convergence* by replacing the asymptotic density δ with an arbitrary *density* μ, that is, a finitely additive set function taking values in $[0,1]$ defined on a field of subsets of $\mathbf{N}$ with $\mu(\mathbf{N}) = 1$ such that if $|A| < \infty$ then $\mu(A) = 0$ and if $A \subset B$ with $\mu(B) = 0$ then $\mu(A) = 0$, cf. [10] and [11]. This notion covers several other variants of statistical convergence that have been considered in the literature.

For an introduction to statistical convergence, see [14], [9], [15], and [11].

The G-continuity for methods of statistical convergence has been considered in two papers (Schoenberg [27], Demirci [12]). The behavior here is markedly different. While G-continuity in the case of regular matrix summability or almost convergence typically leads to Buck-type results, Schoenberg showed that *every* continuous function is 'statistically continuous,' see also Proposition 4 below.

The final class of methods that we want to discuss is given by strong matrix summability. Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a nonnegative matrix with $\lim_n a_{nk} = 0$ for all $k \in \mathbf{N}$, $\sup_n \sum_{k=1}^{\infty} a_{nk} < \infty$ and $\sum_{k=1}^{\infty} a_{nk} \not\to 0$ as $n \to \infty$. Then a sequence $\mathbf{x}$ is *strongly A-convergent* (or *strongly A-summable*) to l if $\sum_{k=1}^{\infty} a_{nk}|x_k - l| < \infty$ for all n and

$$\sum_{k=1}^{\infty} a_{nk}|x_k - l| \to 0 \quad \text{as } n \to \infty.$$

In this case l is called the *strong A-limit of* $\mathbf{x}$. Under the stated assumptions this limit is unique and strong A-convergence defines a regular method, called a *strong matrix method.* The assumptions on A are satisfied in particular if A is a nonnegative regular matrix. For an introduction to strong matrix summability, see [**20**] and [**31**].

It seems that G-continuity for strong matrix methods G will be studied here for the first time, see Theorems 4 and 6, among others.

In the following we let

$$\mathcal{L}(I),\ \mathcal{G}(I) \quad \text{and} \quad \mathcal{C}(I)$$

denote the collections of linear, G-continuous, and continuous functions $f : I \to \mathbf{R}$, and we set $\mathcal{L} = \mathcal{L}(\mathbf{R}), \mathcal{G} = \mathcal{G}(\mathbf{R})$ and $\mathcal{C} = \mathcal{C}(\mathbf{R})$; note that a function f will be called linear if it is of the form $f(u) = au + b$ with constants a and b. We turn our attention to investigating the relationships between these three sets, which is the emphasis of this paper.

2. A sufficient condition for $\mathcal{G} = \mathcal{L}$. We begin with the following simple observation.

Proposition 1. *If G is a regular method, then every linear function $f : I \to \mathbf{R}$ is G-continuous, that is, $\mathcal{L}(I) \subset \mathcal{G}(I)$.*

This shows that $\mathcal{L}$ is the smallest possible space of G-continuous functions, if G is regular. As noted above, Buck [**7**] showed that every Cesàro-continuous function is a linear function, that is, $\mathcal{G} = \mathcal{L}$ for the method G of Cesàro-summability. In [**2**], Antoni and Šalát isolate a

property that is shared by the Cesàro matrix and that implies $\mathcal{G} = \mathcal{L}$ for an arbitrary regular matrix method G. When trying to extend this result to general regular methods one faces the problem that the proof of Antoni and Šalát requires that G-continuity implies continuity, which is no longer necessarily true for non-matrix methods, see Example 3. The following lemma will help us to overcome this problem.

Lemma 1. *Let $\alpha \neq 0, 1$. If $f : I \to \mathbf{R}$ is a function such that*

$$f(\alpha u + (1-\alpha)v) = \alpha f(u) + (1-\alpha)f(v) \tag{2}$$

whenever u, v and $\alpha u + (1-\alpha)v$ belong to I, then there is a dense subset of I on which f is linear.

Proof. First we note that we can assume that $\alpha \in (0,1)$; for if $\alpha > 1$ then we replace α by $(1/\alpha)$ and $\alpha u + (1-\alpha)v$ by u, and if $\alpha < 0$ then we replace α by $(\alpha/(\alpha-1))$ and $\alpha u + (1-\alpha)v$ by v.

We now fix distinct points u_0 and v_0 in I. Then there are $a, b \in \mathbf{R}$ with $f(u) = au + b$ for $u = u_0$ and $u = v_0$. Let $D = \{u \in I : f(u) = au + b\}$. We claim that the closure $\overline{D}$ of D is an interval. Otherwise there is an interval $J = (u_1, v_1)$ with $u_1, v_1 \in \overline{D}$ that meets no points of $\overline{D}$. But by (2), $u_1, v_1 \in \overline{D}$ implies that $\alpha u_1 + (1-\alpha)v_1 \in \overline{D} \cap J$, which is a contradiction. Hence $\overline{D}$ is an interval, and a similar reasoning shows that $\overline{D} = I$. □

Using this lemma we obtain the generalization of the Antoni-Šalát theorem to general methods. With Antoni and Šalát [**2**] we consider the following property for methods G:

(L_1) There exists a G-convergent 0-1-sequence $\mathbf{z}$ such that $G(\mathbf{z}) = \alpha$ with $\alpha \neq 0, 1$.

Theorem 1. *Let G be a regular method with property (L_1). Then every G-continuous function $f : I \to \mathbf{R}$ is linear, that is, $\mathcal{G}(I) = \mathcal{L}(I)$.*

Proof. The first part of the proof is the same as that of Antoni and Šalát. We give it here for the sake of completeness. Let $f : I \to \mathbf{R}$ be a G-continuous function, and let $u, v \in I$. Consider a G-convergent

0-1-sequence $\mathbf{z} = (z_n)$ and a scalar α as in property (L_1), and define

$$x_n = z_n u + (1 - z_n)v,$$

which equals u if $z_n = 1$, and v otherwise. By linearity and regularity of G we have that $\mathbf{x} = (x_n) \in c_G$ with

$$G(\mathbf{x}) = \alpha u + (1 - \alpha)v.$$

On the other hand, $f(x_n)$ equals $f(u)$ if $z_n = 1$, and $f(v)$ otherwise. Hence

$$f(x_n) = z_n f(u) + (1 - z_n) f(v),$$

so that we also have

$$G(f(\mathbf{x})) = \alpha f(u) + (1 - \alpha) f(v).$$

Since f is G-continuous we then have

$$f(\alpha u + (1 - \alpha)v) = f(G(\mathbf{x})) = G(f(\mathbf{x})) = \alpha f(u) + (1 - \alpha) f(v)$$

whenever u, v and $\alpha u + (1 - \alpha)v$ belong to I.

At this point Lemma 1 implies that there is a dense subset D of I and $a, b \in \mathbf{R}$ such that

$$f(u) = au + b \quad \text{for all} \quad u \in D.$$

Now let $u \in I$, and let $x_n \in D$ such that $x_n \to u$. Then also $u = G(\mathbf{x})$, so that $G(f(\mathbf{x})) = G((ax_n + b)_n) = au + b$ by linearity and regularity of G. Since f is G-continuous, we finally have

$$f(u) = G(f(\mathbf{x})) = au + b,$$

which had to be shown. □

Apart from the Antoni-Šalát theorem this result also covers Theorem 1 of Savaş and Das [**26**] who consider methods that generalize the method of almost convergence.

The example in [**2**] shows that the theorem does not remain true if f is only required to be G-continuous at a single point. In Section 6 we will obtain corresponding results under this weaker assumption.

Example 2. The converse of Theorem 1 is not true. There is a regular method G that does not have property (L_1) for which $\mathcal{G}(I) = \mathcal{L}(I)$ for any interval I.

To construct G let $\mathbf{z} = (1, 0, -1, 1, 0, -1, 1, \dots)$ and $c_G = c + \operatorname{span}\{\mathbf{z}\}$ with $G(\mathbf{x}) = \lim_n x_{3n+2}$ for $\mathbf{x} \in c_G$. Then G is a regular method that does not have property (L_1) because, if $G(\mathbf{x}) = l$, then l is a subsequential limit of $\mathbf{x}$.

Let $f : I \to \mathbf{R}$ be a G-continuous function, and let $\mathbf{x} = \mathbf{y} + \lambda \mathbf{z}$ be an I-valued sequence with $\lim \mathbf{y} = u_0$ and $\lambda \in \mathbf{R}$, hence also $G(\mathbf{x}) = u_0$. Then by the G-continuity of f we have that

$$f(\mathbf{x}) = (f(y_1 + \lambda), f(y_2), f(y_3 - \lambda), \dots) \in c_G$$

with

$$G(f(\mathbf{x})) = f(u_0).$$

This implies that there is some $\kappa \in \mathbf{R}$ with

$$\begin{aligned} f(y_{3n+2}) &\longrightarrow f(u_0), \\ f(y_{3n+1} + \lambda) &\longrightarrow f(u_0) + \kappa, \\ f(y_{3n+3} - \lambda) &\longrightarrow f(u_0) - \kappa \end{aligned}$$

as $n \to \infty$. If $\lambda = 0$ and $\mathbf{y}$ is an arbitrary I-valued sequence with $\lim \mathbf{y} = u_0$, then the first limit relation implies that f is continuous at u_0. Taking $\lambda \in \mathbf{R}$ such that $u_0 + \lambda$ and $u_0 - \lambda$ belong to I and letting $y_n = u_0$ for all n the second and third limit relations imply that

$$\frac{f(u_0 + \lambda) + f(u_0 - \lambda)}{2} = f(u_0).$$

By Lemma 1 the linearity of f follows. □

3. A topological view of G-continuity, and a necessary condition for $\mathcal{G} = \mathcal{C}$.

In this section we study the connection between G being a subsequential method and the comparison of $\mathcal{G}$ with $\mathcal{C}$.

Definition. A method is called *subsequential* if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = l$ then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = l$.

For a fixed subsequence (n_k), define the method $G = I_{(n_k)}$ by letting $\mathbf{x}$ be G-convergent if and only if $G(\mathbf{x}) := \lim_k x_{n_k}$ exists. The method $I_{(n_k)}$ is obviously subsequential. The same is true for the method considered in Example 2. Another example is statistical convergence, see [**9**, Corollary 2.4].

In order to link subsequentiality with the size of $\mathcal{G}$ we introduce the following 'topological' notions.

Definition. Let $U \subset \mathbf{R}$ and $l \in \mathbf{R}$. Then l is in the *G-hull* of U if there is a sequence $\mathbf{x} = (x_n)$ of points in U such that $G(\mathbf{x}) = l$. A set is *G-closed* if it contains all of the points in its G-hull.

We let $\overline{U}^G$ denote the G-hull of a set U. If G is a regular method, then $U \subset \overline{U} \subset \overline{U}^G$, and hence U is G-closed if and only if $\overline{U}^G = U$. Note that, depending upon G, one can have either $\overline{U} = \overline{U}^G$ or $\overline{U}$ a proper subset of $\overline{U}^G$; observe that if G is Cesàro summability then the G-hull of $\{0,1\}$ is $[0,1]$, while if G is statistical convergence then the G-hull of $\{0,1\}$ is $\{0,1\}$. Also note that it need not be the case that the G-hull of $\overline{U}^G$ is equal to $\overline{U}^G$; for example, if $\mathbf{x}$ is G-convergent if and only if $G(\mathbf{x}) := \lim_n (x_n + x_{n+1})/2$ exists then for $U = \{0,1\}$ we have $\overline{U}^G = \{0, (1/2), 1\}$, and the G-hull of $\overline{U}^G$ is $\{0, (1/4), (1/2), (3/4), 1\}$.

Proposition 2. *Let G be a regular method. Then $\overline{U} = \overline{U}^G$ for every subset U of $\mathbf{R}$ if and only if G is a subsequential method.*

Proof. Suppose that G is a subsequential method and that $l \in \overline{U}^G$. Then there is a sequence $\mathbf{x} = (x_n)$ in U such that $G(\mathbf{x}) = l$. As G is subsequential, there is a subsequence (x_{n_k}) of $\mathbf{x}$ such that $\lim_k x_{n_k} = l$ and hence $l \in \overline{U}$. As G is regular, it follows that $\overline{U} = \overline{U}^G$.

Now suppose that $\overline{U} = \overline{U}^G$ for every subset U of $\mathbf{R}$. Let $\mathbf{x} = (x_n)$ be a G-convergent sequence with $G(\mathbf{x}) = l$. Observe that, since G is regular, $G(\mathbf{x})$ depends only upon the 'tail' of $\mathbf{x}$ and hence $l \in \overline{\{x_n : n \geq N\}}^G$ for each $N \in \mathbf{N}$. As $\overline{\{x_n : n \geq N\}}^G = \overline{\{x_n : n \geq N\}}$ we obtain that $l \in \cap_N \overline{\{x_n : n \geq N\}}$. Hence there is a subsequence (x_{n_k}) of $\mathbf{x}$ such that $\lim_k x_{n_k} = l$. □

Lemma 2. *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is G-continuous and U is G-closed. Then $f^{-1}(U)$ is G-closed.*

Proof. Let $V = f^{-1}(U)$ and suppose that $l \in \overline{V}^G$. Then there is a V-valued sequence $\mathbf{x}$ such that $G(\mathbf{x}) = l$. Now, since $G(f(\mathbf{x})) = f(l)$, $f(\mathbf{x})$ is U-valued and U is G-closed we obtain that $f(l) \in U$. But now $l \in V$ and hence $\overline{V}^G \subset V$. □

With this lemma we obtain the following. Using a different method we will obtain a stronger result in the next section, see the Corollary to Theorem 5.

Proposition 3. *If G is a regular subsequential method, then every G-continuous function is continuous, that is, $\mathcal{G} \subset \mathcal{C}$.*

Proof. We show that the inverse image of any closed set is closed. Let $U \subset \mathbf{R}$ be closed. As G is subsequential, U is also G-closed. Since f is G-continuous, it follows from Lemma 2 that $f^{-1}(U)$ is G-closed and hence closed. □

Theorem 2. *Let G be a regular method. If every continuous function is G-continuous, that is, if $\mathcal{C} \subset \mathcal{G}$, then G is a subsequential method.*

Proof. We suppose that G is not subsequential and produce a function $f : \mathbf{R} \to \mathbf{R}$ which is continuous but not G-continuous. As G is not subsequential it follows from Proposition 2 that there is a subset U of $\mathbf{R}$ whose closure is properly contained in its G-hull. Let $l \in \overline{U}^G \setminus \overline{U}$. Then there is a continuous function f such that $f(l) = 0$ and $f|_{\overline{U}} = 1$.

We claim that f is not G-continuous. Since $l \in \overline{U}^G$ there is a U-valued sequence $\mathbf{x} = (x_n)$ such that $G(\mathbf{x}) = l$. Now $f(x_n) = 1$ for all n and $f(l) = 0$. As

$$G(f(\mathbf{x})) = 1 \neq 0 = f(l)$$

we have established the claim. □