Demazure crystals, Kirillov–Reshetikhin crystals, and the energy function

Peter Tingley
(joint with Anne Schilling)

Massachusetts Institute of Technology

Wake forest, Sept. 24, 2011

1Slides and notes available at www-math.mit.edu/~ptingley/
Outline

1 Background
   - Highest weight crystals
   - Demazure crystals
   - Kirillov–Reshetikhin crystals
   - Relationship between KR crystals and Demazure crystals.
   - The energy function

2 Results

3 Applications
   - Macdonald polynomials
   - Whittaker functions

4 Future directions
   - Macdonald polynomials from Demazure characters in type $C_n^{(1)}$?
The adjoint crystal for $\mathfrak{sl}_3$

There are 6 one-dimensional weight spaces and one two-dimensional weight space. The generators $F_1$ and $F_2$ act between weight spaces. There are 4 distinguished one-dimensional spaces in the middle. If we use $U_q(\mathfrak{sl}_3)$ and 'rescale' the operators, then "at $q=0$", they match up. You get a colored directed graph.
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Here you see that the graded dimension of the representation is the generating function for semi-standard Young tableaux.
Tensor product rule

For $sl_2$, crystals are just directed segments. For other types, just treat each $sl_2$ independently. Consider $B(\omega_1) \otimes B(\omega_2)$ for $sl_3$. 

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Demazure crystals (for \(\mathfrak{sl}_3\))

For each \(w \in W\), there is a 1-dimensional weight space \(S_w\) of \(V(\lambda)\). The Demazure module \(V_w(\lambda)\) in the \(U^+ q(\mathfrak{g})\) submodule generated by \(S_w\). Kashiwara showed that the global basis restricts to a basis of \(V_w(\lambda)\). Hence, \(V_w(\lambda)\) defines a subset \(B_w(\lambda)\) of \(B(\lambda)\), called the Demazure crystal. \(B_w(\lambda)\) is closed under the \(e_i\) operators, but not the \(f_i\) operators.
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Kirillov–Reshetikhin crystals

For $D_n$ (with $n \geq 7$):

\[ B_5^{(2)} \cong B_5^{(2) \omega_5} \oplus B_5^{(\omega_3 + \omega_5)} \oplus B_5^{(\omega_1 + \omega_5)} \oplus B_5^{(2 \omega_3)} \oplus B_5^{(\omega_1 + \omega_3)} \oplus B_1^{(2)} \]

for $\text{sl}_3$

Classically irreducible

But these usually do not have crystal bases.

In all non-exceptional types, the Kirillov–Reshetikhin crystals $B_r,s$ (for $r \in I$, $s \in \mathbb{N}$) do (Okado, Okado-Schilling).

These have "tableaux" type realizations.

Classical decompositions are known, and are multiplicity free.
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- Classical decompositions are known, and are multiplicity free.
Background

Relationship between KR crystals and Demazure crystals.

Theorem (Fourier-Littelmann, Naito-Sagaki, Fourier-Schilling-Shimozono, ?)

In non-exceptional types, a tensor product $B$ of level $\mathfrak{g}$ KR crystals is isomorphic (as a classical crystal) to a Demazure crystal $B_{\omega}(\Lambda^\tau(0))$. There is a unique isomorphism such that the pullbacks of $\mathfrak{g}$-arrows in the Demazure crystal are $\omega$-arrows in $B$. 

Peter Tingley (MIT)
Relationship between KR crystals and Demazure crystals.

Theorem (Fourier-Littelmann, Naito-Sagaki, Fourier-Schilling-Shimozono, ?)

In non-exceptional types, a tensor product $B$ of level $\tau$ KR crystals is isomorphic (as a classical crystal) to a Demazure crystal $B_{\lambda(\tau)}$. There is a unique isomorphism such that the pullbacks of 0-arrows in the Demazure crystal are 0-arrows in $B$. 

Peter Tingley (MIT)
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- In non-exceptional types, a tensor product $B$ of level $\ell$ KR crystals is isomorphic (as a classical crystal) to a Demazure crystal $B_w(\ell \Lambda_{\tau}(0))$. 
Relationship between KR crystals and Demazure crystals

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Relationship between KR crystals and Demazure crystals

For \( \mathfrak{sl}_3 \):

\[ B_1 \otimes B_2 \otimes B_{s_1} B_{s_2} B_{s_0} (\Lambda_0) \]

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Relationship between KR crystals and Demazure crystals

For $\mathfrak{sl}_3$: \[ B^{1,1} \otimes B^{2,1} \]

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Energy function for a prime KR crystal

The energy function counts the number of vertical dominoes that can be removed. In other types it is similar, but the shape being removed changes a bit.
Energy function for a prime KR crystal

For $D_n^{(1)}$ (non-spin nodes):
Energy function for a prime KR crystal

For $D_n^{(1)}$ (non-spin nodes):

$$B^{5,2} \cong B(2\omega_5) \oplus B(\omega_3 + \omega_5) \oplus B(\omega_1 + \omega_5) \oplus B(2\omega_3) \oplus B(\omega_1 + \omega_3) \oplus B(2\omega_1)$$
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$$E = 4$$
Energy function for a prime KR crystal

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![Diagram of dominoes removed](image)

$$E = 4 \quad 3$$
Energy function for a prime KR crystal

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$E = 4 \quad 3 \quad 2$
Energy function for a prime KR crystal

For $D_n^{(1)}$ (non-spin nodes):

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\[ E = \begin{array}{cccccc}
4 & 3 & 2 & 2 & 1 & 1
\end{array} \]
Energy function for a prime KR crystal

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\[ E = \begin{array}{cccccc}
4 & 3 & 2 & 2 & 1 & 0
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The energy function counts the number of vertical dominoes that can be removed.

In other types it is similar, but the shape being removed changes a bit.
Energy function for a composite KR crystal

There is a unique $H = H_{B_2} \otimes H_{B_1} \rightarrow \mathbb{Z}$ such that

$$H_{B_2}, B_1(u_{B_2} \otimes u_{B_1}) = 0$$

for all $b_2 \in B_2, b_1 \in B_1$.

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} \mathbb{L}L & \text{if } i = 0 \\
\mathbb{R}R & \text{otherwise.} \end{cases} \quad (1)$$

**LL means:** $e_0$ acts on the left in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$.

**RR means:** $e_0$ acts on the left in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$.
Energy function for a composite KR crystal

There is a unique $H = H_{B_2, B_1} : B_2 \otimes B_1 \to \mathbb{Z}$ such that
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Energy function for a composite KR crystal

There is a unique $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$ such that

- $H_{B_2,B_1}(u_{B_2} \otimes u_{B_1}) = 0$
- For all $b_2 \in B_2$, $b_1 \in B_1$, $i \in \{0,1\}$,

$$H(e_i(b_2 \otimes b_1)) = \begin{cases} -1 & \text{if } i = 0 \text{ and LL} \\ 1 & \text{if } i = 0 \text{ and RR} \\ 0 & \text{otherwise.} \end{cases}$$
There is a unique $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$ such that

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-1 & \text{if } i = 0 \text{ and LL} \\
1 & \text{if } i = 0 \text{ and RR} \\
0 & \text{otherwise.}
\end{cases}$$

LL means: $e_0$ acts on the left in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$.
RR means: $e_0$ acts on the left in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$. 
Energy function for a composite KR crystal
For $B = B^r_N,s_N \otimes \cdots \otimes B^r_1,s_1$, $1 \leq i \leq N$ and $i < j \leq N$, set

$$E_i := E_{B^r_i,s_i} \sigma_1 \sigma_2 \cdots \sigma_{i-1} \quad \text{and} \quad H_{j,i} := H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1},$$

where $\sigma_i$ and $H_i$ act on the $i$-th and $(i+1)$-st tensor factors. Then

$$E_B := \sum_{N \geq j > i \geq 1} H_{j,i} + \sum_{i=1}^{N} E_i.$$
Main Theorem

Theorem (Schilling-T-, conjectured by HKOTT)

Fix $g$ of non-exceptional affine type, and let $B = B^{r_1}_{c}, \cdots \otimes B^{r_k}_{c}$ be a composite KR crystal of level $\tilde{\gamma}$. Then the isomorphism between $B$ and the corresponding Demazure crystal $B^w(\tilde{\Lambda}^0(0))$ intertwines the energy function with the affine grading.

Sketch of proof

Using explicit models show that, for all $b \in B^{r_1}_{c}, \cdots$, $E(f_0(b)) \leq E(b) + 1$.

Furthermore, if $\epsilon_i(b) > \tilde{\gamma}$, then this is equality.

An inductive argument gives the same statement for $B$ a composite KR crystal of level $\tilde{\gamma}$.

Since $\phi(b^0_{\Lambda 0}) = \tilde{\gamma}$, the result follows for tensor product rule.

Corollary

$E(b) - E(u)$ records the minimal number of $f_0$ in a sequence of operators taking the ground state path $u$ to $b$. 
Main Theorem

Theorem (Schilling-T-, conjectured by HKOTT)

Fix $g$ of non-exceptional affine type, and let $B = B_{r_1}^{c_1} \otimes \cdots \otimes B_{r_k}^{c_k}$ be a composite KR crystal of level $\Lambda_0$. Then the isomorphism between $B$ and the corresponding Demazure crystal $B_w(\Lambda_{\tau}(0))$ intertwines the energy function with the affine grading.

Sketch of proof

Using explicit models show that, for all $b \in B_{r_i}^{c_i}$,

$$E(f_0(b)) \leq E(b) + 1.$$

Furthermore, if $\epsilon_i(b) > \Lambda_0$, then this is equality.

An inductive argument gives the same statement for $B$ a composite KR crystal of level $\Lambda_0$.

Since $\phi(b_\Lambda(0)) = \epsilon_0$, the result follows for tensor product rule.

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Peter Tingley (MIT)
Main Theorem

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Fix $g$ of non-exceptional affine type, and let $B = B^{r_1,c_{r_1}} \otimes \cdots \otimes B^{r_k,c_{r_k}}$ be a composite KR crystal of level $\ell$. Then the isomorphism between $B$ and the corresponding Demazure crystal $B_w(\ell \Lambda_{\tau(0)})$ intertwines the energy function with the affine grading.

Sketch of proof
Using explicit models show that, for all $b \in B^{r_1,c_{r_1}}$, $E(f_0(b)) \leq E(b) + 1$. Furthermore, if $\epsilon_i(b) > 0$, then this is equality. An inductive argument gives the same statement for $B$ a composite KR crystal of level $\ell$. Since $\phi(b^{\ell \Lambda_{\tau(0)}}) = 0$, the result follows for tensor product rule.

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$E(b) - E(u)$ records the minimal number of $f_0$ in a sequence of operators taking the ground state path $u$ to $b$. 

Peter Tingley (MIT)

Energy function

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Sketch of proof

- Using explicit models show that, for all $b \in B^{r, c_r \ell}$, $E(f_0(b)) \leq E(b) + 1$. Furthermore, if $\varepsilon_i(b) > \ell$, then this is equality.
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- Since $\varphi(b_{\ell\Lambda_0}) = \ell$, the result follows for tensor product rule.
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Corollary
Main Theorem

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Corollary

$E(b) - E(u)$ records the minimal number of $f_0$ in a sequence of operators taking the ground state path $u$ to $b$. 
Work of Sanderson and Ion shows that, in types $A_1^n$, $D_1^n$ and $E_1^n$, the non-symmetric Macdonald polynomials satisfy

$$E_\lambda(q, 0) = q^{c_{\text{ch}}(V_w(\Lambda_\tau(0)))} |_{e_\delta = q, e\Lambda_0 = 1}.$$ 

Our results imply that, in types $A_1^n$ and $D_1^n$, the symmetric Macdonald polynomials satisfy

$$P_\lambda(q, 0) = \sum_{b \in B} q^{-E(b)} e^{\text{wt}(b)},$$

where $E$ is the combinatorial energy function (called $D$ in our paper).

We also see the non-symmetric Macdonald polynomials as partial sums over KR crystals.
Work of Sanderson and Ion shows that, in types \( A_n^{(1)} \), \( D_n^{(1)} \) and \( E_n^{(1)} \), the non-symmetric Macdonald polynomials satisfy

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Macdonald polynomials

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We also see the non-symmetric Macdonald polynomials as partial sums over KR crystals.
Example

In \text{sl}_3, B_1 \otimes B_1,1 \rightarrow B_1,1 \otimes B_1,1 \rightarrow B_1,1 \otimes B_1,2 \rightarrow B_2,1 \otimes B_2,2 \rightarrow B_2,2 \otimes B_2,3 \rightarrow B_3,2 \otimes B_3,3,

where to simplify the diagram we also show the 0 arrows that survive in the corresponding Demazure crystal.
Example

\[ P_{-2\omega_2}(x; q, 0) = x_1^2 + (q + 1)x_1x_2 + x_2^2 + (q + 1)x_1x_3 + (q + 1)x_2x_3 + x_3^2 \]
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In \( \mathfrak{sl}_3 \), \( B^{1,1} \otimes B^{1,1} \) looks like:

\[
\begin{array}{c}
2 \otimes 1 \rightarrow 3 \otimes 1 \\
0 \rightarrow 1 \otimes 1 \rightarrow 1 \otimes 2 \\
1 \rightarrow 2 \otimes 2 \rightarrow 2 \otimes 3 \rightarrow 3 \otimes 3,
\end{array}
\]
Example

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In \( \mathfrak{sl}_3 \), \( B^{1,1} \otimes B^{1,1} \) looks like:

\[
\begin{align*}
2 \otimes 1 & \xrightarrow{2} 3 \otimes 1 \\
& \xrightarrow{0} 1 \otimes 1 \xrightarrow{1} 1 \otimes 2 \\
& \xrightarrow{1} 2 \otimes 2 \xrightarrow{2} 2 \otimes 3 \xrightarrow{2} 3 \otimes 3, \\
& \xrightarrow{1} 3 \otimes 2 \\
& \xrightarrow{2} 1 \otimes 3 \\
& \xrightarrow{1} 1 \otimes 3 \\
& \xrightarrow{1} 3 \otimes 3,
\end{align*}
\]

where to simplify the diagram we also show the 0 arrows that survive in the corresponding Demazure crystal.
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\[ = q(x_1x_2 + x_1x_3 + x_2x_3) \]

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\[
\begin{array}{c}
2 \otimes 1 \xrightarrow{2} 3 \otimes 1 \xrightarrow{0} 1 \otimes 1 \xrightarrow{1} 1 \otimes 2 \xrightarrow{1} 2 \otimes 2 \xrightarrow{2} 2 \otimes 3 \xrightarrow{2} 3 \otimes 3,
\end{array}
\]

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In \( \mathfrak{sl}_3 \), \( B^{1,1} \otimes B^{1,1} \) looks like:

\[
\begin{array}{cccccccc}
2 \otimes 1 & \xrightarrow{2} & 3 \otimes 1 & \xrightarrow{0} & 1 \otimes 1 & \xrightarrow{1} & 1 \otimes 2 & \xrightarrow{1} & 2 \otimes 2 & \xrightarrow{2} & 2 \otimes 3 & \xrightarrow{2} & 3 \otimes 3, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
3 \otimes 2 & & 1 \otimes 3 & & 2 \otimes 3 & & 3 \otimes 3, & & & & & & \\
\end{array}
\]

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\[
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\[
\begin{array}{ccccccccc}
2 \otimes 1 & \rightarrow & 3 \otimes 1 & \rightarrow & 1 \otimes 1 & \rightarrow & 1 \otimes 2 & \rightarrow & 2 \otimes 2 & \rightarrow & 2 \otimes 3 & \rightarrow & 3 \otimes 3,
\end{array}
\]

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Example

\[ P_{-2\omega_2}(x; q, 0) = x_1^2 + (q + 1)x_1x_2 + x_2^2 + (q + 1)x_1x_3 + (q + 1)x_2x_3 + x_3^2 \]
\[ = q(x_1x_2 + x_1x_3 + x_2x_3) + x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 \]

In \( \mathfrak{sl}_3 \), \( B^{1,1} \otimes B^{1,1} \) looks like:

\[
\begin{array}{ccccccc}
2 \otimes 1 & \rightarrow & 3 \otimes 1 & \rightarrow & 1 \otimes 1 & \rightarrow & 1 \otimes 2 & \rightarrow & 2 \otimes 2 & \rightarrow & 2 \otimes 3 & \rightarrow & 3 \otimes 3,
\end{array}
\]

where to simplify the diagram we also show the 0 arrows that survive in the corresponding Demazure crystal.
Gerasimov, Lebedev, Oblezin showed that $q$-deformed $\mathfrak{gl}_n$-Whittaker functions are Macdonald polynomials specialized at $t = 0$. So, by Sanderson and Ion, they can be expressed using Demazure characters. Hence, by our results they can be expressed in terms of KR crystals and the energy function.
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Future directions

Macdonald polynomials from Demazure characters in type $C_n^{(1)}$.

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$$B_{1,1} \otimes B_{1,1} \otimes b_{\Lambda_0} = B_{s_1 s_2 s_1 s_2 s_0} (\Lambda_0) \quad \text{and} \quad B_{s_2 s_1 s_2} (\Lambda_2)$$

These tensor products seem to break up as unions of Demazure modules. Via Lenart's results, this would give a formula for Macdonald polynomials as sums of Demazure characters.
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Peter Tingley (MIT) 
Energy function 
Wake forest, Sept. 24, 2011 14 / 14
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