Demazure crystals, Kirillov–Reshetikhin crystals, and the energy function[1]

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(joint with Anne Schilling)

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Wake forest, Sept. 24, 2011

[1]Slides and notes available at www-math.mit.edu/∼ptingley/

Outline

1. Background
   - Highest weight crystals
   - Demazure crystals
   - Kirillov–Reshetikhin crystals
   - Relationship between KR crystals and Demazure crsystals.
   - The energy function
2. Results
3. Applications
   - Macdonald polynomials
   - Whittaker functions
4. Future directions
   - Macdonald polynomials from Demazure characters in type $C_n^{(1)}$?

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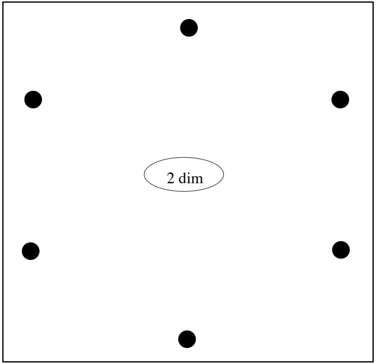

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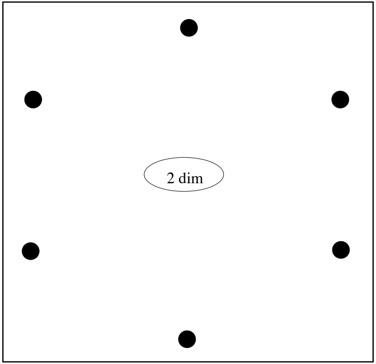

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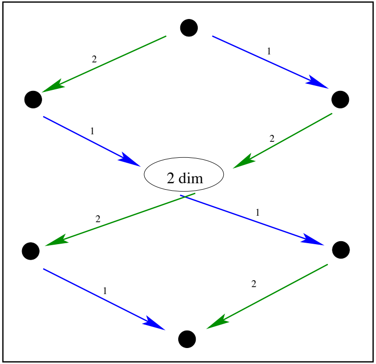

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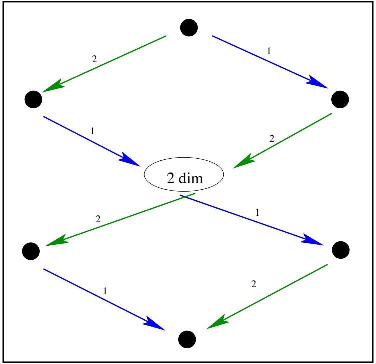

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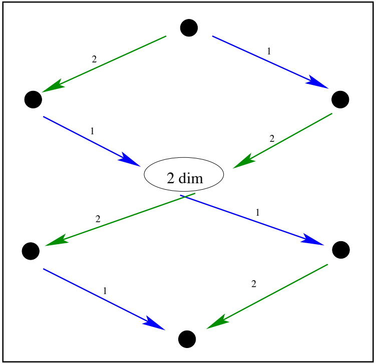

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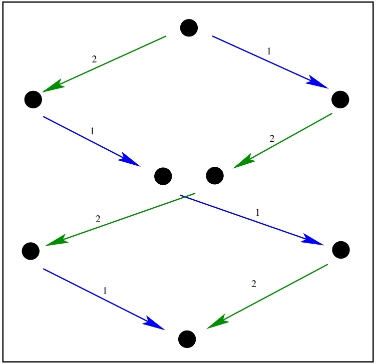

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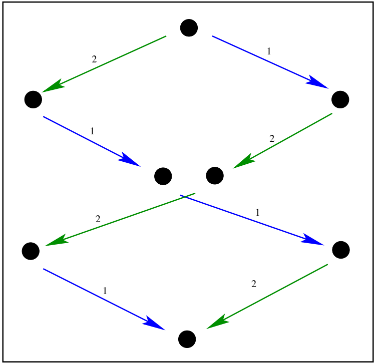

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- There are 4 distinguished one dimensional spaces in the middle.
- If we use $U_q(\mathfrak{sl}_3)$ and 'rescale' the operators, then "at $q = 0$", they match up. You get a colored directed graph.

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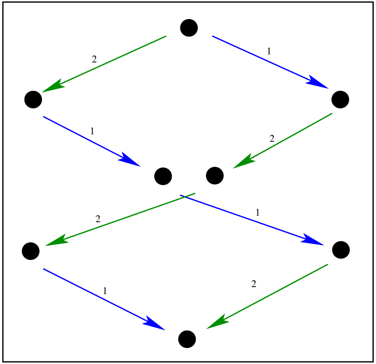

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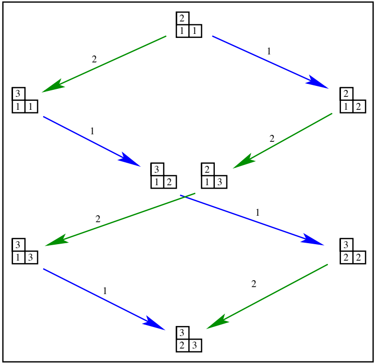

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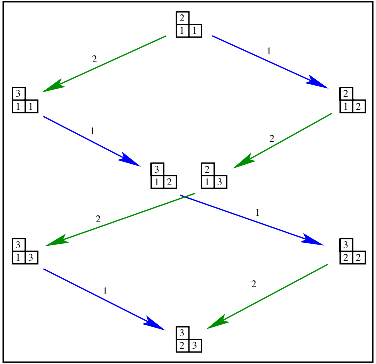

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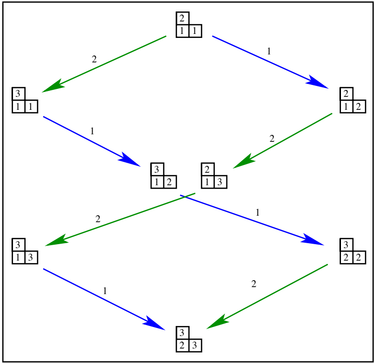

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- Here you see that the graded dimension of the representation is the generating function for semi-standard Young tableaux.

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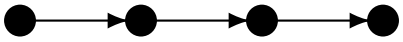

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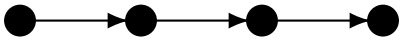

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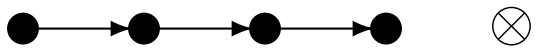

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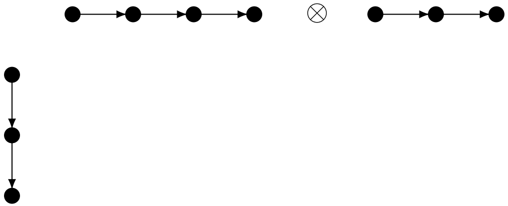

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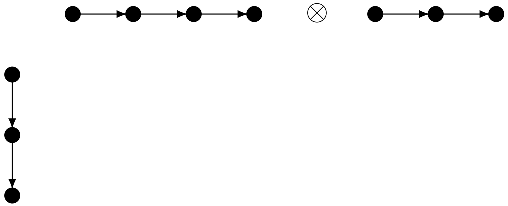

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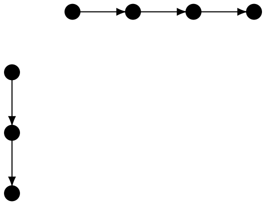

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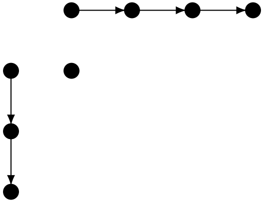

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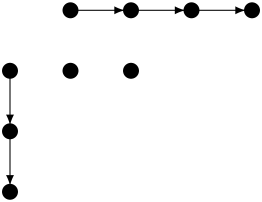

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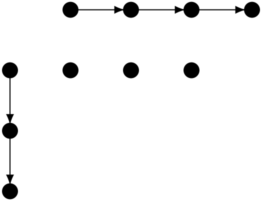

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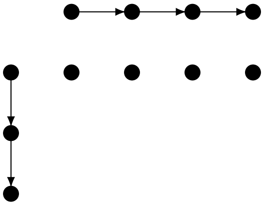

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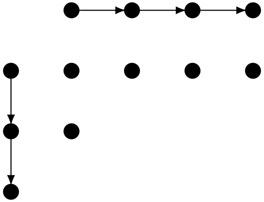

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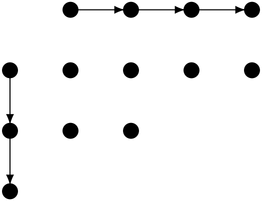

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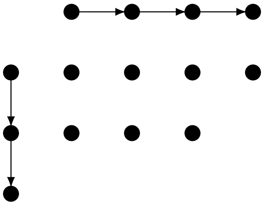

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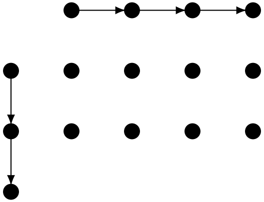

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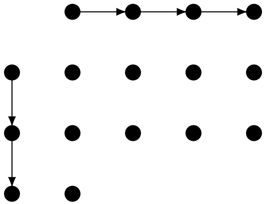

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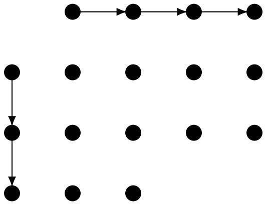

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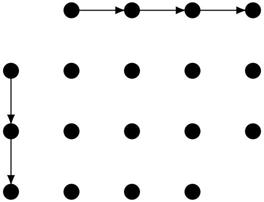

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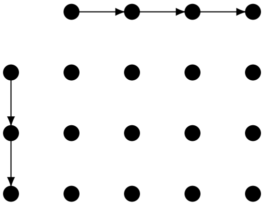

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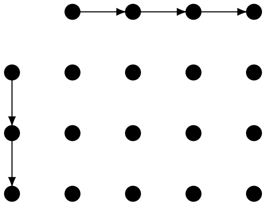

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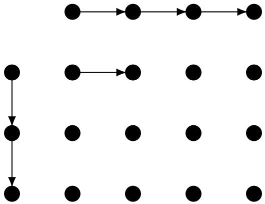

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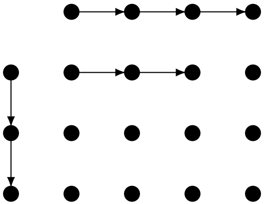

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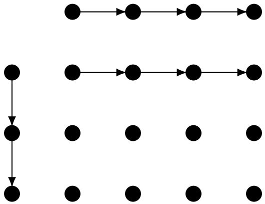

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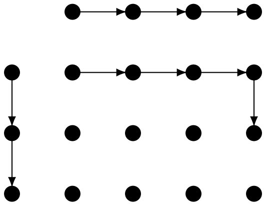

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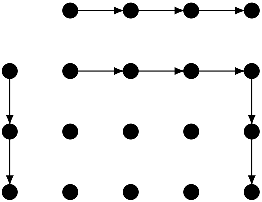

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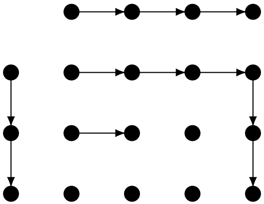

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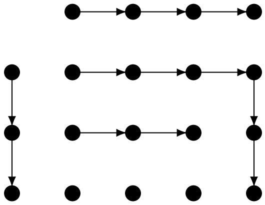

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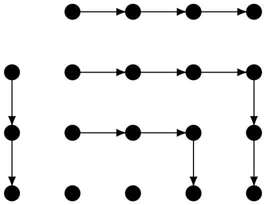

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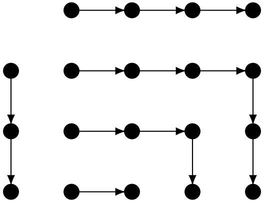

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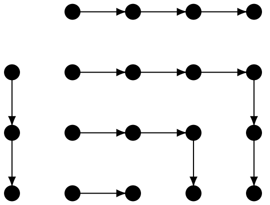

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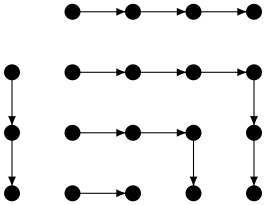

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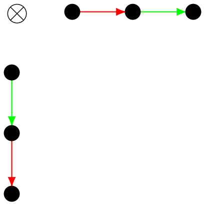

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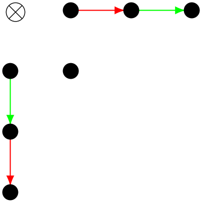

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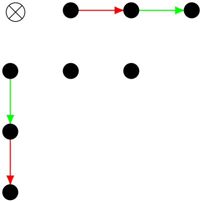

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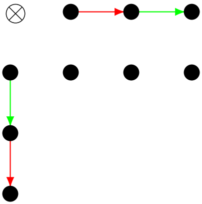

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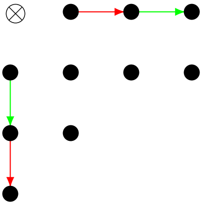

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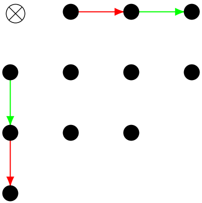

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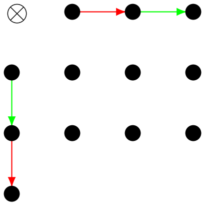

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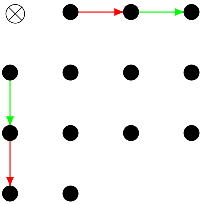

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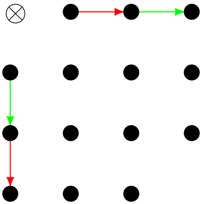

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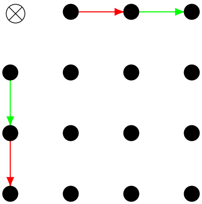

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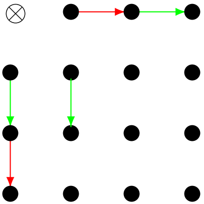

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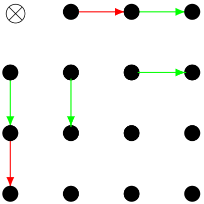

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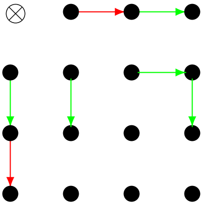

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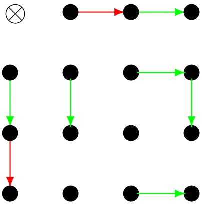

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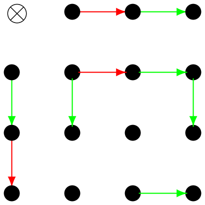

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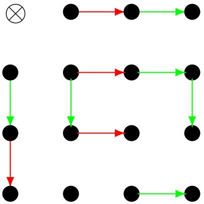

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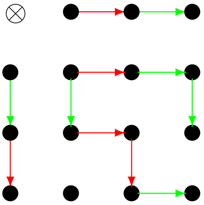

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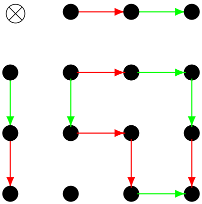

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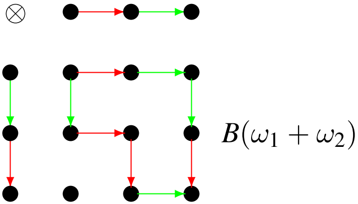

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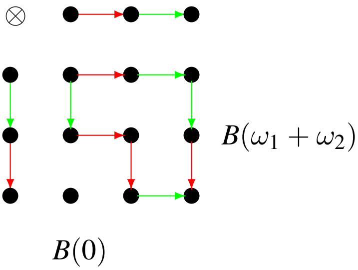

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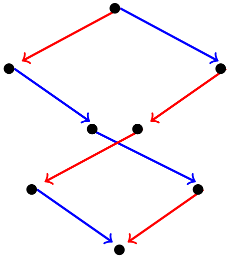

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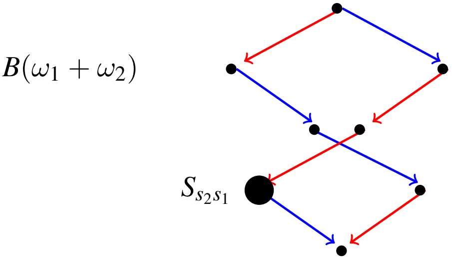

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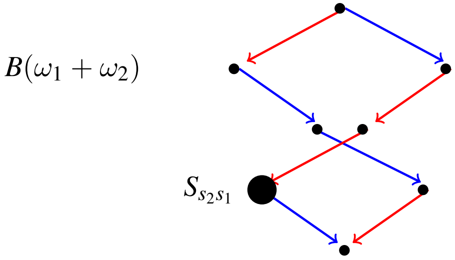

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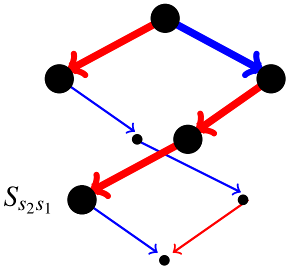

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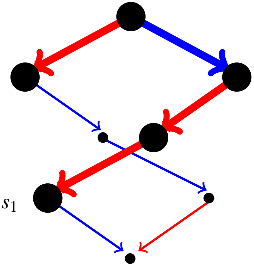

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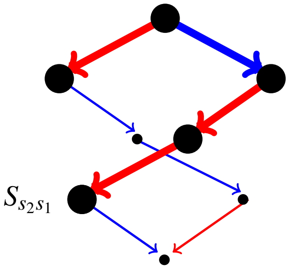

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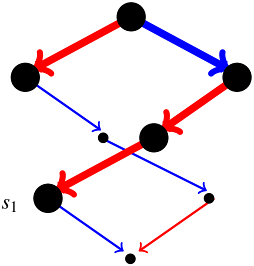

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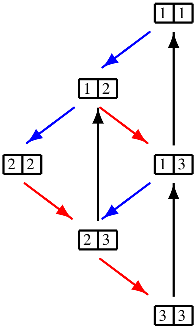

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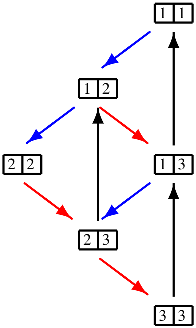

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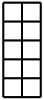

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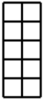

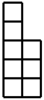

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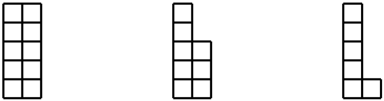

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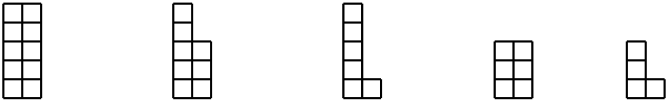

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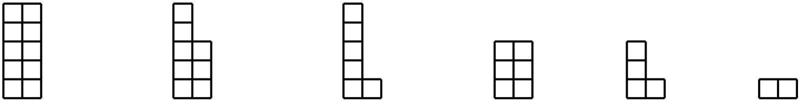

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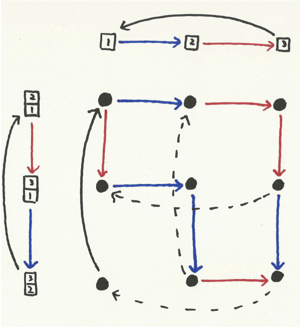

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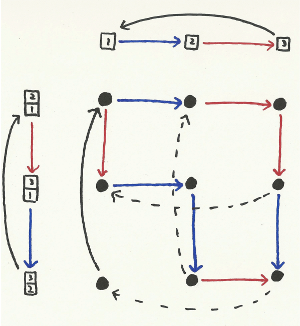

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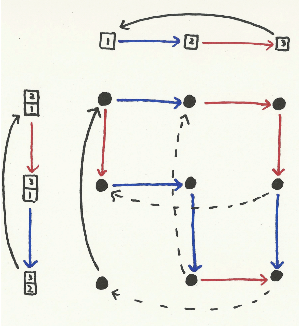

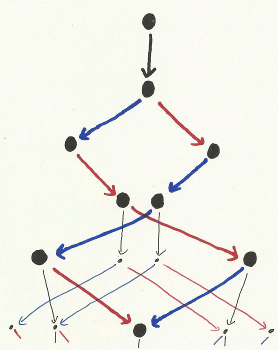

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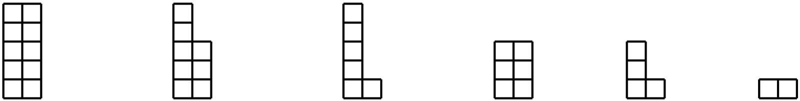

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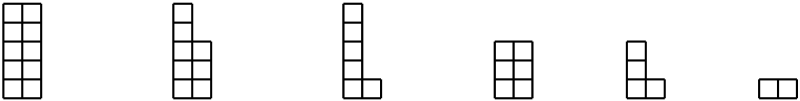

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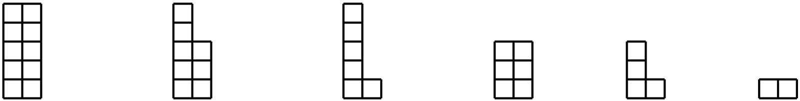

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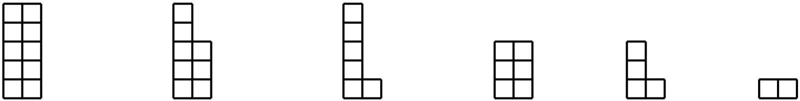

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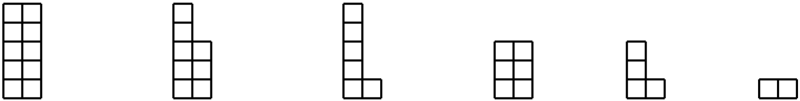

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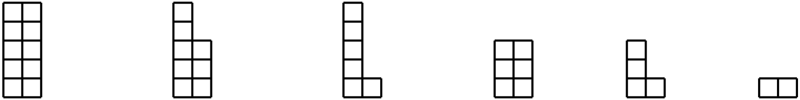

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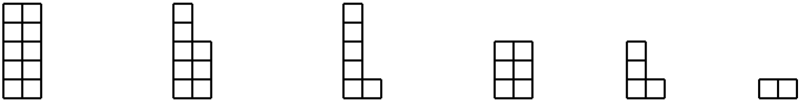

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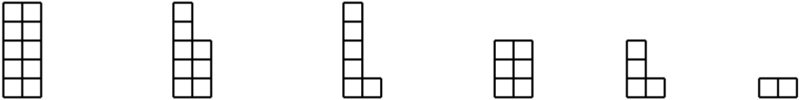

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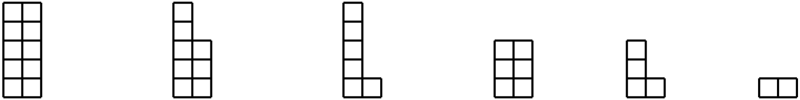

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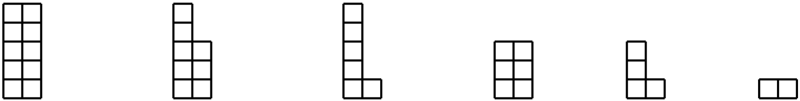

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- The energy function counts the number of vertical dominoes that can be removed.
- In other types it is similar, but the shape being removed changes a bit.

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LL means: e_0 acts on the left in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$.
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For $B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, $1 \leq i \leq N$ and $i < j \leq N$, set

$$E_i := E_{B^{r_i,s_i}} \sigma_1 \sigma_2 \cdots \sigma_{i-1} \quad \text{and} \quad H_{j,i} := H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1},$$

where σ_i and H_i act on the i-th and $(i+1)$-st tensor factors . Then

$$E_B := \sum_{N \geq j > i \geq 1} H_{j,i} + \sum_{i=1}^{N} E_i.$$

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Corollary

$E(b) - E(u)$ *records the minimal number of* f_0 *in a sequence of operators taking the ground state path* u *to* b*.*

Macdonald polynomials

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- Work of Sanderson and Ion shows that, in types $A_n^{(1)}$, $D_n^{(1)}$ and $E_n^{(1)}$, the non-symmetric Macdonald polynomials satisfy

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- We also see the non-symmetric Macdonald polytomials as partial sums over KR crystals.

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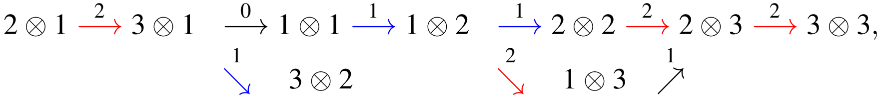

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- So, by Sanderson and Ion, they can be expressed using Demazure characters.

Whittaker functions

- Gerasimov, Lebedev, Oblezin showed that q-deformed $\mathfrak{gl}_n$-Whittaker functions are Macdonald polynomials specialized at $t = 0$.
- So, by Sanderson and Ion, they can be expressed using Demazure characters.
- Hence, by our results they can be expressed in terms of KR crystals and the energy function.

Future directions

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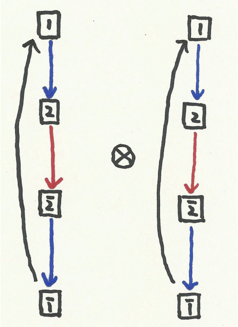

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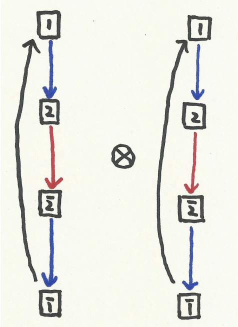

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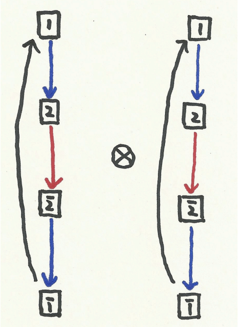

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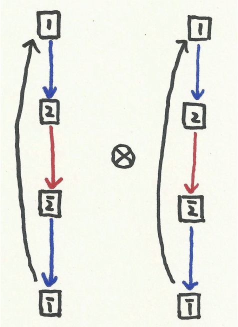

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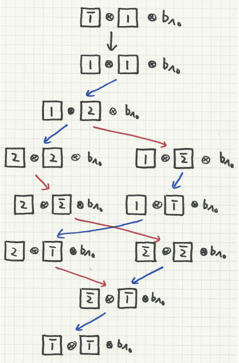

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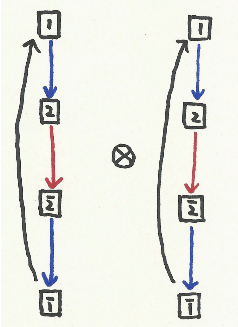

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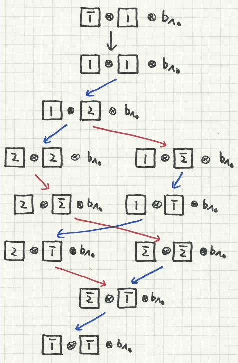

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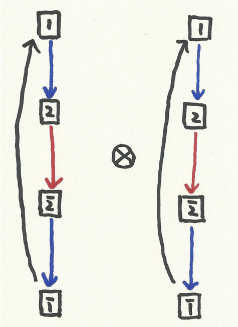

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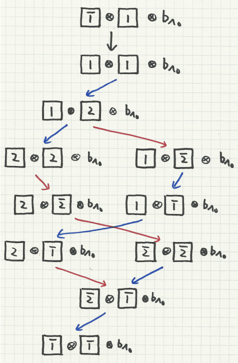

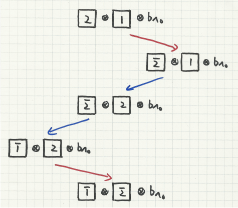

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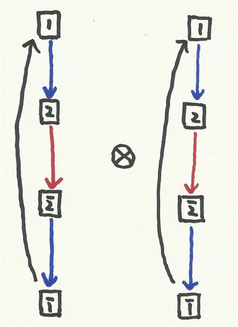

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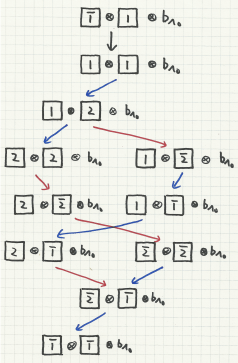

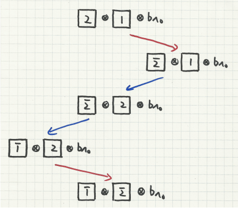

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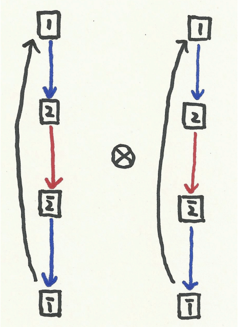

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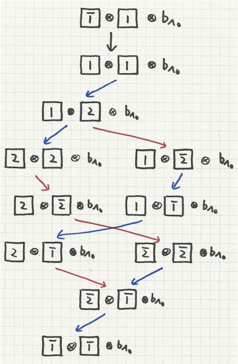
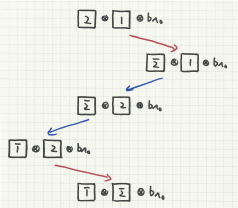


$B_{s_2 s_1 s_2}(\Lambda_2)$



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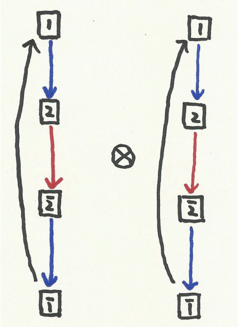

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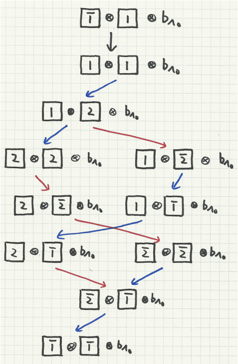

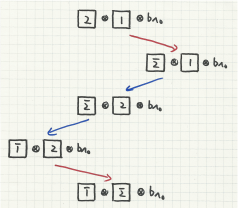

- These tensor products seem to break up as unions of Demazure modules.

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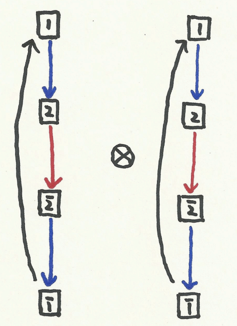

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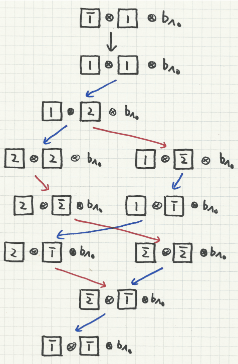

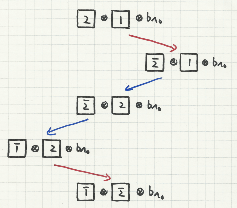

- These tensor products seem to break up as unions of Demazure modules.
- Via Lenart's results, this would give a formula for Macdonald polynomials as sums of Demazure Characters.