NOTES ON FOCK SPACE

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Abstract. These notes are intended as a fairly self contained explanation of Fock space and various algebras that act on it, including a Clifford algebras, a Weyl algebra, and an affine Kac-Moody algebra. We also discuss how the various algebras are related, and in particular describe the celebrated boson-fermion correspondence. We finish by briefly discussing a deformation of Fock space, which is a representation for the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{sl}}_\ell)$.

Contents

1. Introduction

The term Fock space comes from particle physics, where it is the state space for a system of a variable number of elementary particles. There are two distinct types of elementary particles, bosons and fermions, and their Fock spaces look quite different. Fermionic Fock spaces are naturally representations of a Clifford algebra, where the generators correspond

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to adding/removing a particle in a given pure energy state. Similarly, bosonic Fock space is naturally a representation of a Weyl algebra.

Here we focus on one example of this construction each for Bosons and Fermions. Our fermionic Fock space $\mathbf{F}$ corresponds to a system of fermionic particles with pure energy states indexed by $\mathbb{Z}+1/2$. Our space $\mathbf{B}^{(0)}$ corresponds to a system of bosons with pure energy states indexed by $\mathbb{Z}_{>0}$, and our full bosonic Fock space $\mathbf{B}$ is a superposition of $\mathbb{Z}$ shifted copies of $\mathbf{B}^{(0)}$. There is a natural embedding of the Weyl algebra for $\mathbf{B}^{(0)}$ into a completion of the Clifford algebra for $\mathbf{F}$, which leads to the celebrated boson-fermion correspondence.

We begin by discussing $\mathbf{F}$ and $\mathbf{B}$ as vector spaces. We present various ways of indexing the standard basis of $\mathbf{F}$, and describe the boson-fermion correspondence as a vector space isomorphism between $\mathbf{F}$ and $\mathbf{B}$. Since $\mathbf{F}$ and $\mathbf{B}$ are both vector spaces with countable bases it is obvious that such an isomorphism exists, so the isomorphism we present is only interesting in that it has nice properties with respect to the actions of the Clifford and Weyl algebras. A more intuitive development of the theory (and also how it is presented in [4, Chapter 14]) is probably to first notice the relationship between the Clifford and Weyl algebras, then derive the corresponding relationship between the vector spaces. However, we find it useful to have the map explicitly described in an elementary way.

We next discuss the various algebras that naturally act on our Fock space, and how these actions are related. These include an infinite rank matrix algebra and various affine Kac-Moody algebras, as well as the Weyl and Clifford algebras. In particular the algebra of $(\mathbb{Z}+1/2)\times(\mathbb{Z}+1/2)$ matrices naturally acts on $\mathbf{F}$. The affine Kac-Moody algebras $\widehat{\mathfrak{sl}}_\ell$ naturally embeds into a central extension of a completion of this matrix algebra, as does the larger affine algebra $\widehat{\mathfrak{gl}}_\ell$, leading to actions of these algebras on $\mathbf{F}$. We finish by presenting the Misra-Miwa action of the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}\otimes_{\mathbb{C}}\mathbb{C}[q,q^{-1}]$. Understanding the relationship between the actions of these various algebras on $\mathbf{F}$ has proven very useful (see [4, Chapter 14]). It is really this representation theory that we are interested in, not the physics motivations. There will be very little physics beyond this introduction.

These notes are intended to be quite self contained in the sense that they should be comprehensible independent of other references on Fock space. We do however refer to other sources for many important proofs. Much of our presentation loosely follows [4, Chapter 14], which we highly recommend to anyone who in looking for a deeper understanding of Fock space and it's relation to the representation theory of Kac-Moody algebras.

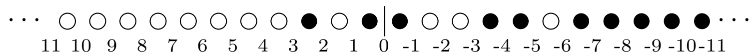
1A. **Acknowledgments.** This notes were one consequence of a long series of discussions Arun Ram, so I would like to thank Arun for being so generous with his time. I would also like to thank Gus Schrader and A.J. Tolland for a helpful comments on earlier drafts.

2. Fock space as a vector space

Fermionic Fock space $\mathbf{F}$ is an infinite dimensional vector space. This has a standard basis, which can be indexed by a variety of objects. In this section we discuss indexing by Maya diagrams, charged partitions, and normally ordered wedge products. Bosonic Fock space is essentially a space of polynomials in infinitely many variables. This has a "standard" basis constructed using Schur symmetric functions. Using the fact that Schur symmetric functions are indexed by partitions, we define a bijection between the standard bases of $\mathbf{F}$ and $\mathbf{B}$, which can be extended to a vector space isomorphism. Of course there are many isomorphisms of vector spaces between **B** and **F**. Our choice is justified by the representation theoretic results in the next section.

2A. Maya diagrams.

Definition 2.1. A *Maya diagram* is a placement of a white or black bead at each position in $\mathbb{Z} + 1/2$, subject to the condition that at all but finitely many positions $m < 0$ are filled with a black bead and all but finitely many positions $m > 0$ are filled with a white bead. For instance,



Note that we label the real line from right to left. This is done so that later on we more closely match the conventions of [4]. It is sometimes convenient to think of the black beads as "filled positions" and the white beads as "empty positions" in a "Dirac sea".

2B. Charged partitions.

Definition 2.2. A **charged partition** λ is a pair (λ', k) consisting of a partition λ' and an integer k (the charge).

There is a natural bijection between Maya diagrams and charged partitions: draw a line above the Maya diagram by placing a segment sloping down and to the right over every white bead, and a segment sloping down and to the right over every black bead. The result is the outer boundary of a partition. The charge is determined by superimposing axes with the origin above position 0 in the Maya diagram, and so that far to the right the axis follows the diagram. The charge is the signed distance between the diagram and the axis far to the left. See Figure 1.

We use the notation $|\lambda\rangle$ to denote the standard basis element of F corresponding to the charged partition λ.

2C. Semi-infinite wedge space.

Definition 2.3. Let $V_{\mathbb{Z}+1/2}$ be the free span over $\mathbb{C}$ of $\{e_m\}_{m\in\mathbb{Z}+1/2}$.

Definition 2.4. *Semi-infinite wedge space* is $V_{\mathbb{Z}+1/2} \wedge V_{\mathbb{Z}+1/2} \wedge \cdots$

Definition 2.5. A semi-infinite wedge product $e_{m_1} \wedge e_{m_2} \wedge e_{m_3} \wedge \cdots$ is called *normally ordered* if $m_1 > m_2 > m_3 > \cdots$.

Definition 2.6. A semi-infinite wedge product $e_{m_1} \wedge e_{m_2} \wedge e_{m_3} \wedge \cdots$ is called *regular* if, for all large enough k, one has $m_{k+1} = m_k - 1$.

There is a bijection between regular normally ordered semi infinite wedges and Maya diagrams which takes $e_{m_1} \wedge e_{m_2} \wedge \cdots$ to the Maya diagram with black beads exactly in positions $m_1, m_2, \ldots$. For instance, the normally ordered wedge

$$e_{2.5} \wedge e_{0.5} \wedge e_{-0.5} \wedge e_{-3.5} \wedge e_{-4.5} \wedge e_{-6.5} \wedge e_{-7.5} \wedge e_{-8.5} \cdots \tag{1}$$

corresponds to the Maya diagram shown in Definition 2.1.

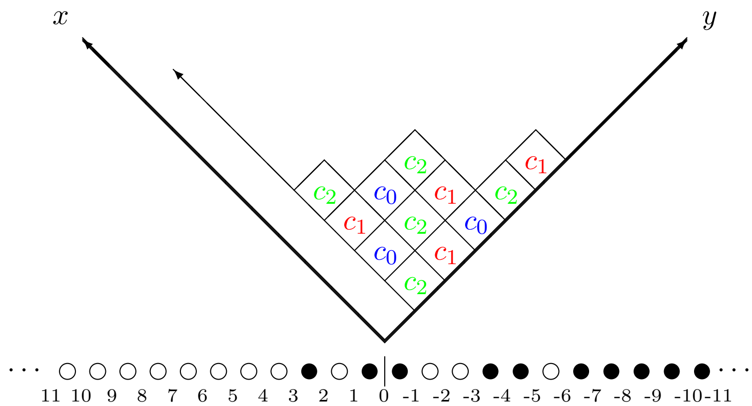


FIGURE 1. The bijection between Maya diagrams and charged partitions. The black beads in the Maya diagram correspond to those positions where the rim of the partition slopes up and to the right. The parts of the finite component λ' are the lengths of all the finite "rows" of boxes sloping up at to the left. Here $\lambda' = (4, 3, 3, 1, 1)$. The charge in this example is -1, because far to the left the line described by the Maya diagram ends up 1 step to the right of the axis. The charged partition λ corresponding to the Maya diagram is the pair (λ', k). We often choose a "level" ℓ and color the squares of λ' with $c_{\bar{j}}$ for residues $\bar{j}$ mod ℓ. The color of a square b is the position of that square in the horizontal direction, mod ℓ (reading right to left).

2D. **Fermionic Fock space F.** We define fermionic Fock space $\mathbf{F}$ to be the free span over $\mathbb{C}$ of all Maya diagrams. The basis of $\mathbf{F}$ consisting of Maya diagrams is called the standard basis. Using the bijections from Section 2B and 2C, the standard basis of $\mathbf{F}$ is also indexed by charged partitions or regular normally ordered semi-infinite wedges.

The *charge* m part of $\mathbf{F}$ is

$$\mathbf{F}^{(m)} := \mathit{span}\{\text{charged partitions with charge } m\}.$$

In particular, $\mathbf{F}^{(0)}$ is just spanned by ordinary partitions. Clearly $\mathbf{F} = \oplus_{m\in\mathbb{Z}}\mathbf{F}^{(m)}$

2E. **Bosonic Fock space B.** The *bosonic Fock space* $\mathbf{B}$ is

$$\mathbf{B} := \mathbb{C}[x_1, x_2, x_3, \ldots; q, q^{-1}].$$

The *charge* m part of $\mathbf{B}$ is

$$\mathbf{B}^{(m)} := q^m\mathbb{C}[x_1, x_2, \ldots].$$

Notice that $\mathbf{B} = \oplus_{m\in\mathbb{Z}}\mathbf{B}^{(m)}$.

2F. **The boson-fermion correspondence as a map of vector spaces.** There is a natural identification between $\mathbf{F}$ and $\mathbf{B}$. This is part of the celebrated boson-fermion correspondence. The importance of this maps arises because it gives a non-trivial relationship between certain algebras that act on $\mathbf{F}$ and $\mathbf{B}$. We discuss these relationships in Section 3C, but for now we are content to simply define the map of vector spaces. This uses the theory of Schur symmetric functions.

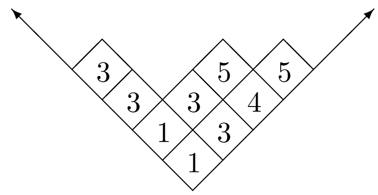


FIGURE 2. A column strict filing of $\lambda = (4, 2, 2, 1)$. The function t takes b to the integer in box b. Recall that "rows" slope up and to the left, and "columns" slope up and to the right.

Definition 2.7. Fix an ordinary partition λ'. A *column strict filling* of λ is a function t from the set of boxes in the diagram of λ' to $\mathbb{Z}_{>0}$, which is weakly increasing along rows and strictly increasing along columns. See Figure 2.

Definition 2.8. Let λ' be an ordinary (i.e. uncharged) partition. The *Schur symmetric function* $s_{\lambda'}$ in infinitely many variables $y_1, y_2, \ldots$ corresponding to λ' is

$$s_{\lambda'}(\mathbf{y}) := \sum_{\substack{t \text{ a column strict} \\ \text{filling of } \lambda'}} \prod_{\text{boxes } b \text{ of } \lambda'} y_{t(b)}. \tag{2}$$

Comment 2.9. The Schur symmetric functions are symmetric in the sense that they are invariant under permutations of the variables y_i (this can be shown using some combinatorics, or by appealing to representation theory). They are polynomials in the sense that, if all but finitely many of the variables are set to 0, the result is a polynomial. They have many important properties, most of which arise because (once all but finitely many variables are set to 0) they can be thought of as the characters of the irreducible representations of $\mathfrak{gl}_n$.

Definition 2.10. For any integer $k > 0$, let $p_k = y_1^k + y_2^k + \cdots$ be the k^{th} power symmetric function (in infinitely many variables).

Definition 2.11. The character polynomial $\chi_{\lambda'}$ is the unique polynomial such that

$$\chi_{\lambda'}(p_1, \frac{1}{2}p_2, \frac{1}{3}p_3, \frac{1}{4}p_4 \ldots) = s_{\lambda'}.$$

Comment 2.12. To find $\chi_{\lambda'}$, one may set $y_j = 0$ for all j larger than the longest column of λ'. As well, $\chi_{\lambda'}$ cannot depend on p_k for any k bigger than the number of boxes in λ', so finding any given $\chi_{\lambda'}$ is a finite problem.

Proposition 2.13. *There is an isomorphism of vector spaces $\sigma : \mathbf{F} \to \mathbf{B}$ given by, for any charged partition $\lambda = (\lambda', k)$, $\sigma(|\lambda\rangle) = q^k \chi_{\lambda'}(x_1, x_2, \ldots)$. Furthermore, this restricts to an isomorphism $\mathbf{F}^{(m)} \to B^{(m)}$ for all $m \in \mathbb{Z}$.*

Proof. It is well known that Schur polynomials are a basis for the space of all symmetric functions in infinitely many variables. From this it is clear that $\{q^k s_{\lambda'}\}$ is a basis for $\mathbf{B}$. By definition, the set of all charged partitions is a basis for $\mathbf{F}$. So the result is immediate. That it restricts to an isomorphism $\mathbf{F}^{(m)} \to \mathbf{B}^{(m)}$ is also trivial. $\square$

2G. **The inner product.** It is often convenient to put an inner product on $\mathbf{F}$, where we declare $\{|\lambda\rangle\}$ to be an orthonormal basis.

Comment 2.14. For the physics applications discussed in the introduction, it may be desirable to use a completion of $\mathbf{F}$. It is not currently clear to me which completion is best. For instance one may want to take $\ell^2(\{\text{charged partitions}\})$ instead of just the $\mathbb{C}$ span. Then one would have a Hilbert space, but the algebras a_∞ and $\mathcal{W}$ as we define them below no longer act. For us these algebras are more important, so we do not take any completion.

3. Fock space as a representation

Nothing in Section 2 explains why Fock space is of interest to so many people. The answer lies in the fact that several important algebras act naturally on it. We now discuss (some of) these algebras, how they act, and how they are related.

3A. **F as a representation of a Clifford algebra.** The Clifford algebra is the associative algebra $\mathbf{C}l$ generated by ψ_m, ψ_m^* for $m \in \mathbb{Z}+1/2$ subject to the relations

$$\psi_n\psi_m + \psi_m\psi_n = 0 \tag{3}$$

$$\psi_n^*\psi_m^* + \psi_m^*\psi_n^* = 0 \tag{4}$$

$$\psi_n\psi_m^* + \psi_m^*\psi_n = \delta_{m,n}. \tag{5}$$

It is well known and straightforward to check that $\mathbf{C}l$ acts on Fock space as follows: Use the description of $\mathbf{F}$ in terms of semi-infinite wedges from Section 2C. For $m \in \mathbb{Z}+1/2$ and v a normally ordered semi-infinite wedge product,

$$\psi_m \cdot v = e_m \wedge v. \tag{6}$$

ψ_n^* is the adjoint of ψ^* with respect to the inner product from Section 2G. Explicitly,

$$\psi_m^* \cdot v = \begin{cases} 0 & \text{if } e_m \text{ does not appear as a factor in } v \\ v' & \text{if } v \text{ can be expressed as } e_m \wedge v' \text{ for a normally order wedge } v'. \end{cases} \tag{7}$$

Clearly one can obtain any normally ordered wedge v from any other normally ordered wedge w by applying a finite number of operators ψ_m and ψ_n^*. Thus $\mathbf{F}$ is irreducible as a representation of $\mathbf{C}l$.

Definition 3.1. $\widetilde{\mathbf{C}l}$ is the completion of $\mathbf{C}l$ in the topology generated by the open sets $X + \mathbf{C}l\psi_m^*$ and $X + \mathbf{C}l\psi_{-m}$ for all $X \in \mathbf{C}l$, $m > 0$. Explicitly, an element of $\widetilde{\mathbf{C}l}$ is an infinite sum

$$z + \sum_{m<N} X_m\psi_m + \sum_{m>M} Y_m\psi_m^*, \tag{8}$$

where $z \in \mathbb{C}$, $N, M \in \mathbb{Z}$, and all X_m, Y_m are elements of $\mathbf{C}l$.

It is clear that $\widetilde{\mathbf{C}l}$ acts in a well defined way on $\mathbf{F}$, since, for each normally ordered wedge v, all but finitely many of ψ_m, ψ_{-m} for $m > 0$ act trivially on v.

Comment 3.2. When it seems prudent to distinguish $X \in \widetilde{\mathbf{C}l}$ from it's action of $\mathbf{F}$, we use the notation $\pi_{\mathbf{F}}(X)$ to denote the action.

3B. **B as a representation of a Weyl algebra.**

Definition 3.3. The infinite dimensional Heisenberg algebra $\mathbf{H}$ is generated by α_k for $k \in \mathbb{Z}\backslash\{0\}$ and a central element c subject to the relations:

$$\alpha_j \alpha_k - \alpha_k \alpha_j = j \delta_{j,-k} c.$$

Definition 3.4. The Weyl algebra $\mathcal{W}$ is

$$\mathcal{W} := \mathbf{H}/(c-1).$$

Proposition 3.5. *$\mathcal{W}$ acts on $\mathbf{B}$ by the follows formulas. Each $B^{(m)}$ is preserved under this action, and forms an irreducible representation for $\mathcal{W}$.*

$$\begin{aligned} &\alpha_k \rightarrow \frac{\partial}{\partial x_k} && \text{if } k > 0 \\ &\alpha_k \rightarrow \text{ multiplication by } -kx_{-k} && \text{if } k < 0 \end{aligned}$$

Proof. This is a very straightforward exercise. □

Comment 3.6. When it seems prudent to distinguish $Y \in \mathcal{W}$ from it's action of $\mathbf{B}$, we use the notation $\pi_{\mathbf{B}}(Y)$ to denote the action.

3C. **The boson-fermion correspondence.** Proposition 2.13 gives an isomorphism of vector spaces $\sigma : \mathbf{F} \rightarrow \mathbf{B}$. This isomorphism was chosen because it reveals an important relationship between the algebras $\mathbf{C}l$ and $\mathcal{W}$, which is known as the boson-fermion correspondence. We now explain that relationship, referring to [4, Chapter 14] for rigorous proofs.

Definition 3.7. For $Y \in \mathcal{W}$, let $\pi_{\mathbf{F}}(Y) = \sigma^{-1} \circ \pi_{\mathbf{B}}(Y) \circ \sigma$. That is, $\pi_{\mathbf{F}}(Y)$ is the operator on $\mathbf{F}$ induced from $\pi_{\mathbf{B}}(Y)$ by the vector space isomorphism $\sigma : \mathbf{F} \rightarrow \mathbf{B}$.

Proposition 3.8. *(see [4]) For all $k \neq 0$,*

$$\pi_{\mathbf{F}}(\alpha_k) = \pi_{\mathbf{F}} \sum_{m \in \mathbb{Z}+1/2} \psi_m \psi^*_{m+k}, \qquad \pi_{\mathbf{F}}(q) = \pi_{\mathbf{F}}(s).$$

Since for $k \neq 0$ we have $\psi_m \psi^*_{m+k} = \psi^*_{m+k} \psi_m$, it is clear that $\pi_{\mathbf{F}}(\alpha_k) \in \widetilde{\mathbf{C}l}$. Thus Proposition 3.8 gives is a simple imbedding of $\mathcal{W}$ into $\widetilde{\mathbf{C}l}$. It is natural to ask if one can go the other way, and express $\mathbf{C}l$ in terms of $\mathcal{W}$. All elements of $\mathcal{W}$ preserve the subspaces $\mathbf{F}^{(m)}$ of $\mathbf{F}$, and the generators of $\mathbf{C}l$ clearly do not preserve these subspaces. So we will clearly need to introduce some new operators on the $\mathcal{W}$ side. It turns out that it suffices to introduce the following simple "shift" operator.

Definition 3.9. s is the operator of $\mathbf{F}$ defined by $s|\lambda', k\rangle = |\lambda', k+1\rangle$. Note that s corresponds to multiplication by q under the vector space isomorphism $\sigma : \mathbf{F} \rightarrow \mathbf{B}$.

Introduce the following power series, noticing that the coefficient of each power of n in each expression is a well defined operator on $\mathbf{F}$, since all but finitely many terms act as zero on any fixed $|\lambda\rangle$.

$$\psi(z) = \sum_{m \in \mathbb{Z}+1/2} z^m \pi_F(\psi_m) \tag{9}$$

$$\psi^*(z) = \sum_{m \in \mathbb{Z}+1/2} z^{-m} \pi_F(\psi_m^*) \tag{10}$$

$$\Gamma_+(z) = \exp \sum_{k \in \mathbb{Z}_{>0}} \frac{z^{-k}}{k} \pi_F(\alpha_k) \tag{11}$$

$$\Gamma_-(z) = \exp \sum_{k \in \mathbb{Z}_{>0}} \frac{z^{k}}{k} \pi_F(\alpha_{-k}). \tag{12}$$

Definition 3.10. $\mathrm{ch} : \mathbf{F} \to \mathbb{C}$ is the linear functional which takes $|\lambda\rangle = (\lambda', k)$ to the charge k.

The following is [4, Theorem 14.10], adjusted slightly to match our conventions.

Proposition 3.11.

$$\psi(z) = s z^{ch+1/2} \Gamma_-(z) \Gamma_+(z)^{-1}$$
$$\psi^*(z) = s^{-1} z^{-ch-1/2} \Gamma_-(z)^{-1} \Gamma_+(z).$$

This in principle expresses the fermionic operators ψ_m and ψ_m^* in terms of the α_k. Due to the appearance of the factor z^{ch} in these formulas, the expression for the action of ψ_m or ψ_m^* on $|\lambda\rangle$ in terms of the α_k depends on the charge of λ.

The proof of Proposition 3.11 is not trivial, although does not use much difficult machinery. It proceeds roughly as follows. Kac first shows that the right sides of the equations in Proposition 3.8 generate an algebra of operators on $\mathbf{F}$ which is isomorphic to $\mathcal{W}$ (this is not hard once one understands the imbedding of the matrix algebra a_∞ into $\widetilde{\mathbf{C}l}$, see Comment 3.28). This immediately implies the existence of some vector space isomorphism from $\mathbf{B}$ to $\mathbf{F}$ which satisfies Proposition 3.8. One then studies commutation relations between the α_k and the generating functions $\psi(z)$ and $\psi^*(z)$ to prove that this isomorphism also satisfies Proposition 3.11. The final step is to show that the isomorphism is given by Proposition 2.13. So really these notes are completely backwards!

Example 3.12. The action of α_k on the standard basis has a very simple description when the standard basis of $\mathbf{F}$ is indexed by Maya diagrams. If v is a Maya diagram, then $\alpha_k v$ is the sum of $(-1)^{j(v,v')} v'$ over all v' obtained from v by moving a single black bead to the right exactly k places, where $j(v, v')$ is the number of black beads that are "jumped" by the bead that moves. For example,

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(13)
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and

α_2(··· ○ ○ ○ ○ ○ ○ ○ ○ ● ○ ●|● ○ ○ ● ● ○ ● ● ● ● ● ···) =

(14)
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Comment 3.13. Fix $n > 0$. If the standard basis for $\mathbf{F}$ is indexed by charged partitions, the basis vectors which have non-zero coefficient in $\alpha_n|\lambda\rangle$ are exactly those charged partitions which are obtained from $|\lambda\rangle$ by adding a "ribbon" or "rim-hook" of length n to $|\lambda\rangle$. This is part of the reason that ribbons appear in the context of Fock space (see for instance [5]).

Comment 3.14. It may at first seem unfortunate that we need to introduce the power series in Equations (9)-(12), but it turns out that these power series themselves are very useful. Each has the property that it acts on any $v \in \mathbf{F}$ to give a Laurent power series is z with coefficients in $\mathbf{F}$. It follows that each of the products below is well defined as a map from $\mathbf{F}$ to Laurent power series in two variables with coefficients in $\mathbf{F}$. It follows from results in [4, Chapter 14] (see [7, Appendix B2] for this exact statement) that these power series satisfy the following simple commutation relations.

$$\Gamma_+(x)\Gamma_-(y) = (1 - xy)\Gamma_-(y)\Gamma_+(x) \tag{15}$$

$$\Gamma_+(x)\psi(z) = (1 - z^{-1}x)^{-1}\psi(z)\Gamma_+(x) \tag{16}$$

$$\Gamma_-(x)\psi(z) = (1 - xz)^{-1}\psi(z)\Gamma_-(x) \tag{17}$$

$$\Gamma_+(x)\psi^*(z) = (1 - z^{-1}x)\psi^*(z)\Gamma_+(x) \tag{18}$$

$$\Gamma_-(x)\psi^*(z) = (1 - xz)\psi^*(z)\Gamma_-(x). \tag{19}$$

Here $(1-a)^{-1}$ is always expanded as $1 + a + a^2 + \cdots$. Equality is interpreted as saying that, once each side is applied to any fixed $v \in \mathbf{F}$, all coefficients of the resulting power series agree. These facts have been put to great use in, for example, [8]. There Okounkov that Reshetikhin interpret the $\Gamma_\pm(z)$ as transition functions, and use the above commutation relations to find limit shapes and correlation function for systems of random plane partitions.

3D. **Fock space as a representation of $\mathfrak{gl}_{\mathbb{Z}+1/2}$ and a_∞.** Here we construct actions of the matrix Lie algebras $\mathfrak{gl}_{\mathbb{Z}+1/2}$ and a_∞ on $\mathbf{F}$. The actions of $\mathfrak{gl}_{\mathbb{Z}+1/2}$ on Fock space is constructed by imbedding it into the Clifford algebra $\mathbf{C}l$, and the action of a_∞ is constructed is the same way, except that one must imbed it in the completion $\widetilde{\mathbf{C}l}$. These actions can also easily be defined directly.

Definition 3.15. $M_{\mathbb{Z}+1/2}$ is the algebra of matrices with rows and columns indexed by $\mathbb{Z}+1/2$, in which all but finitely many entries are 0. Let $E_{m,n}$ denote the matrix with a single 1 in position (m,n) and zeros everywhere else.

Definition 3.16. $\mathfrak{gl}_{\mathbb{Z}+1/2}$ is the Lie algebra corresponding to $M_{\mathbb{Z}+1/2}$. That is, $\mathfrak{gl}_{\mathbb{Z}+1/2}$ is equal to $M_{\mathbb{Z}+1/2}$ as a vector space, and the Lie bracket is defined by

$$[X,Y] = XY - YX.$$

Definition 3.17. A Lie-associative map from a Lie algebra $\mathfrak{g}$ to an associative algebra A is a map σ such that, for all $X,Y \in \mathfrak{g}$, $\sigma([X,Y]) = \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X)$.

The following can easily be verified by directly checking relations.

Proposition 3.18. *(see [4]) There is a Lie-associative embedding of $\mathfrak{gl}_{\mathbb{Z}+1/2}$ into $\mathbf{C}l$ given by $E_{m,n} \to \psi_m \psi_n^*$.*

Hence $\mathbf{F}$ carries an action of $\mathfrak{gl}_{\mathbb{Z}+1/2}$. We would like to extend this to an action of the larger algebra of matrices with non-zero entries on only finitely many diagonals, but where infinitely many non-zero entries are allowed on each of those diagonals. This is almost possible since, for any standard basis vector $|\lambda\rangle$ and any $k \neq 0$, only finitely many of $\{E_{m,m+k}\}$ act on $|\lambda\rangle$ non-trivially. However, at $k = 0$ infinitely many of these terms act non-trivially, so the action of $\mathfrak{gl}_{\mathbb{Z}+1/2}$ on $\mathbf{F}$ does not extend to this larger algebra. To fix the problem, we must introduce a central extension.

Definition 3.19. Let $\mathfrak{gl}^c_{\mathbb{Z}+1/2}$ be the central extension of $\mathfrak{gl}_{\mathbb{Z}+1/2}$ by a central element c, with Lie bracket defined as follows. We use the notation $\overline{X}$ to mean the matrix X thought of as an element of $\mathfrak{gl}^c_{\mathbb{Z}+1/2}$.

$$[\overline{E}_{m,n}\overline{E}_{p,q}] = \begin{cases} E_{m,n}E_{p,q} - E_{p,q}E_{m,n} + \delta_{m,q}\delta_{n,p}c & \text{if } m > 0 \text{ and } n < 0 \\ E_{m,n}E_{p,q} - E_{p,q}E_{m,n} - \delta_{m,q}\delta_{n,p}c & \text{if } m < 0 \text{ and } n > 0 \\ E_{m,n}E_{p,q} - E_{p,q}E_{m,n} & \text{if } m \text{ and } n \text{ have the same sign.} \end{cases}$$

It is straightforward to check that this is in fact a central extension.

Comment 3.20. This central extension is trivial, since there is an isomorphism of Lie algebras $\mathfrak{gl}^c_{\mathbb{Z}+1/2} \to \mathfrak{gl}_{\mathbb{Z}+1/2} \oplus \mathbb{C}c$ given by

$$\overline{E}_{m,n} \to \begin{cases} E_{m,n} \text{ if } m \neq n \text{ or } m < 0 \\ E_{m,n} - c \text{ if } m = n \text{ and } m > 0. \end{cases}$$

However, it will be non-trivial once we allow certain matrices with infinitely many non-zero entries.

Proposition 3.21. *There is a Lie associative imbedding of $\mathfrak{gl}^c_{\mathbb{Z}+1/2}$ into $\mathbf{C}l$ defined by*

$$\overline{E}_{m,n} \to \psi_m \psi_n^* \quad \text{if } m \neq n \text{ or } m > 0 \tag{20}$$

$$\overline{E}_{n,n} \to -\psi_n^* \psi_n \quad \text{if } n < 0 \tag{21}$$

$$c \to 1. \tag{22}$$

In particular, the action of $\mathbf{Cl}$ *on* $\mathbf{F}$ *introduces an action of* $\mathfrak{gl}^c_{\mathbb{Z}+1/2}$ *on* $\mathbf{F}$.

Proof. Consider the action of $\mathfrak{gl}_{\mathbb{Z}+1/2} \oplus \mathbb{C}c$ on $\mathbf{F}$ by using the normal action of $\mathfrak{gl}_{\mathbb{Z}+1/2}$, and allowing c to act as 1. Then use the isomorphism from Comment 3.20. For $m \neq n$ or $m > 0$, the action of $\overline{E}_{m,n}$ is immediate. For n negative, $\overline{E}_{n,n}$ acts as $\psi_n\psi_n^* - 1$. By Equation (5) this is equal to $-\psi_n^*\psi_n$. □

Proposition 3.22. *For any standard basis vector* $|\lambda\rangle$ *of* $\mathbf{F}$ *and any fixed* $k \in \mathbb{Z}$, $\overline{E}_{m+k,m} \in M^c_{\mathbb{Z}+1/2}$ *satisfies* $E_{m+k,m}|\lambda\rangle = 0$ *for all but finitely many* m. □

Proof. This is obvious from Proposition 3.21, and the action of $\mathbf{C}l$ on $\mathbf{F}$. □

Definition 3.23. $\widetilde{\mathfrak{gl}}_{\mathbb{Z}+1/2}$ is the Lie algebra of $(\mathbb{Z}+1/2) \times (\mathbb{Z}+1/2)$ matrices such that, for all but finitely many $k \in \mathbb{Z}$ and all $m \in \mathbb{Z}+1/2$, $E_{m+k,m} = 0$. The Lie bracket is the standard bracket for matrices, $[X,Y] = XY - YX$.

Comment 3.24. In words, $\widetilde{\mathfrak{gl}}_{\mathbb{Z}+1/2}$ consists of matrices which may have infinitely many non-zero entries, but where all non-zero entries lie on a finite number of diagonals. It is clear that matrix multiplication is well defined on the set of such matrices.

Definition 3.25. a_∞ is the central extension of $\widetilde{\mathfrak{gl}}_{\mathbb{Z}+1/2}$, defined by

$$[E_{m,n}E_{p,q}] = \begin{cases} E_{m,n}E_{p,q} - E_{p,q}E_{m,n} + \delta_{m,q}\delta_{n,p}c & \text{if } m > 0 \text{ and } n < 0 \\ E_{m,n}E_{p,q} - E_{p,q}E_{m,n} - \delta_{m,q}\delta_{n,p}c & \text{if } m < 0 \text{ and } n > 0 \\ E_{m,n}E_{p,q} - E_{p,q}E_{m,n} & \text{if } m \text{ and } n \text{ have the same sign.} \end{cases}$$

Certain infinite sums of these matrix elements are present in a_∞, but it is straightforward to check that the central extension is well defined in the sense that the coefficient of c that appears in any bracket is finite. That this is a Lie algebra then follows from the fact that $\mathfrak{gl}^c_{\mathbb{Z}+1/2}$ is a Lie algebra (and can also be directly verified).

Proposition 3.26. *The imbedding of* $\mathfrak{gl}^c_{\mathbb{Z}+1/2}$ *into* $\mathbf{C}l$ *extends to an embedding of* a_∞ *into* $\widetilde{\mathbf{C}l}$. *In particular,* $\mathbf{F}$ *is a representation of* a_∞.

Proof. This follows immediately from the definition of $\widetilde{\mathbf{C}l}$. □

Comment 3.27. a_∞ is a non-trivial central extension of $\widetilde{\mathfrak{gl}}_{\mathbb{Z}+1/2}$. The action of $\mathfrak{gl}_{\mathbb{Z}+1/2}$ on $\mathbf{F}$ does not extend to an action of $\widetilde{\mathfrak{gl}}_{\mathbb{Z}+1/2}$ on $\mathbf{F}$, so the extension is crucial.

Comment 3.28. The images of the α_m in $\widetilde{\mathbf{C}l}$ from Proposition 3.8 are naturally contained in the image of a_∞; they are matrices with all 1 on one diagonal, and all 0 everywhere else. It is easy to see that these matrices have the same commutation relations as the generators of $\mathcal{W}$, which in fact proves that the map from Proposition 3.8 gives an action of $\mathcal{W}$ on $\mathbf{F}$.

3E. **Fock space as a representation of $\widehat{\mathfrak{sl}}_\ell$ and $\widehat{\mathfrak{gl}}_\ell$.** It turns out that each $\mathbf{F}^{(m)}$ carries an action of $\widehat{\mathfrak{sl}}_\ell$. $\mathbf{F}^{(m)}$ is not irreducible under this action, but all the irreducible components are very similar; they are isomorphic as representations of the derived algebra $\widehat{\mathfrak{sl}}'_\ell$. This is explained by the fact that there is a larger affine algebra $\widehat{\mathfrak{gl}}_\ell$ which acts on $\mathbf{F}^{(m)}$, and $\mathbf{F}^{(m)}$ is irreducible under the action of $\widehat{\mathfrak{gl}}_\ell$.

Definition 3.29. $\mathfrak{sl}_\ell$ is the Lie algebra of $\ell \times \ell$ matrices with trace 0. $\mathfrak{gl}_\ell$ is the Lie algebra of all $\ell \times \ell$ matrices. We use the notation $X_{i,j}$ to denote the matrix with a 1 in position (i,j) and zeros everywhere else (this is to distinguish from $E_{m,n} \in \mathfrak{gl}_{\mathbb{Z}+1/2}$).

Definition 3.30. The affine Lie algebra $\widehat{\mathfrak{sl}}_\ell$ is, as a vector space $(\mathfrak{sl}_\ell \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$. The Lie bracket is defined by

$$[X \otimes t^n, Y \otimes t^m] = (XY - YX) \otimes t^{m+n} + m\delta_{m-n}(X,Y)c \tag{23}$$

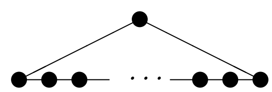
$$[d, X \otimes t^n] = nX \otimes t^n \tag{24}$$

$$c \text{ is central.} \tag{25}$$

Here (X,Y) is the killing form $tr(ad(X)ad(Y))$, re-normalized so that $(X_{1,\ell}, X_{\ell,1}) = 1$.

Definition 3.31. $\widehat{\mathfrak{sl}}'_\ell$ is the derived algebra of $\widehat{\mathfrak{sl}}_\ell$. Note that as a vector space over $\mathbb{C}$, $\widehat{\mathfrak{sl}}'_n$ is $(\mathfrak{sl}_\ell \otimes \mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}c$.

Proposition 3.32. *(see* [4, Chapter 7]*) $\widehat{\mathfrak{sl}}_\ell$ is isomorphic to the affine Kac-Moody algebra with the ℓ-node Dynkin diagram*



In particular, $\widehat{\mathfrak{sl}}'_\ell$ is generated by Chevalley generators $\tilde{E}_{\bar{i}}, \tilde{F}_{\bar{i}}$ for $i \in \mathbb{Z}/\ell\mathbb{Z}$. Furthermore, the isomorphism can be chosen such that the following hold:

$$\tilde{E}_{\bar{i}} \to X_{i+1,i} \otimes t^0 \quad i = 1, \dots \ell - 1 \tag{26}$$

$$\tilde{F}_{\bar{i}} \to X_{i,i+1} \otimes t^0 \quad i = 1, \dots \ell - 1 \tag{27}$$

$$\tilde{E}_{\bar{0}} \to X_{1,\ell} \otimes t^1 \tag{28}$$

$$\tilde{F}_{\bar{0}} \to X_{\ell,1} \otimes t^{-1}. \tag{29}$$

Comment 3.33. For $1 \in \mathbb{Z}/\ell\mathbb{Z}$, $\tilde{E}_{\bar{i}}$ is sent to a lower triangular matrix, not an upper triangular one. This is done so that the natural way of letting $\widehat{\mathfrak{sl}}_\ell$ act on $\mathbf{F}$ will give a positive level representation instead of a negative level representation.

Proposition 3.34. *(see* [4, Chapter 14]*)*

(i) *There is a Lie-associative embedding of $\widehat{\mathfrak{sl}}'_\ell/(c-1)$ into $a_\infty \subset \widetilde{Cl}$ given by, for $X_{i,j} \in \widehat{\mathfrak{sl}}_\ell$,*

$$X_{i,j} \otimes t^m \to \sum_{k \in \mathbb{Z}+1/2} \psi_{i-1/2+m\ell+k\ell} \psi^*_{j-1/2+k\ell}$$

Thus $\mathbf{F}$ carries an action of $\widehat{\mathfrak{sl}}'_\ell$.

(ii) *Each $\mathbf{F}^{(m)}$ is preserved by this action.*

(iii) *Every irreducible component of $\mathbf{F}^{(m)}$ under this action is isomorphic to the highest weight representation $V_{\Lambda_{\bar{m}}}$ as a representation of $\widehat{\mathfrak{sl}}'_\ell$.*

Proof. Parts (i) and (ii) are proven in [4]. We delay the proof of Part (iii) since it follows immediately from the discussion of the $\widehat{\mathfrak{gl}}_\ell$ case below. □

The action of the Chevalley generators of $\widehat{\mathfrak{sl}}_\ell$ on $\mathbf{F}$ can be described combinatorially:

Proposition 3.35. *Let λ be a charged partition. Then for all $i \in \mathbb{Z}/\ell\mathbb{Z}$,*

$$\tilde{E}_{\bar{i}}|\lambda\rangle := \sum_{\substack{\lambda\setminus\mu \text{ is an} \\ i \text{ colored box}}} |\mu\rangle \qquad \tilde{F}_{\bar{i}}|\lambda\rangle := \sum_{\substack{\mu\setminus\lambda \text{ is an} \\ i \text{ colored box}}} |\mu\rangle. \tag{30}$$

Here the boxes are colored as in Figure 1.

Proof. This is immediate from the definition of the actions of $\tilde{E}_{\bar{i}}$ and $\tilde{F}_{\bar{i}}$. □

Proposition 3.36. *The action of $\widehat{\mathfrak{sl}}'_\ell$ on $\mathbf{F}$ can be extended to an action of $\widehat{\mathfrak{sl}}_\ell$ on $\mathbf{F}$ by letting d act on $|\lambda\rangle$ as multiplication by*

$$k(\lambda) := \#\ \textit{squares in the finite part of } \lambda \textit{ colored } c_0,$$

where the coloring is as in Figure 1.

Proof. It suffices to prove that d commutes with E_i, F_i for $i \neq 0$, $E_0 d = (d+1)E_0$, and $F_0 d = (d-1)F_0$. These are all straightforward. □

The action of $\widehat{\mathfrak{sl}}_\ell$ on $\mathbf{F}^{(m)}$ can in fact be extended to an action of a larger affine algebra $\widehat{\mathfrak{gl}}_\ell$ as follows

Definition 3.37. The affine Lie algebra $\widehat{\mathfrak{gl}}_\ell$ is, as a vector space $(\mathfrak{gl}_\ell \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$. The Lie bracket is defined by

$$[X \otimes t^n, Y \otimes t^m] = (XY - YX) \otimes t^{m+n} + m\delta_{m-n}(X,Y)c \tag{31}$$

$$[d, X \otimes t^n] = nX \otimes t^n \tag{32}$$

$$c \text{ is central.} \tag{33}$$

Here (X,Y) is the killing form $tr(ad(X)ad(Y))$, renormalized so that $(X_{1,\ell}, X_{\ell,1}) = 1$. Let $\widehat{\mathfrak{gl}}'_\ell$ denote $(\mathfrak{gl}_\ell \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c$ (which is clearly a Lie-subalgebra).

Proposition 3.38.

(i) *There is an embedding of $\widehat{\mathfrak{gl}}'_\ell/(c-1)$ into $\widetilde{Cl}$ given by, for $X_{i,j} \in \widehat{\mathfrak{gl}}_\ell$,*

$$X_{i,j} \otimes t^m \to \sum_{k \in \mathbb{Z}+1/2} \psi_{i-1/2+m\ell+k\ell}\psi^*_{j-1/2+k\ell}$$

Thus $\mathbf{F}$ carries an action of $\widehat{\mathfrak{gl}}'_\ell$ which agrees with from Proposition 3.34 on the subalgebra $\widehat{\mathfrak{sl}}'_\ell$.

(ii) *The action can be extended as in Proposition 3.36 to give an action of the whole affine algebra $\widehat{\mathfrak{gl}}_\ell$.*

(iii) *Each $\mathbf{F}^{(m)}$ is preserved by this action, and forms a single irreducible representation of $\widehat{\mathfrak{gl}}_\ell$.*

Proof. Parts (i) and (ii) follow exactly as in the $\widehat{\mathfrak{sl}}_\ell$ case. It is also clear that each $\mathbf{F}^{(m)}$ is preserved under the action of $\widehat{\mathfrak{gl}}_\ell$. It remains to show that each $\mathbf{F}^{(m)}$ is irreducible. For this, we use that fact that $\mathbf{F}^{(m)}$ is irreducible as a representation of $\mathcal{W}$. Thus it suffices to show that for all k, $\pi_{\mathbf{F}}\alpha_k$ is in the algebra of operators generated by $\widehat{\mathfrak{gl}}_\ell$. For each residue $\bar{j}$ mod ℓ, let

$$C_{\bar{j}} = \sum_{a=1}^{\ell} X_{a+j,a}. \tag{34}$$

For example, if $\ell = 4$ then

$$C_{\bar{0}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C_{\bar{3}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{35}$$

It is clear from definitions that for all $k \neq 0$ the actions of $C_{\bar{k}} \otimes t^{\lfloor k/\ell \rfloor}$ and α_k on $\mathbf{F}$ agree. □

Comment 3.39. One can show that $\widehat{\mathfrak{gl}}'_\ell$ is isomorphic as a Lie algebra to $\widehat{\mathfrak{sl}}'_\ell \oplus \mathcal{W} \oplus \mathbb{C}\mathbf{1}/(c_{\widehat{\mathfrak{sl}}_\ell} = c_{\mathcal{W}})$, where $c_{\widehat{\mathfrak{sl}}_\ell} = c_{\mathcal{W}}$ means the central elements c in $\widehat{\mathfrak{sl}}'_\ell$ and $\mathcal{W}$ are identified. Here the copy of $\mathcal{W}$ is the span of $\mathrm{Id} \otimes z^k$ for all $k \neq 0$, and $\mathbf{1}$ is $\mathrm{Id} \otimes t^0$. For this reason, Proposition 3.38 implies that there is an action of $\mathcal{W}$ on $\mathbf{F}$ which commutes with the action of $\widehat{\mathfrak{sl}}'_\ell$. This commuting action has been noticed by many people.

Note that Proposition 3.38 along with Comment 3.39 completes the proof of Proposition 3.34 part (iii).

3F. **The q-deformed Fock space $\mathbf{F}_q$; a representation of $U_q(\widehat{\mathfrak{sl}}_\ell)$.** We now describe the Misra-Miwa Fock space for $U_q(\widehat{\mathfrak{sl}}_\ell)$. This is a representation of $U_q(\widehat{\mathfrak{sl}}_\ell)$ originally developed by Misra and Miwa [6] using work of Hayashi [3] (see also [1, Chapter 10]). It can be thought of as a q-deformation of the action of $\widehat{\mathfrak{sl}}_\ell$ on $\mathbf{F}$ described in Section 3E. We refer to [2] for the definition of $U_q(\widehat{\mathfrak{sl}}_\ell)$. To fit with our conventions, we use $\tilde{E}_{\bar{i}}$ and $\tilde{F}_{\bar{i}}$ to denote the Chevalley generators, which in [2] are denoted by X_i^+ and X_i^-.

Definition 3.40. Let $\mathbf{F}_q := \mathbf{F} \otimes_{\mathbb{C}} \mathbb{C}(q)$.

Definition 3.41. Let λ and μ be charged partitions such that λ is contained in μ, and $\mu\backslash\lambda$ is a single box. Set

(i) $A_i(\lambda) := \{c_i$ colored boxes $n |\ \lambda \cup n$ is a partition$\}$.
(ii) $R_i(\lambda) := \{c_i$ colored boxes $n |\ \lambda\backslash n$ is a partition$\}$.
(iii) $N_i^a(\mu\backslash\lambda) := |\{n \in A_i(\lambda) |$n is to the left of $\mu\backslash\lambda\}| - |\{n \in R_i(\lambda) |$ n is to the left of $\mu\backslash\lambda\}|$.
(iv) $N_i^r(\mu\backslash\lambda) := |\{n \in A_i(\lambda) |$n is to the right of $\mu\backslash\lambda\}| - |\{n \in R_i(\lambda) |$ n is to the right of $\mu\backslash\lambda\}|$.

Theorem 3.42. *(See [1, Theorem 10.6]) There is an action of $U_q(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_q$ defined by*

$$\tilde{E}_{\bar{i}}|\lambda\rangle := \sum_{\substack{\lambda\backslash\mu \text{ is an} \\ i \text{ colored box}}} q^{-N_i^r(\lambda\backslash\mu)}|\mu\rangle \qquad \tilde{F}_{\bar{i}}(b_\lambda) := \sum_{\substack{\lambda\backslash\mu \text{ is an} \\ i \text{ colored box}}} q^{N_i^a(\mu\backslash\lambda)}|\mu\rangle. \tag{36}$$

$\mathbf{F}_q$ is not irreducible, but decomposes in the same way as $\mathbf{F}$ decomposes into irreducible representations of $\widehat{\mathfrak{sl}}_\ell$. It is clear by comparing with Proposition 3.35 that, when q is set to 1, one recovers the normal action of $\widehat{\mathfrak{sl}}_\ell$ on Fock space.

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